EECS 451 POLES & ZEROS AND FREQUENCY RESPONSE

Given: Locations of zeros $\{z_1 \dots z_M\}$ and poles $\{p_1 \dots p_N\}$ of a filter.
Goal: *Shape* of gain function=$|H(\omega)|$=magnitude of frequency response.

H(z): $H(z) = \frac{(z-z_1)(z-z_2)\dots(z-z_M)}{(z-p_1)(z-p_2)\dots(z-p_N)}$ (ignore any constant in front).
H(w): $|H(\omega)| = [|e^{j\omega} - z_1| \dots |e^{j\omega} - z_M|]/[|e^{j\omega} - p_1| \dots |e^{j\omega} - p_N|]$.
What does each of these terms contribute to gain $|H(\omega)|$?

Zeros: Let n^{th} zero $z_n = e^{j\omega_n}$. Then $|e^{j\omega} - z_n| = 0$ at $\omega = \omega_n$.

- Gain $|H(\omega_o)| = 0$ if there is a zero at $e^{j\omega_o}$ (on the unit circle $|z| = 1$).
- Gain $|H(\omega_o)| \approx 0$ if there's a zero at $Ae^{j\omega_o}, A \approx 1$ (near the unit circle).

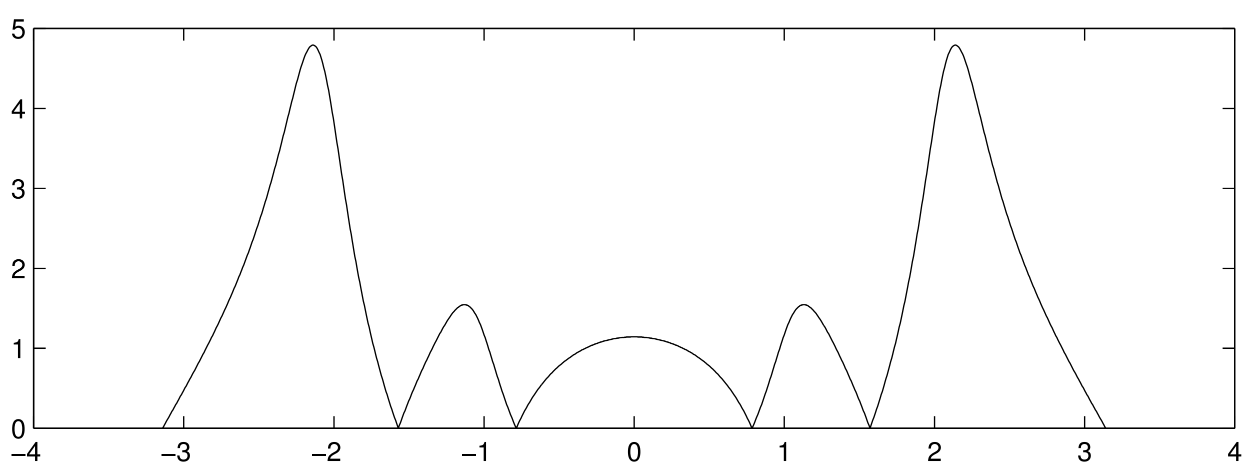
Poles: Let n^{th} pole $p_n = Ae^{j\omega_n}, A \approx 1$. Then $\frac{1}{|e^{j\omega}-p_n|} = \frac{1}{|A-1|}$ at $\omega = \omega_n$.

- Gain $|H(\omega_o)|$ is large if there is a pole at $Ae^{j\omega_o}$ (near $|z| = 1$).

1. Start on the unit circle at $\omega = 0 \rightarrow z = e^{j\omega} = 1$.
2. Trace along the unit circle counterclockwise (increasing ω)
3. When pass a zero at $Ae^{j\omega_m}, A \approx 1$, gain dips at ω_m.
4. When pass a pole at $Ae^{j\omega_n}, A \approx 1$, gain peak at ω_n.
5. On unit circle $z = e^{j\omega}$: $z = 1 \rightarrow DC; z = j \rightarrow \omega = \frac{\pi}{2}; z = -1 \rightarrow \omega = \pi$.

EX: **Zeros:** $\{e^{\pm j\pi/4}, e^{\pm j\pi/2}, e^{j\pi}\}$. **Poles:** $\{0.8e^{\pm j\pi/3}, 0.8e^{\pm j2\pi/3}\}$. Gain:

1. Note gain dips to zero at $\omega = \pm\frac{\pi}{4} = \pm 0.785$ and $\omega = \pm\frac{\pi}{2} = \pm 1.57$.
2. Note peaks at $\omega = \pm\frac{\pi}{3} = \pm 1.05$ and $\omega = \pm\frac{2\pi}{3} = \pm 2.10$
3. The closer a zero or pole is to unit circle, the sharper the peak or dip.
4. The closer pole is to unit circle, the longer $h[n]$ takes to decay to zero.
5. Need all poles inside the unit circle for the system to be BIBO stable.



SIMPLE EXAMPLE OF A DSP SYSTEM

Given: Continuous-time signal $\tilde{x}(t) = \sin(250\pi t) + \sin(400\pi t)$ (125 & 200 Hz).

Goal: Use DSP to **keep** the 125 Hz tone and **filter out** the 200 Hz tone.

Sample: **Nyquist** rate=2(200 Hz)=400 Hz. **Oversample** at 1000 Hz.
1000 samples/second $\Leftrightarrow$ sample every 0.001 second $\Leftrightarrow t = 0.001n$.
$t = 0.001n \to x(n) = \tilde{x}(0.001n) = \sin(0.25\pi n) + \sin(0.4\pi n)$ (see below)

Inter-polate: Since output is sum of sinusoids, can interpolate using sinusoids:
$t = 0.001n \to n = 1000t$. Substitute $n = 1000t$ to perform D/A.

Note: In general, we must use the interpolation formula

$$\tilde{y}(t) = \sum_{-\infty}^{\infty} y(n) \frac{2B}{S} \frac{\sin B(t-nT)}{B(t-nT)} = \left[\sum y(n)\delta(t-nT)\right] * \left(\frac{\sin(Bt)}{\pi t} T\right).$$

where: $T = 0.001, \quad B = 2\pi 200, \quad S = \frac{2\pi}{T} = 2000\pi$ (see *Sampling* handout).

System: $\tilde{x}(t) \to \underbrace{\overline{|\text{LPF : 500Hz}|}}_{\text{antialias filter}} \to \underbrace{\overline{|\text{A/D}|}}_{\text{sampler}} \to \underbrace{\overline{|h(n)|}}_{\text{filter}} \to \underbrace{\overline{|\text{D/A}|}}_{\text{inter-polate}} \to \tilde{y}(t)$

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h(n): Want to **keep** $\omega = 0.25\pi$ and **eliminate** $\omega = 0.4\pi$.

Zeros: Eliminate $\omega = 0.4\pi \to Zeros : z = e^{j\omega} = e^{\pm j0.4\pi}$.

Poles: Almost cancel zeros$\to Poles : z = 0.9e^{\pm j0.4\pi}$.

Why? Need poles inside the unit circle for stable and causal $h(n)$.

H(z): Transfer function: $\frac{(z-e^{j0.4\pi})(z-e^{-j0.4\pi})}{(z-0.9e^{j0.4\pi})(z-0.9e^{-j0.4\pi})} = \frac{z^2-2z\cos(0.4\pi)+1}{z^2-2z0.9\cos(0.4\pi)+0.81}$.

Difference Equation: $\frac{Y(z)}{X(z)} = H(z) = \frac{z^2-0.62z+1}{z^2-0.56z+0.81}[\cos(0.4\pi) = 0.31]$. Cross-multiply$\to$
$y(n+2) - 0.56y(n+1) + 0.81y(n) = x(n+2) - 0.62x(n+1) + x(n)$