DEF: $X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt; \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$

Note: (1) Sign change in $e^{\pm j\omega t}$; (2) Integrate over t vs. ω; (3) Factor of $\frac{1}{2\pi}$.

Need: $\int_{-\infty}^{\infty} |x(t)|dt < \infty \Leftrightarrow x(t)$ "absolutely integrable" for $X(\omega)$ to exist.

Huh? Splits $x(t)$ into frequency components~*prism*; recombine into $x(t)$.

Also: Regard as Fourier series expansion of $x(t)$ having period $T \to \infty$. Then $\Delta\omega = \frac{2\pi}{T} \to 0$: continuous spectrum. Units of $X(\omega)$: seconds.

Also: $\mathcal{F}\{x(t)\} = \mathcal{L}\{x(t)\}|_{s=j\omega}$ where $\mathcal{L}$ is the **2**-sided Laplace transform.

BASIC PROPERTIES

1. $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \to \boxed{h(t)} \to y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t}d\omega.$
2. **Linear:** $\mathcal{F}\{ax(t) + by(t)\} = a\mathcal{F}\{x(t)\} + b\mathcal{F}\{y(t)\}$ for constants a, b.

EX: $\mathcal{F}\{\delta(t)–ae^{-at}u(t)\}=1–a\frac{1}{j\omega+a}=\frac{j\omega+a}{j\omega+a} - \frac{a}{j\omega+a}=\frac{j\omega}{j\omega+a} \approx \frac{j\omega}{a}$ for $\omega << a$.

3. **Convolution:** $\mathcal{F}\{\int_{-\infty}^{\infty} x(u)y(t-u)du\} = \mathcal{F}\{x(t)\}\mathcal{F}\{y(t)\}$.

EX: $e^{-t}u(t)*e^{-2t}u(t)=\mathcal{F}^{-1}\{\frac{1}{j\omega+1}\frac{1}{j\omega+2}\} = \mathcal{F}^{-1}\{\frac{1}{j\omega+1}-\frac{1}{j\omega+2}\}=[e^{-t}–e^{-2t}]u(t)$.

4. **Time scaling:** $\mathcal{F}\{x(at)\} = \frac{1}{|a|}X(\frac{\omega}{a})$ for any constant a. Try $\cos(\omega_o t)$.

Also: **Time reversal:** $\mathcal{F}\{x(-t)\} = X(-\omega)$ and $\mathcal{F}\{x(t)^*\} = X(-\omega)^*$.

5. **Time delay:** $\mathcal{F}\{x(t-D)\} = X(\omega)e^{-j\omega D}$. Note same signs.

Also: (1) Magnitude same; (2) Linear (in ω) phase shift: "linear phase."

6. **Modulation:** $\mathcal{F}\{x(t)e^{jat}\} = X(\omega - a)$. Note different signs.

EX: $\mathcal{F}\{x(t)\cos(\omega_o t)\} = \frac{1}{2}X(\omega-\omega_o) + \frac{1}{2}X(\omega+\omega_o)$. Shifts spectrum. This one equation will form the basis of our study of communications.

7. **Differentiation:** $\mathcal{F}\{\frac{dx}{dt}\} = j\omega X(\omega)$. Assumes no initial condition.

Why? $\frac{dx}{dt}=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{de^{j\omega t}}{dt}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)e^{j\omega t}d\omega=\mathcal{F}^{-1}\{j\omega X(\omega)\}$.

Note: If $x(t)$ includes $0.000001\sin(10^{12}t)$ then $\frac{dx}{dt}$ includes $1000000\cos(10^{12}t)$.

Use: Note h(t)=$\delta(t)–ae^{-at}u(t)$ has frequency response $H(\omega)$ computed above.

8. **Time multiplication:** $\mathcal{F}\{tx(t)\} = -\frac{dX(j\omega)}{d(j\omega)}$. $\quad \mathcal{F}\{te^{-at}u(t)\} = [\frac{1}{j\omega+a}]^2$.
9. **DC:** (frequency=0) X(0)=$\int_{-\infty}^{\infty} x(t)dt$=average; x(0)=$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$.
10. **Duality:** If $\mathcal{F}\{x(t)\} = X(\omega)$, then $\mathcal{F}\{X(t)\} = 2\pi x(-\omega)$.

CONJUGATE SYMMETRY AND EVEN AND ODD FUNCTIONS

1. $x(t)$ real$\to$ $X(-\omega) = X^*(\omega)$: Called "conjugate symmetry" of $X(\omega)$.
   a. $Re[X(\omega)] = +Re[X(-\omega)] = +\int_{-\infty}^{\infty} x(t)\cos(\omega t)dt$: even function.
   b. $Im[X(\omega)] = -Im[X(-\omega)] = -\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$: odd function.
   c. $|X(\omega)| = |X(-\omega)|$: Fourier magnitude is an even function of ω.
   d. $Arg[X(\omega)] = -Arg[X(-\omega)]$: Argument (phase) is odd function.

2. Let $x(t) = x_e(t) + x_o(t)$ where $x_e(t) = \frac{x(t)+x(-t)}{2}$=even part of $x(t)$. (note can always do this) and $x_o(t) = \frac{x(t)-x(-t)}{2}$=odd part of $x(t)$.

Then: $Re[X(\omega)] = \mathcal{F}\{x_e(t)\}$ and $j\,Im[X(\omega)] = \mathcal{F}\{x_o(t)\}$=pure imaginary.

3. $x(t)$ real and even function $\Leftrightarrow$ $X(\omega)$ real and even function.
4. $x(t)$ discrete/periodic $\Leftrightarrow$ $X(\omega)$ periodic/discrete (Fourier series).

5a. **Parseval:** $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)d\omega$; note $y(t) = x(t)$:

5b. **Energy:** $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(2\pi f)|^2 df$.

EX: Show energy of $e^{-at}u(t)$ is same in the time and frequency domains:
$\int_0^{\infty} e^{-2at}dt = \frac{1}{2a} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\frac{1}{j\omega+a}|^2 d\omega = \frac{1}{2\pi}\frac{1}{a}\tan^{-1}\frac{\omega}{a}|_{-\infty}^{\infty} = \frac{1}{2\pi}\frac{1}{a}2\frac{\pi}{2} = \frac{1}{2a}$.

EXAMPLES OF CONTINUOUS FOURIER TRANSFORMS

1. $\mathcal{F}\{\delta(t)\} = 1$; $\mathcal{F}\{1\} = 2\pi\delta(\omega)$ (not absolutely integrable) (duality).
$\mathcal{F}\{3\delta(t)+\delta(t-1)+4\delta(t-2)\} = 3+e^{-j\omega}+4e^{-j2\omega} = [1+7\cos(\omega)-j\sin\omega]e^{-j\omega}$.

Note: Discrete in $t \to$ periodic (period 2π) in ω (DTFT).

2a. $\mathcal{F}\{\cos(\omega_o t)\} = \frac{1}{2}\mathcal{F}\{e^{j\omega_o t}\} + \frac{1}{2}\mathcal{F}\{e^{-j\omega_o t}\} = \pi\delta(\omega+\omega_o) + \pi\delta(\omega-\omega_o)$.

2b. $\mathcal{F}\{\sin(\omega_o t)\} = \frac{1}{2j}\mathcal{F}\{e^{j\omega_o t}\} - \frac{1}{2j}\mathcal{F}\{e^{-j\omega_o t}\} = j\pi\delta(\omega+\omega_o) - j\pi\delta(\omega-\omega_o)$.

Note: Periodic in $t \to$ discrete in ω (Fourier series–line spectrum).

3. $\mathcal{F}\{e^{-at}1(t)\} = \frac{1}{j\omega+a}$ for constant $a > 0$. Here $1(t) = 1, t > 0; 0, t < 0$.
4. $\mathcal{F}\{e^{-a|t|}\} = \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{\omega^2+a^2}$ for constant $a > 0$ (reverse time).

Note: $\mathcal{F}$\{real and even function\}=real and even function.

5. $\mathcal{F}\left\{\begin{cases} 1 & \text{for } |t| < t_o; \\ 0 & \text{for } |t| > t_o \end{cases}\right\} = 2t_o\frac{\sin(t_o\omega)}{t_o\omega}$; $\mathcal{F}^{-1}\left\{\begin{cases} 1 & \text{for } |\omega| < \omega_o; \\ 0 & \text{for } |\omega| > \omega_o \end{cases}\right\} = \frac{\sin(\omega_o t)}{\pi t}$.

Note: In Matlab's SP toolbox, `sinc(x)`$=\frac{\sin(\pi x)}{\pi x}$ for $x \neq 0$ and 1 for $x = 0$.

So: $h(t) = \frac{\sin(\omega_o t)}{\pi t} = \frac{\omega_o}{\pi}\text{sinc}(\frac{\omega_o t}{\pi})$ and $H(\omega) = 2t_o\frac{\sin(t_o\omega)}{t_o\omega} = 2t_o\,\text{sinc}(\frac{t_o\omega}{\pi})$. sinc?

But: Let $f_o = \frac{\omega_o}{2\pi}$=cutoff frequency in Hertz. Then $h(t) = (2f_o)\text{sinc}(2f_o t)$.

And: Let $f = \frac{\omega}{2\pi}$=Fourier frequency in Hertz. Then $H(f) = (2t_o)\text{sinc}(2t_o f)$.

Note: Impulse response $h(t)$ for "brick-wall" low-pass filter is sinc function.

6. $\mathcal{F}^{-1}\left\{\begin{cases} 1, & B-a < |\omega| < B+a \\ 0, & \text{otherwise} \end{cases}\right\} = \frac{\sin(at)}{\pi t}2\cos(Bt)$. Noncausal h(t).
for constants $0 < a < B$ (brick-wall band-pass filter) (modulation).

7. $\mathcal{F}\{1(t)\} = \mathcal{F}\{\frac{1}{2}(1+\text{SGN}(t))\} = \frac{1}{j\omega} + \pi\delta(\omega)$. NOT: $\mathcal{F}\{e^{-0t}u(t)\} = \frac{1}{j\omega+0}$!

Note: $1(t)$ *not* absolutely integrable $\to \mathcal{F}\{x(t)\} \neq \mathcal{L}\{x(t)\}|_{s=j\omega}$.

8. $\mathcal{F}\{e^{-t^2/2}\} = \sqrt{2\pi}e^{-\omega^2/2}$ ($\mathcal{F}$\{Gaussian\}=Gaussian)
9. $\mathcal{F}\{e^{jt^2/2}\} = \sqrt{2\pi}e^{j\pi/4}e^{-j\omega^2/2}$ (chirp signal)(compare to #8).