Recitation 4 - EECS 451, Winter 2009

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OUTLINE

- Review of important concepts (Lecture 5-6)
- Practise problems

Concepts: 2-sided z-transforms

1. Definition of z-transform
   - For a given DT signal $x(n)$, $X(z) := \sum_{n=-\infty}^{\infty} x(n)z^{-n}$, where z is complex valued.
   - If $x(n)r^{-n}$ is absolutely summable, then $X(z)$ has a finite value where $|z| = r$.
2. ROC (Region of Convergence)
   - The set of values of z for which the sequence $x(n)z^{-n}$ is absolutely summable, i.e. $\{z \in \mathbf{C} : \sum_{n=-\infty}^{\infty} |x(n)z^{-n}| < \infty\}$, where $\mathbf{C}$ is the set of complex numbers.
   - Simply put, ROC indicates the region of z where $X(z)$ is finite.
   - By definition, ROC cannot contain any poles.
3. The shape of ROCs:
   - The ROC of an anti-causal signal is of the form $|z| < |a|$.
   - The ROC of a causal signal is of the form $|z| > |a|$.
   - The ROC of a two sided signal is of the form $|a| < |z| < |b|$.
   - The ROC of a finite length signal is the entire $z-$space except for $z = 0$ and/or $z = \infty$.
4. Useful z-transformation pairs:
   - If $x(n) = a^n u(n)$, then $X(z) = \frac{z}{z-a}$, ROC $= |z| > |a|$.
   - If $x(n) = -a^n u(-n-1)$, then $X(z) = \frac{z}{z-a}$, ROC $= |z| < |a|$.
5. Properties of z-transform: We have $x(n) \overset{Z}{\leftrightarrow} X(z)$ and $ROC_X = r_2 < |z| < r_1$
   - Linearity: $a_1x_1(n) + a_2x_2(n) \overset{Z}{\leftrightarrow} a_1X_1(z) + a_2X_2(z)$, $ROC \geq ROC_{X_1} \cap ROC_{X_2}$

- Time shifting: $x(n-k) \overset{Z}{\leftrightarrow} z^{-k}X(z)$, $ROC = ROC_X$ except $z = 0$ or $z = \infty$.
- Scaling in the z-domain: $a^n x(n) \overset{Z}{\leftrightarrow} X(a^{-1}z)$, $ROC = |a|r_2 < |z| < |a|r_1$
- Time reversal: $x(-n) \overset{Z}{\leftrightarrow} X(z^{-1})$, $ROC = \frac{1}{r_1} < |z| < \frac{1}{r_2}$
- Differentiation in the z-domain: $nx(n) \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}$, $ROC = ROC_X$
- Convolution: $x_1(n) * x_2(n) \overset{Z}{\leftrightarrow} X_1(z)X_2(z)$, $ROC \geq ROC_{X_1} \cap ROC_{X_2}$
- Correlation: $x_1(n) * x_2(-n) \overset{Z}{\leftrightarrow} X_1(z)X_2(z^{-1})$, $ROC \geq ROC_{X_1(z)} \cap ROC_{X_2(z^{-1})}$

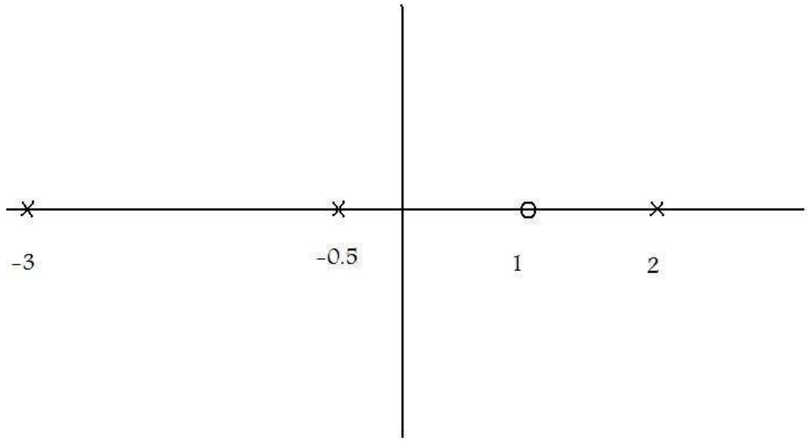
6. Useful theorems on z-transform
   - Initial value theorem: If $x(n)$ is causal, then $x(0) = lim_{z\to\infty} X(z)$
   - Multiplication of two sequences and Parseval's relation

7. LTI systems and z-transforms
   - The z-transform of the impulse response $h(n)$ is called the system function $[H(z)]$
   - DT LTI systems described by LCCDE have a rational z-transform, i.e. $H(z) = \frac{B(z)}{A(z)}$
   - If a signal $y(n)$ is outputted by the system when the input signal is $x(n)$, then their z-transforms are related as $Y(z) = H(z)X(z)$
   - Causality/Stability of the system can be determined from the ROC of $H(z)$

Concepts: Inverse z-transform (Partial Fraction Expansions)

1. $\frac{X(z)}{z} = \frac{B(z)}{A(z)}$ and $M < N$
   - Distinct simple roots: $\frac{X(z)}{z} = \frac{A_1}{z-a} + \frac{A_2}{z-b}$
   - Multiple simple roots: $\frac{X(z)}{z} = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2}$
   - Distinct complex roots: $\frac{X(z)}{z} = \frac{A_1 z + B_1}{z^2+a^2} + \frac{A_2 z + B_2}{z^2+b^2}$
   - Multiple complex roots: $\frac{X(z)}{z} = \frac{A_1 z + B_1}{z^2+a^2} + \frac{A_2 z + B_2}{(z^2+a^2)^2}$

2. $\frac{X(z)}{z} = \frac{B(z)}{A(z)}$ and $M \geq N$
   - Divide $B(z)$ by $A(z)$ to express it as $\frac{X(z)}{z} = Q(z) + \frac{R(z)}{A(z)}$ where the degree of $R(z)$ is less than the degree of $A(z)$
   - Now apply the appropriate partial fraction expansion to $\frac{R(z)}{A(z)}$



Problems

1. Express the z-transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

in terms of $X(z)$.

2. Using appropriate properties of the z-transforms, determine $x(n)$ for the following transformation

$$X(z) = log(1 - \frac{1}{2}z^{-1}), |z| > \frac{1}{2}$$

Hint: Differentiate $X(z)$

3. Determine the causal signal $x(n)$ if its z-transform is

$$X(z) = \frac{2 - 1.5z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

4. Consider an LTI system with a pole-zero pattern shown above

   (a) Determine the ROC of the system function $H(z)$ if the system is known to be stable.

   (b) Is it possible for the pole-zero plot to correspond to a causal and stable system?

   (c) How many possible systems can be associated with this pole-zero pattern?