EECS 451 NUMERICAL ANALYSIS USING THE DTFT

Given: A continuous-time signal $x(t)$ bandlimited to B Hz.

Goal: To compute its derivative $\frac{dx}{dt}$ and indefinite integral $\int_0^t x(u)du$.

Plan: Sample $x(t) \to x(n\Delta)$; *digital filter* $x(n\Delta) \to \boxed{h(n\Delta)} \to y(n\Delta)$.

Sample: Sample every $\Delta \Leftrightarrow$sampling rate $\frac{1}{\Delta} >> 2B \Leftrightarrow 2B\Delta << 1$.

System: $x(t) \to \underbrace{\boxed{LPF : B\,\text{Hz}}}_{\text{antialias filter}} \to \underbrace{\boxed{A/D}}_{\text{sampler}} \to \underbrace{\boxed{h(n)}}_{\text{digital filter}} \to \underbrace{\boxed{D/A}}_{\text{inter-polate}} \to \frac{dx}{dt}$.

Computation of Derivative of $x(t)$ from $x(n\Delta)$

Forward: $y(n\Delta) \approx \frac{\Delta x}{\Delta t} = [x((n+1)\Delta) - x(n\Delta)]/\Delta$. Compute $y(t) = \frac{dx}{dt}$

Backward: $y(n\Delta) \approx \frac{\Delta x}{\Delta t} = [x(n\Delta) - x((n-1)\Delta)]/\Delta$. using *finite differences*

Central: $y(n\Delta) \approx \frac{\Delta x}{\Delta t} = [x((n+1)\Delta) - x((n-1)\Delta)]/2\Delta$. Ref: Math 116.

DTFT: Compute DTFTs of each difference operator, then scale by Δ:

Forward: $H_F(e^{j\omega}) = (e^{j\omega} - 1) \quad \to H_F(e^{j\omega\Delta})/\Delta = (e^{j\omega\Delta} - 1)/\Delta$.

Backward: $H_B(e^{j\omega}) = (1 - e^{-j\omega}) \quad \to H_B(e^{j\omega\Delta})/\Delta = (1 - e^{-j\omega\Delta})/\Delta$.

Central: $H_C(e^{j\omega}) = (e^{j\omega} - e^{-j\omega})/2 \to H_C(e^{j\omega\Delta})/\Delta = (e^{j\omega\Delta} - e^{-j\omega\Delta})/2\Delta$.

Plots: See plots below of frequency responses $|H(e^{j\omega})|$ vs. ω for all of these. All approximate *ideal differentiator* $j\omega$ for small ω (zero at origin).

Lesson: Plots show amount of oversampling necessary for good results.

Series: $\frac{1}{\Delta}H_F(e^{j\omega\Delta}) \approx \frac{1}{\Delta}(1+j\omega\Delta-1) = j\omega = \frac{1}{\Delta}(1-(1-j\omega\Delta)) \approx \frac{1}{\Delta}H_B(e^{j\omega\Delta})$.

Central: $\omega\Delta << 1 \to H_C(e^{j\omega\Delta})/\Delta \approx ((1+j\omega\Delta) - (1-j\omega\Delta))/2\Delta = j\omega$.

Error: Forward and backward: $O(\omega^2\Delta)$. Central: $O(\omega^3\Delta^2)$ (better).

But: See below! Note that $H_C(z)$ has a zero at $z = -1 \Leftrightarrow \omega = \pi$!

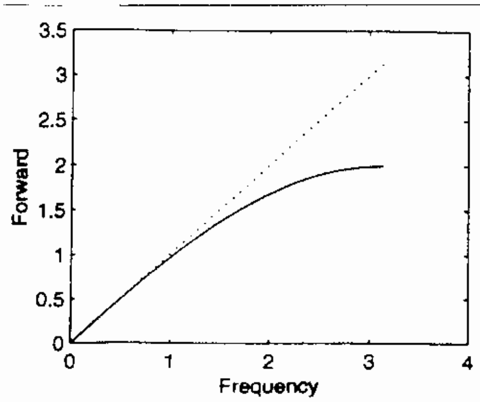
IIR filter: Let $H_{IIR}(e^{j\omega}) = j\omega, |\omega| < \pi \to H_{IIR}(e^{j\omega\Delta})/\Delta = j\omega\frac{\Delta}{\Delta}, |\omega| < \frac{\pi}{\Delta}$:

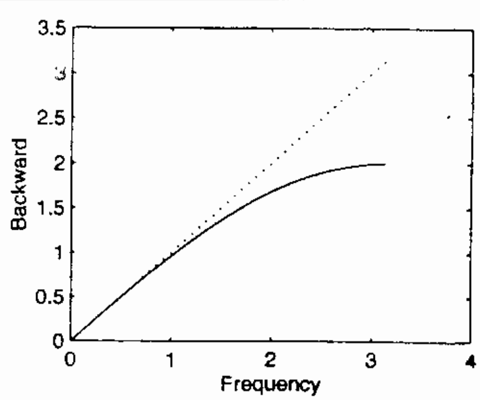
DTFT: $h_{IIR} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(j\omega)e^{j\omega n}d\omega = (-1)^n/n, n \neq 0$. Implements $\frac{d}{dt}$ exactly.

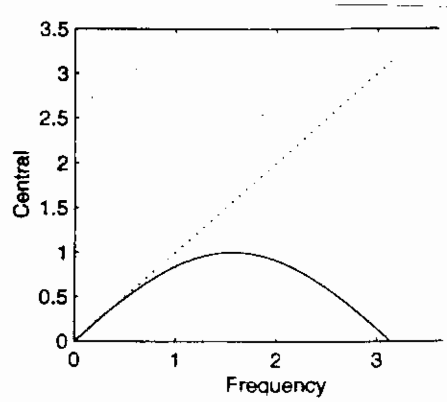
But: IIR filter$\to$need *all* data to implement: trouble with end effects.

Truncate: Truncating$\to$ $h_{IIR}(n) \approx (\delta(n+1) - \delta(n-1)) \to$2(central difference)!

FIR filter: **>>H=remez(N-1,[0,1],[0,1],'differentiator');freqz(H,1).**







EECS 451 COMPUTATION OF $\int x(t)dt$ FROM $x(n\Delta)$

Compute integral $y(t) = \int_0^t x(u)du$ from $x(n\Delta)$ using integration "rules" from Math 116. Then the definite integral $\int_a^b x(t)dt = y(b) - y(a)$.

Rectangle: $y(n\Delta) = \sum_{i=0}^{n-1} x(i\Delta)\Delta \Leftrightarrow y(n\Delta) = y((n-1)\Delta) + x((n-1)\Delta)\Delta$.

Trapezoid: Same as rectangle rule except halve $x(0)$ and $x(n)$ (now include $x(n)$).
Difference equation: $y(n\Delta) = y((n-1)\Delta) + [x(n\Delta) + x((n-1)\Delta)]\Delta/2$.

Simpson's: $y(n\Delta) = y((n-2)\Delta) + [x(n\Delta) + 4x((n-1)\Delta) + x((n-2)\Delta)]\Delta/3$.
Note alternate weightings by 4 and 2; this is a *quadratic* approximation

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DTFT: Compute DTFTs of integration operators, then scale by Δ (omitted):

Rectangle: $H_R(e^{j\omega}) = DTFT\{h(n) = u(n-1)\} = e^{-j\omega}/(1 - e^{-j\omega}) \approx 1/(j\omega)$.

Trapezoid: $H_T(e^{j\omega}) = \frac{1}{2}\frac{1+e^{-j\omega}}{1-e^{-j\omega}} \approx 1/(j\omega)$. **Poles:** +1. **Zeros:** -1.

Simpson's: $H_S(e^{j\omega}) = \frac{1}{3}\frac{1+4e^{-j\omega}+e^{-j2\omega}}{1-e^{-j2\omega}} \approx \frac{1}{j\omega}$. **Poles:** ± 1. **Zeros:** -3.732,-0.268.

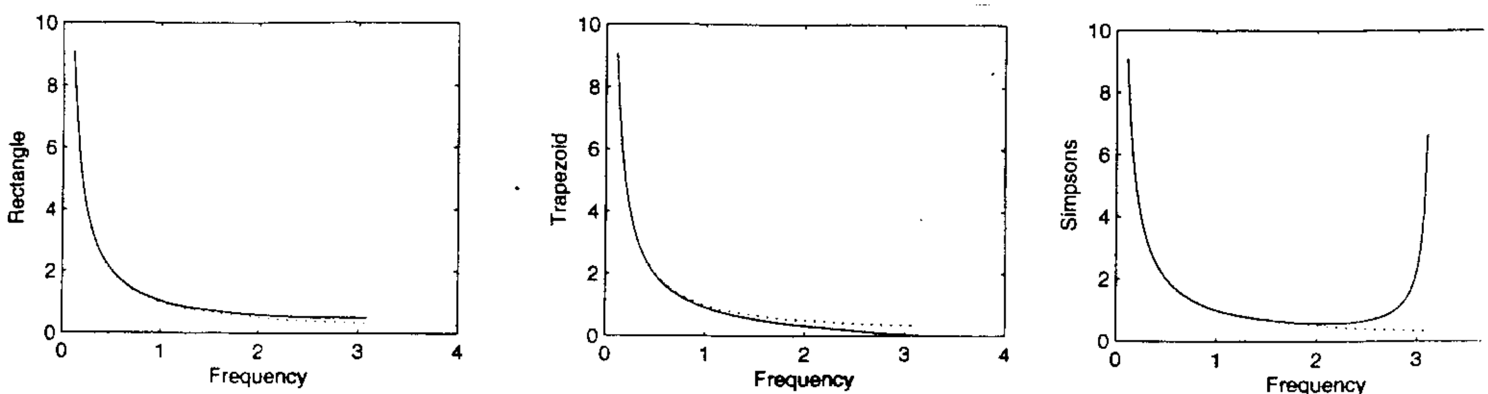
Note: Now little insight gained from series approximations.

Performance: See plots below of frequency responses $|H(e^{j\omega})|$ vs. ω for all of these.
All approximate the ideal integrator $\frac{1}{j\omega}$ for small ω (pole at origin).

Lesson: Simpson's rule can **blow up** if don't oversample enough!

Why? $H_S(z)$ has a *pole* at $z = -1 \Leftrightarrow \omega = \pi$! (Highest discrete-time freq.)
(All methods have a pole at the origin, as does the ideal integrator.)

EX: Use Simpson's rule to compute $\int_0^{10} \cos(\pi t)dt = 0$ using $\Delta = 1$:
$y(10) = \frac{1}{3}(1-4+2-4+2-4+2-4+2-4+1) = -3.33!$ $y(100) = -33.3!$

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clear; W=linspace(0,pi); HF=abs(exp(j*W)-1);
HB=abs(1-exp(-j*W)); HC=abs(exp(j*W)-exp(-j*W))/2;
subplot(2,2,1),plot(W,HF,'-',W,W,':'),xlabel('Frequency'),ylabel('Forward')
subplot(2,2,2),plot(W,HB,'-',W,W,':'),xlabel('Frequency'),ylabel('Backward')
subplot(2,2,3),plot(W,HC,'-',W,W,':'),xlabel('Frequency'),ylabel('Central')

clear; W=linspace(0.11,pi-0.05); %Avoid very small omega for display purposes.
HR=1./abs(1-exp(-j*W)); HT=abs((1+exp(-j*W))./(1-exp(-j*W)))/2;
HS=abs((1+4*exp(-j*W)+exp(-2*j*W))./(1-exp(-2*j*W)))/3;
subplot(2,2,1),plot(W,HR,'-',W,1./W,':'),xlabel('Frequency'),ylabel('Rectangle')
subplot(2,2,2),plot(W,HT,'-',W,1./W,':'),xlabel('Frequency'),ylabel('Trapezoid')
subplot(2,2,3),plot(W,HS,'-',W,1./W,':'),xlabel('Frequency'),ylabel('Simpsons')
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