DEF: $H(z) = N(z)/D(z) = Y(z)/X(z) = B(z)/A(z)$ as defined below.

Poles: roots of $D(z) = 0$. **ARMA:** $\sum a(i)y(n-i) = \sum b(j)x(n-j)$.

Zeros: roots of $N(z) = 0$. **Impulse response:** $\delta(n) \to \boxed{h(n)} \to h(n)$.

Transfer functions associated with zero initial conditions (ZSR).

I/O: Input $x(n) \to \boxed{h(n)} \to$output $y(n)$ $\underbrace{\Longleftrightarrow}_{Y(z)/X(z)}$ $\mathbf{H(z)}$ $\underbrace{\Longleftrightarrow}_{\mathcal{Z}\{h(n)\}}$ $h(n)$

ARMA: $\sum a(i)y(n-i) = \sum b(j)x(n-j)$ $\underbrace{\Longleftrightarrow}_{B(z)/A(z)}$ $\mathbf{H(z)}$ $\underbrace{\Longleftrightarrow}_{C\prod\frac{z-z_i}{z-p_i}}$ P-Z plot

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EX #1: $x(n) = (-2)^n u(n) \to y(n) = \frac{2}{3}(-2)^n u(n) + \frac{1}{3}u(n)$. Find $h(n)$.

Soln: $X(z) = \frac{z}{z+2}.Y(z) = \frac{2z/3}{z+2} + \frac{z/3}{z-1} = \frac{z^2}{(z+2)(z-1)}.H(z) = \frac{z}{z-1}.h(n) = u(n)$.

First term of $y(n)$=forced response. Second term=natural response.

Difference eqn.: $\frac{Y(z)}{X(z)} = H(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}} \to y(n)-y(n-1) = x(n)$.

EX #2: Find difference equation implementing $H(z) = \frac{z^3+7z^2}{z^3-z^2+2z-3}$

Soln: Write $H(z) = \frac{Y(z)}{X(z)} = \frac{1+7z^{-1}}{1-z^{-1}+2z^{-2}-3z^{-3}}$ and **cross-multiply**:

$Y(z)(1 - z^{-1} + 2z^{-2} - 3z^{-3}) = X(z)(1 + 7z^{-1}) \to$difference equation:

$\mathcal{Z}^{-1}$ $y(n) - y(n-1) + 2y(n-2) - 3y(n-3) = x(n) + 7x(n-1)$.

EX #3: Find step response of system with zero at 1, pole at 3, and $H(0) = 1$.

Soln: $H(z) = 3\frac{z-1}{z-3}$. $U(z) = \frac{z}{z-1} \to Y(z) = 3\frac{z}{z-3} \to y(n) = 3 \cdot 3^n u(n)$.

$\frac{Y(z)}{X(z)} = H(z) = 3\frac{z-1}{z-3} = 3\frac{1-z^{-1}}{1-3z^{-1}} \to y(n)-3y(n-1) = 3x(n)-3x(n-1)$.

Modes≠ $y(n) - 3y(n-1) + 2y(n-2) = x(n) - x(n-1)$. Modes: 1,2.

Poles: $H(z) = \frac{1-z^{-1}}{1-3z^{-1}+2z^{-2}} = \frac{1}{1-2z^{-1}}$. Poles: 2. $\{poles\} \subset \{modes\}$.

ZIR: $y(n) = C_1 2^n u(n) + C_2 1^n u(n)$, depending on initial conditions.

ZSR: $y(n) = C_3 2^n u(n)$ (natural response)+forced response$\sim u(n)$.

If no *pole-zero cancellation*, then $\{poles\} = \{modes\}$.

BIBO LTI system BIBO stable **iff** $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$ (is finite).

Stable Same as: $\{\text{unit circle } |z| = 1\} \subset ROC$ of $H(z) = \mathcal{Z}\{h(n)\}$.

Causal Causal LTI BIBO stable **iff** poles inside unit circle.

Anticausal BIBO stable **iff** poles outside unit circle.

APPLICATIONS OF THE z-TRANSFORM

Given: $x[n] = 3^n u[n] \rightarrow \boxed{y[n] - 2y[n-1] = x[n-1] - x[n-2]} \rightarrow y[n]$

Goal: Compute the response $y[n]$ of the system to this particular input.

$\mathcal{Z}$: $Y(z) - 2z^{-1}Y(z) = z^{-1}X(z) - z^{-2}X(z)$ and $X(z) = \frac{z}{z-3}$ here

$\rightarrow Y(z) = \frac{z^{-1}-z^{-2}}{1-2z^{-1}}\frac{z}{z-3} = \frac{z-1}{(z-2)(z-3)} = \frac{2}{z-3} - \frac{1}{z-2} \quad [2 = \frac{3-1}{3-2}; -1 = \frac{2-1}{2-3}]$

$\rightarrow y[n] = [2(3)^{n-1} - (2)^{n-1}]u[n-1] =$ FORCED RESPONSE (like $x[n]$) + NATURAL RESPONSE (like $h[n]$).

Given: $x[n] = (\frac{1}{2})^n u[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = \{\underline{0}, 0, 1\} = \delta[n-2]$

Goal: Compute the response of this system to $2\cos(\frac{\pi}{3}n)$.

H(z): TRANSFER FUNCTION $= H(z) = \mathcal{Z}\{y[n]\}/\mathcal{Z}\{x[n]\} = z^{-2}/[z/(z-\frac{1}{2})] = (z-\frac{1}{2})/z^3$.

h[n]: IMPULSE RESPONSE $= h[n] = \mathcal{Z}^{-1}\{(z-\frac{1}{2})/z^3\} = \mathcal{Z}^{-1}\{z^{-2}-\frac{1}{2}z^{-3}\} = \{\underline{0}, 0, 1, -\frac{1}{2}\}$.

H(w): FREQUENCY RESPONSE $= H(\omega) = H(z)|_{z=e^{j\omega}} = (e^{j\omega} - \frac{1}{2})/e^{j3\omega}$.

$\omega = \frac{\pi}{3}$: $H(\frac{\pi}{3}) = [e^{j\pi/3} - \frac{1}{2}]/e^{j3\pi/3} = (\frac{1}{2} + j\frac{\sqrt{3}}{2} - \frac{1}{2})/(-1) = -j\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}e^{-j\pi/2}$.

Sol'n: $2\cos(\frac{\pi}{3}n) \rightarrow \boxed{\textbf{LTI}} \rightarrow \sqrt{3}\cos(\frac{\pi}{3}n - \frac{\pi}{2}) = \sqrt{3}\sin(\frac{\pi}{3}n)$.

Given: $x[n] \rightarrow \boxed{y[n] = x[n] - \frac{3}{4}x[n-1] + \frac{1}{8}x[n-2]} \rightarrow y[n]$

Huh? $x[n]$=cell phone signal. $y[n]$=multipath due to buildings.

Goal: Compute the **inverse filter** that recovers $x[n]$ from $y[n]$:

Huh? $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] \rightarrow \boxed{g[n]} \rightarrow x[n]$. That is, $g[n]$ *undoes* $h[n]$.

Idea: Systems in cascade (series) $\Leftrightarrow h[n] * g[n] = \delta[n] \Leftrightarrow H(z)G(z) = 1$.

Here: $h[n] = \{\underline{1}, -\frac{3}{4}, \frac{1}{8}\} \rightarrow H(z) = 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = (z^2 - \frac{3}{4}z + \frac{1}{8})/z^2$.

$\rightarrow G(z) = 1/H(z) = z^2/[z^2 - \frac{3}{4}z + \frac{1}{8}] = z^2/[(z - \frac{1}{2})(z - \frac{1}{4})]$.

$\mathcal{Z}^{-1}$: $\frac{G(z)}{z} = \frac{z}{(z-1/2)(z-1/4)} = \frac{2}{z-1/2} - \frac{1}{z-1/4} \rightarrow G(z) = 2\frac{z}{z-1/2} - 1\frac{z}{z-1/4}$.

using: residues $2 = (1/2)/[(1/2) - (1/4)]$ and $-1 = (1/4)/[(1/4) - (1/2)]$.

g[n]: $g[n] = 2(\frac{1}{2})^n u[n] - (\frac{1}{4})^n u[n]$=inverse filter for original system.

Note: Stable since zeros of H(z)=poles of G(z) are inside unit circle.

Note: $g[0] \neq 0$ since both numerator & denominator of $G(z)$ have the same degrees.