Review Session II - EECS 451, Winter 2009

March 25th, 2009

OUTLINE

- Review of Important concepts

Concepts from Chapter 4

1. Discrete Time Fourier series (DTFS)

   (a) For a discrete time periodic signal with period N, we have

   $$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{kn}{N}}, \qquad x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi\frac{kn}{N}}$$

   where c_k is also periodic with period N

   (b) If $x(n)$ is real and periodic, we have $c_k = c^*_{-k}$ (i.e. $c_k = c^*_{N-k}$)

   (c) Parseval's relations yields

   $$\text{Average Power} = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

   where $|c_k|^2$ is the Power spectrum density.

2. Discrete Time Fourier Transform (DTFT)

   (a) For a discrete time aperiodic signal, we have

   $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}, \qquad x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

   where $X(\omega)$ is periodic with period 2π

   (b) If $x(n)$ is real and aperiodic , we have $X(\omega) = X^*(-\omega)$

   (c) Parseval's relation yields

   $$\text{Average Energy} = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

   where $|X(\omega)|^2$ is the Energy density spectrum

3. Symmetry properties: For a discrete signal $x(n)$ with Fourier transform $X(\omega)$
   (a) If $x(n)$ is real, $X(\omega)$ is conjugate symmetric i.e. $X^*(\omega) = X(-\omega)$
   (b) If $x(n)$ is real and even, $X(\omega)$ is real and even
   (c) If $x(n)$ is real and odd, $X(\omega)$ is imaginary and odd
   (d) If $x(n)$ is imaginary and even, $X(\omega)$ is imaginary and even
   (e) If $x(n)$ is imaginary and odd, $X(\omega)$ is real and odd

   These symmetry properties hold for DTFS and DFT as well

4. Properties of DTFT: Many follow from the fact $X(\omega) = X(z)|_{z=e^{j\omega}}$
   (a) Linearity: $a_1x_1(n) + a_2x_2(n) \overset{DTFT}{\leftrightarrow} a_1X_1(\omega) + a_2X_2(\omega)$
   (b) Time shifting: $x(n-k) \overset{DTFT}{\leftrightarrow} e^{-j\omega k}X(\omega)$
   (c) Time Reversal: $x(-n) \overset{DTFT}{\leftrightarrow} X(-\omega)$
   (d) Frequency shifting: $e^{j\omega_0 n}x(n) \overset{DTFT}{\leftrightarrow} X(\omega - \omega_0)$
   (e) Convolution: $x_1(n) * x_2(n) \overset{DTFT}{\leftrightarrow} X_1(\omega)X_2(\omega)$
   (f) Multiplication: $x_1(n)x_2(n) \overset{DTFT}{\leftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2(\omega - \lambda)d\lambda$
   (g) Differentiation in the frequency domain: $nx(n) \overset{DTFT}{\leftrightarrow} j\frac{dX(\omega)}{d\omega}$
   (h) Conjugation: $x^*(n) \overset{DTFT}{\leftrightarrow} X^*(-\omega)$

Concepts from Chapter 5

1. Linear phase response is defined as $\Theta(\omega) = n_0\omega$, where n_0 is an integer. The linear phase results in a pure delay that is often desired in filter design

2. Pole zero placement and magnitude response: Assuming ROC includes the unit circle, i.e. system is BIBO stable,
   (a) when $e^{j\omega}$ is close to a zero, $|H(\omega)|$ is small
   (b) when $e^{j\omega}$ is close to a pole, $|H(\omega)|$ is large

3. Digital filter design
   (a) For a causal system - Number of poles $\geq$ Number of zeros
   (b) For a stable system - All poles should be within the unit circle
   (c) For a real filter - All poles and zeros should occur in conjugate pairs
   (d) For a low pass filter - Place poles near low frequencies and zeros near high frequencies
   (e) For a high pass filter - Place poles near high frequencies and zeros near low frequencies
   (f) For a band pass filter - Place pairs of conjugate poles near the unit circle at the frequencies we wish to pass. The peak of the magnitude response of the filter becomes sharper as the poles approach the unit circle

(g) For a notch filter - Place pairs of complex conjugate zeros on the unit circle at the frequencies we wish to stop

(h) For an all pass filter - Place a pole at $\frac{1}{z_0^*}$ if a zero is present at z_0 and vice versa

(i) For a linear phase filter - Place a pole at $\frac{1}{z_0}$ if a pole is present at z_0 and vice versa

(j) For a minimum phase filter - All poles and zeros should be within the unit circle

4. Inverse systems

   (a) A system $\mathcal{T}$ is said to be invertible iff each possible output signal is the response to only one input signal

   (b) If $\mathcal{T}$ is LTI and invertible, then $\mathcal{T}^{-1}$ is also LTI and invertible

   (c) If $\mathcal{T}$ is LTI, causal and stable, then $\mathcal{T}^{-1}$ is causal iff # of poles of $\mathcal{T}$ = # of zeros of $\mathcal{T}$

   (d) If $\mathcal{T}$ is LTI, causal and stable, then $\mathcal{T}^{-1}$ is stable iff all zeros of $\mathcal{T}$ are strictly inside the unit circle

   (e) A system with a minimum phase has a stable inverse that is also minimum phase

Concepts from Chapter 7

1. For a time limited signal $x(n)$ of length N, the N-point DFT is defined by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n}$$

   where $k = 0, \cdots, N-1$ and the inverse DFT formula is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi n}{N}k}$$

   where $n = 0, \cdots, N-1$

   (a) Note that the indices range from 0 to $N-1$ for $x(n)$ and $X(k)$

   (b) The DFT is efficiently implemented by the Fast Fourier transform (FFT)

2. Relationship to other transformations

   (a) If $x_p(n)$ is the N-periodic superposition of $x(n)$ (a time-limited signal of length N), then the DTFS of $x_p(n)$ is obtained by

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi k}{N}n},$$

   thus leading to $X(k) = Nc_k$

(b) The DFT of $x(n)$ has the following relationship with $X(\omega)$ (DTFT) and $X(z)$ (z-transformation)

$$X(k) = X(\omega)|_{\omega=\frac{2\pi k}{N}}, \qquad X(k) = X(z)|_{z=e^{j\frac{2\pi k}{N}}}$$

where $x(n)$ is a time-limited N-point signal.

3. Circular Symmetries

   (a) N-point circular even is defined by $x(N-n) = x(n)$ for $1 \leq n \leq N-1$

   (b) N-point circular odd is defined by $x(N-n) = -x(n)$ for $1 \leq n \leq N-1$

   (c) If $x(n)$ is real, we have $X(k) = X^*((-k) \mod N)$ (circular Hermitian symmetry)

4. Properties of the DFT (Suppose $x(n) \stackrel{N-DFT}{\longleftrightarrow} X(k)$)

   (a) Linearity: $a_1x_1(n) + a_2x_2(n) \stackrel{N-DFT}{\longleftrightarrow} a_1X_1(k) + a_2X_2(k)$

   (b) Circular time shift: $x((n-l) \mod N) \stackrel{N-DFT}{\longleftrightarrow} e^{-j2\pi lk/N}X(k)$

   (c) Time reversal: $x((-n) \mod N) \stackrel{N-DFT}{\longleftrightarrow} X((-k) \mod N)$

   (d) Circular frequency shift: $x(n)e^{j2\pi k_0 n/N} \stackrel{N-DFT}{\longleftrightarrow} X((k-k_0) \mod N)$

   (e) Complex conjugate: $x^*(n) \stackrel{N-DFT}{\longleftrightarrow} X^*((-k) \mod N)$ and $x^*((-n) \mod N) \stackrel{N-DFT}{\longleftrightarrow} X^*(k)$

   (f) Convolution: $x_1(n) \circledN x_2(n) \stackrel{N-DFT}{\longleftrightarrow} X_1(k)X_2(k)$

   (g) Multiplication: $x_1(n)x_2(n) \stackrel{N-DFT}{\longleftrightarrow} \frac{1}{N}X_1(k) \circledN X_2(k)$, where $\circledN$ denotes N-point circular convolution

   (h) Parseval's theorem: $\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$

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