

1. $y(0) = y(-1) + y(-2) + x(0) = 0 + 0 + 1 = 1$. $y(2) = y(1) + y(0) = 1 + 1 = 2$. Clearly $y(1) = y(0) + y(-1) + x(1) = 1 + 0 + 0 = 1$. $y(3) = y(2) + y(1) = 2 + 1 = 3$. works.

Take 2-sided $\mathcal{Z} : H(z) = z^{-1}H(z) + z^{-2}H(z) + 1 \rightarrow H(z) = \frac{1}{1-z^{-1}-z^{-2}}$ (see p.202).

Partial fraction: $H(z) = \frac{\phi/\sqrt{5}}{1-\phi z^{-1}} - \frac{\phi'/\sqrt{5}}{1-\phi' z^{-1}} \rightarrow h(n) = \frac{1}{\sqrt{5}}(\phi^{n+1} - (\phi')^{n+1})u(n)$

where $\phi = \frac{1+\sqrt{5}}{2} = 1.618$ =golden ratio and $\phi' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\phi} = 1 - \phi$ (see why?)

Note: For $n \geq 10$ or so, $h(n) \approx \frac{1}{\sqrt{5}}\phi^{n+1} \rightarrow \frac{h(n+1)}{h(n)} \approx \phi$; hence "golden ratio."

- 2b. $Y(z) - 1.5(z^{-1}Y(z) + 1) + 0.5(z^{-2}Y(z) + 0 + 1z^{-1}) = 0 \rightarrow Y(z) = \frac{1.5-0.5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}$
 $Y(z) = \frac{2}{1-z^{-1}} - \frac{0.5}{1-0.5z^{-1}} \rightarrow y(n) = 2u(n) - 0.5(0.5)^n u(n) = (2 - (0.5)^{n+1})u(n)$.

- 2c. $Y(z) = \frac{1}{2}(z^{-1}Y(z) + 1) + X(z); X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \rightarrow Y(z) = \frac{(3/2)-(1/6)z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$.
 Partial fraction: $Y(z) = \frac{7/2}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}} \rightarrow y(n) = \frac{7}{2}(\frac{1}{2})^n u(n) - 2(\frac{1}{3})^n u(n)$.

- 3b. $Y(z) = X(z)(1 - 0.1z^{-1}) \rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 - 0.1z^{-1}$. No common factors.

- 3a. $Y(z)(1 - .2z^{-1}) = X(z)(1 - .3z^{-1} + .02z^{-2}) \rightarrow H(z) = \frac{1-.3z^{-1}+.02z^{-2}}{1-.2z^{-1}} = 1 - .1z^{-1}$.

The two systems have the same transfer function $H(z) \rightarrow$ equivalent.

Can distinguish them only if they have nonzero *initial conditions*.

4. $H(z) = \mathcal{Z}\{h(n)\} = \frac{1}{1-\frac{2}{5}z^{-1}} - \frac{1}{1-3z^{-1}}$. Step response: $S(z) = H(z)U(z); U(z) = \frac{1}{1-z^{-1}}$.

Partial fraction: $S(z) = \frac{1}{1-\frac{2}{5}z^{-1}} \frac{1}{1-z^{-1}} - \frac{1}{1-3z^{-1}} \frac{1}{1-z^{-1}} = \frac{-2/3}{1-\frac{2}{5}z^{-1}} + \frac{-3/2}{1-3z^{-1}} + \frac{\frac{5}{3}+\frac{1}{2}}{1-z^{-1}}$

$\rightarrow s(n) = \frac{13}{6}u(n) - \frac{2}{3}(\frac{2}{5})^n u(n) + \frac{3}{2}3^n u(-n-1)$. Note anticausal term sign

- 5a. $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{(1/4)z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}}$. $Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$. $H(z) = \frac{Y(z)}{X(z)}$.

$H(z) = \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{3}{1-\frac{1}{4}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}} \rightarrow h(n) = 3(\frac{1}{4})^n u(n) - 2(\frac{1}{3})^n u(n)$.

- 5b. $H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}} \rightarrow Y(z)(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}) = X(z)(1 - \frac{1}{2}z^{-1}) \rightarrow$
 $y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) - \frac{1}{2}x(n-1)$ (by cross-multiplying above).

- 5d. $h(n)$ clearly absolutely summable \rightarrow stable. OR: Poles inside unit circle.

- 6a. $H(z) = C \frac{z-1}{(z+0.5)(z-2)(z+3)}$ for some constant C . Stable $\rightarrow ROC = \{z : 0.5 < |z| < 2\}$.

- 6b. NO. Causal $\rightarrow ROC = \{z : |z| > 3\}$. $\{z : |z| > 3\} \cap \{0.5 < |z| < 2\} = \emptyset$.

- 6c. 4. ROCs: $|z| > 3$ (causal); $2 < |z| < 3$; $0.5 < |z| < 2$ (stable); $|z| < 0.5$.

NOTES: the zero at 1 is irrelevant. Possible ROCs: annuli between pole magnitudes.

- 7a. The command is: **filter([1 zeros(1,1023) -0.8],[1],Y);**

The reverberation amounts to convolution with an upsampled geometric $h(n)$ with

$H(z) = \sum_{n=0}^{31} 0.8^n z^{-1024n}$. Note $(1 - 0.8z^{-1024})H(z) = 1 - 0.8^{32}z^{-32768} \approx 1$

since $0.8^{32} \approx 0.0008 \approx 0$, so this is a good inverse filter. See overleaf for plots.

