Given: $X(z) = \frac{b_0+b_1z+\ldots+b_Mz^M}{a_0+a_1z+\ldots+a_Nz^N}$. Assume $M \le N$ (else divide; see p.189).
$\frac{X(z)}{z} = \frac{b_0+b_1z+\ldots+b_Mz^M}{a_0z+a_1z^2+\ldots+a_Nz^{N+1}} = \frac{\text{RATIO OF TWO}}{\text{POLYNOMIALS}} = \frac{\text{RATIONAL}}{\text{FUNCTION}}$.

Poles: $\{0, p_1 \ldots p_N\}$ are roots of $a_0z + \ldots + a_Nz^{N+1} = 0$; assume p_n distinct. Compute using "roots" in Matlab. a_n real→complex conjugate pairs.

Partial fraction expansion $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-p_1} + \ldots + \frac{A_N}{z-p_N}$ since distinct poles: $0 \ne p_1 \ne \ldots \ne p_N$.
$A_n = (z-p_n)X(z)/z$ evaluated at $z = p_n$. OR: "residuez" in Matlab.
NOTE: $p_{n+1} = p_n^* \rightarrow A_{n+1} = A_n^*$: Coeffs also complex conjugate pairs.

Causal signal $X(z) = A_0 + A_1\frac{z}{z-p_1} + \ldots + A_N\frac{z}{z-p_N}$. Term-by-term, compute $\mathcal{Z}^{-1}$:
$x[n] = A_0\delta[n] + A_1p_1^n u[n] + \ldots + A_Np_N^n u[n]$ is sum of geometric signals.

Complex conjugate: $Ap^n + A^*(p^*)^n = 2|A||p|^n \cos(\omega_0 n + \theta)$ where $A = |A|e^{j\theta}; p = |p|e^{j\omega_0}$.
This is *much* easier than trying to use sines and cosines directly!

EX #1: *Simple real example:* Compute causal inverse z-xform of $\frac{z-3}{z^2-3z+2}$.

1. Write $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_2}{z-2}$ since $z^2 - 3z + 2 = (z-1)(z-2)$.
2. $A_0 = \frac{(0-3)}{(0-1)(0-2)} = -\frac{3}{2}$. $A_1 = \frac{(1-3)}{(1-0)(1-2)} = 2$. $A_2 = \frac{(2-3)}{(2-0)(2-1)} = -\frac{1}{2}$.
3. $X(z) = -\frac{3}{2} + 2\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-2} \rightarrow x[n] = -\frac{3}{2}\delta[n] + 2u[n] - \frac{1}{2}(2)^n u[n]$.

EX #2: *Simple complex example:* Compute causal inverse z-xform of $\frac{2z}{z^2-2z+2}$.

1. $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-(1+j)} + \frac{A_1^*}{z-(1-j)}$ since $z^2-2z+2 = (z-(1+j))(z-(1-j))$.
2. $A_0 = \frac{2(0)}{(0-(1+j))(0-(1-j))} = 0$. $A_1 = \frac{2(1+j)}{(1+j)((1+j)-(1-j))} = -j$.
$\frac{-j}{z-(1+j)} + \frac{j}{z-(1-j)} \rightarrow -j(1+j)^n u[n] + j(1-j)^n u[n] = 2(\sqrt{2})^n \sin(\frac{\pi}{4}n)$.

EX #3: What if there are multiple poles at the origin $z = 0$? Use this trick:

$$X(z) = \frac{z^3+2z^2+3z+4}{z^2(z-1)} = \frac{z^3+2z^2+3z+4}{z^3}\frac{z}{z-1} = (1+2z^{-1}+3z^{-2}+4z^{-3})\frac{z}{z-1}$$

$$\rightarrow x[n] = \{\underline{1}, 2, 3, 4\} * u[n] = u[n] + 2u[n-1] + 3u[n-2] + 4u[n-3].$$

EX #4: $X(z) = 120/[(z-1)(z-2)(z-3)(z-4)(z-5)]$. ROC: $\{z : |z| > 5\}$.

Partial fraction expansion coefficients computation

$\frac{X(z)}{z} = \frac{-1}{z} + \frac{5}{z-1} - \frac{10}{z-2} + \frac{10}{z-3} - \frac{5}{z-4} + \frac{1}{z-5}$. Computed as follows:

$A_0 = (z-0)X(z)/z|_{z=0} = 120/[(0-1)(0-2)(0-3)(0-4)(0-5)] = -1$.
$A_1 = (z-1)X(z)/z|_{z=1} = 120/[(1-0)(1-2)(1-3)(1-4)(1-5)] = 5$.
$A_2 = (z-2)X(z)/z|_{z=2} = 120/[(2-0)(2-1)(2-3)(2-4)(2-5)] = -10$.
$A_3 = (z-3)X(z)/z|_{z=3} = 120/[(3-0)(3-1)(3-2)(3-4)(3-5)] = 10$.
$A_4 = (z-4)X(z)/z|_{z=4} = 120/[(4-0)(4-1)(4-2)(4-3)(4-5)] = -5$.
$A_5 = (z-5)X(z)/z|_{z=5} = 120/[(5-0)(5-1)(5-2)(5-3)(5-4)] = 1$.

Inverse z-xform $x[n] = -\delta[n] + 5u[n] - 10(2)^n u[n] + 10(3)^n u[n] - 5(4)^n u[n] + 1(5)^n u[n]$.
Note this is an unstable signal, since it blows up as $n \rightarrow \infty$.

PARTIAL FRACTION EXPANSIONS: COMPLEX POLES

Given: $X(z)=\frac{z-1}{z^3+4z^2+8z+8}$ (Chen p. 257). **Find:** Causal inverse z-transform.

Poles: $z^3+4z^2+8z+8=(z+2)(z-2e^{j2.09})(z-2e^{-j2.09})$ (from **roots**)

Form: $\frac{X(z)}{z}=\frac{z-1}{z(z+2)(z-2e^{j2.09})(z-2e^{-j2.09})}=\frac{A}{z}+\frac{B}{z+2}+\frac{C}{z-2e^{j2.09}}+\frac{C^*}{z-2e^{-j2.09}}$

Residues: $A=(z-0)\frac{X(z)}{z}|_{z=0}=\frac{0-1}{(0+2)(0-2e^{j2.09})(0-2e^{-j2.09})}=-\frac{1}{8}$

$B=(z+2)\frac{X(z)}{z}|_{z=-2}=\frac{-2-1}{(-2)(-2-2e^{j2.09})(-2-2e^{-j2.09})}=\frac{3}{8}$

$C=(z-2e^{j2.09})\frac{X(z)}{z}|_{z=2e^{j2.09}}=\frac{(2e^{j2.09}-1)/2e^{j2.09}}{(2e^{j2.09}+2)(2e^{j2.09}-2e^{j-2.09})}=0.19e^{-j2.29}$

$-\frac{1}{8}\delta[n]$ $+\frac{3}{8}(-2)^nu[n]+(0.19)2^ne^{j(2.09n-2.29)}u[n]+(0.19)2^ne^{-j(2.09n-2.29)}u[n]$

Using: $Ap^n+A^*(p^*)^n=2|A||p|^n\cos(\omega_0n+\theta)$ where $A=|A|e^{j\theta}; p=|p|e^{j\omega_0}$,

Simplify: $x(n)=-\frac{1}{8}\delta[n]+\frac{3}{8}(-2)^nu[n]+(0.38)2^n\cos(2.09n-2.29)u[n]$.

NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS

Given: $X(z)=A_1\frac{z}{z-p_1}+\ldots+A_N\frac{z}{z-p_N}$. From before, can compute $\mathcal{Z}^{-1}$:

Any of: $x(n)=\begin{cases}A_1p_1^nu(n) & \text{if ROC}\subset\{|z|>|p_1|\}\\ -A_1p_1^nu(-n-1) & \text{if ROC}\subset\{|z|<|p_1|\}\end{cases}+$

$\ldots+\begin{cases}A_Np_N^nu(n) & \text{if ROC}\subset\{|z|>|p_N|\}\\ -A_Np_N^nu(-n-1) & \text{if ROC}\subset\{|z|<|p_N|\}\end{cases}.$

ROC: Recall ROC has the form ROC$=\bigcap_{n=1}^N\{|z|>|p_n|\text{ OR }|z|<|p_n|\}$.

Thus: $N+1$ possible inverse two-sided z-transforms of $X(z)$, not 2^N.
Individual choices above must be consistent to yield a valid ROC.

EX: $X(z)=120/[(z-1)(z-2)(z-3)(z-4)(z-5)]$; ROC: $3<|z|<4$.

PARTIAL FRACTION: $\frac{X(z)}{z}=\frac{-1}{z}+\frac{5}{z-1}-\frac{10}{z-2}+\frac{10}{z-3}-\frac{5}{z-4}+\frac{1}{z-5}$.

INVERSE Z−XFORM: $x(n)=-\delta(n)+5\begin{cases}u(n)\\-u(-n-1)\end{cases}-10\begin{cases}2^nu(n)\\-2^nu(-n-1)\end{cases}+\ldots$

$\rightarrow-\delta(n)$ $+5u(n)-10(2)^nu(n)+10(3)^nu(n)+5(4)^nu(-n-1)-5^nu(-n-1)$

since: ROC$=\{3<|z|<4\}\subset\{|z|>1,|z|>2,|z|>3,|z|<4,|z|<5\}$.

Note: 6 different inverse z-xforms of $X(z)$! Need ROC to determine $x(n)$.
ROCs: $\{|z|<1\};\{1<|z|<2\};\{2<|z|<3\}\ldots\{4<|z|<5\};\{5<|z|\}$.