Recall: $\lim_{n\to\infty} x_n = x \Leftrightarrow$ For any $\epsilon > 0, \exists N$ such that $|x_n - x| < \epsilon \quad \forall n > N$.
Given: A *sequence* of *random variables* $\{x_1, x_2 \ldots\}$. Need *not* be iidrv.

DEF: $x_n \to x$ *in probability* $\Leftrightarrow \lim_{n\to\infty} Pr[|x_n - x| > \epsilon] = 0 \Leftrightarrow$ $\substack{\text{stochastic}\\ \text{convergence}}$.
EX1: If $\{x_n\}$ iidrv, then sample mean $\hat{M}_n = \frac{1}{n}\sum_{i=1}^{n} x_i \to E[x]$ in prob.
Proof: See Estimators handout. Requires both $E[x]$ and σ_x^2 to be finite.
Note: This is *weak law of large numbers*, since convergence in prob. is *weak*.

EX2: If $\{x_n\}$ iidrv, $f_{x_i} = \frac{1}{A}, 0 < X < A$, then $\max[x_1 \ldots x_n] \to A$ in prob.
Note: Each of these shows *consistency* of an estimator (#1 of prob. set #7).

DEF: $x_n \to x$ *in mean square* $\Leftrightarrow \lim_{n\to\infty} E[(x_n - x)^2] = 0 \Leftrightarrow \underset{n\to\infty}{\text{L.I.M.}}\, x_n = x$.
EX: If $\{x_n\}$ iidrv, then $\hat{M}_n \to E[x]$ in mean square=in quadratic mean.
Note: This is *Mean Ergodic Thm.*: $\underset{n\to\infty}{\text{L.I.M.}}\, \hat{M}_n = E[x]$. Use $\hat{M}_n$ unbiased:
Proof: $E[(\hat{M}_n - E[x])^2] = E[(\hat{M}_n - E[\hat{M}_n])^2] = \sigma^2_{\hat{M}_n} = \frac{\sigma^2}{n} \to 0$ if $\sigma^2 < \infty$.

DEF: $x_n \to x$ *with prob. one* $\Leftrightarrow Pr[\{\omega \in \Omega : \lim_{n\to\infty} x_n(\omega) = x(\omega)\}] = 1$.
Huh? Pr[set of sample functions that converge to sample point of x]=1.
Aliases: *Convergence a.s.* (almost surely), *converg. a.e.* (almost everywhere).
How? To show convergence with prob. one, usually use Thm. 3 below.
EX: If $\{x_n\}$ iidrv, then $\hat{M}_n \to E[x]$ a.s. (*strong law of large numbers*).

Thm. 1: Convergence in mean square→convergence in probability.
Proof: Suppose $\underset{n\to\infty}{\text{L.I.M.}}\, x_n = x$. Use Markov inequality: As $n \to \infty$,
$Pr[|x_n - x| > \epsilon] = Pr[(x_n - x)^2 > \epsilon^2] \leq \frac{E[(x_n - x)^2]}{\epsilon^2} \to 0$. QED.

Thm. 2: Convergence with probability one→convergence in probability.
Proof: Let $A_n = \{\omega \in \Omega : |x_n(\omega) - x(\omega)| > \epsilon\}$ and $F_n = \cup_{i=n}^{\infty} A_i$ (so limsup).
Huh? A_n=set of ω s.t. $x_n(\omega)$ not yet converged within ϵ at time n.
F_n=set of ω s.t. $x_n(\omega)$ not yet converged within ϵ at *any* time $\geq n$.
Note: Convergence in probability $\Leftrightarrow \lim_{n\to\infty} Pr[A_n] = 0$ (see above).

Then: $x_n \to x$ a.e. $\Leftrightarrow 0 = Pr[\{\omega : \lim_{n\to\infty} |x_n(\omega) - x(\omega)| > \epsilon\}] = Pr[\lim_{n\to\infty} F_n]$
$= \lim_{n\to\infty} Pr[F_n]$ using *cont. of prob.* since $\{F_n\}$ is decreasing sequence.

But: $(\lim_{n\to\infty} Pr[F_n] = 0) \to (\lim_{n\to\infty} Pr[A_n] = 0) \to$convergence in prob. QED.

Q: Why $(\lim_{n\to\infty} Pr[F_n]=0) \to (\lim_{n\to\infty} Pr[A_n]=0)$ but not vice-versa?

A: $A_n \subset F_n \to Pr[A_n] \le Pr[F_n] \to (\lim Pr[F_n]=0 \to \lim Pr[A_n]=0)$.

But: $\lim_{n\to\infty} Pr[\cup_{i=n}^{\infty} A_i] \neq \lim_{n\to\infty} Pr[A_n]$! Why not?

Lemma: $\lim_{n\to\infty} \sum_{i=n}^{\infty} Pr[A_i]=0$ if $\sum_{i=0}^{\infty} Pr[A_i]<\infty$ (doesn't blow up; is finite).

Huh? Remainder term in infinite series goes to zero if the series converges.

Proof: $\sum_{i=n}^{\infty} Pr[A_i]=\sum_{i=0}^{\infty} Pr[A_i]-\sum_{i=0}^{n-1} Pr[A_i]$ if $\sum_{i=0}^{\infty} Pr[A_i]$ bounded. Now take the limit of this as $n\to\infty$. QED.

Thm: *Borel-Cantelli Lemma*: $Pr[\lim_{n\to\infty} \cup_{i=n}^{\infty} A_i]=0$ if $\sum_{i=0}^{\infty} Pr[A_i]<\infty$.

EX: $A_i=A$. Then $Pr[\lim_{n\to\infty} \cup_{i=n}^{\infty} A_i]=Pr[A]\neq 0$ since $\sum_{i=0}^{\infty} Pr[A]\to\infty$.

Proof: $Pr[\lim \cup_{i=n}^{\infty} A_i]=\lim Pr[\cup_{i=n}^{\infty} A_i]\le \lim \sum_{i=n}^{\infty} Pr[A_i]=0$ by Lemma.

Note: Using cont. of prob. and union bound. Use Borel-Cantelli to prove:

Thm. 3: $(\sum_{n=0}^{\infty} Pr[|x_n-x|>\epsilon]<\infty) \to (x_n \to x \text{ with probability one})$.

Proof: Apply Borel-Cantelli lemma with $A_n=\{\omega : |x_n(\omega)-x(\omega)|>\epsilon\}$.

$\sum Pr[|x_n-x|>\epsilon]<\infty \to Pr[\lim \cup_{i=n}^{\infty}\{\omega : |x_n-x|>\epsilon\}]=0$.

Now reverse the proof of Thm. 2. QED.

Note: This condition is sufficient, but not necessary, for convergence a.e.

Thm. 4: $(\sum_{n=0}^{\infty} Pr[|x_n-x|>\epsilon]<\infty) \to (x_n \to x \text{ in probability})$.

Proof: $(\sum_{n=0}^{\infty} Pr[A_n]<\infty) \to (\lim_{n\to\infty} Pr[A_n]=0)$. QED.

Huh? Infinite series converges→ its general term converges to zero.

$\sum$: $\boxed{\sum Pr[|x_n-x|>\epsilon]<\infty} \xrightarrow[\text{Thm.3}]{} \boxed{\text{Convg. prob.1}} \xrightarrow[\text{Thm.2}]{} \boxed{\text{Convg. inprob.}} \xleftarrow[\text{Thm.1}]{} \boxed{L.I.M.}$

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1. **In prob** is weaker than both **mean-square** and **with prob. 1**.
2. **With prob. 1** does *not* imply **mean-square**; nor the converse.
3. To show **with prob. 1**, use Thm. 3 (except for *your* homework!)