DEF: A *random vector* is a vector of random variables $\vec{x} = [x_1 \dots x_N]'$.
Note: Unless otherwise stated, a random vector is a *column* vector.
DEF: The *mean vector* of random vector $\vec{x}$ is $\vec{\mu} = E[\vec{x}] = [E[x_1] \dots E[x_N]]'$.

DEF: The *covariance matrix* $K_x = \Lambda_x$ of $\vec{x}$ is the $N \times N$ matrix whose $(i,j)^{th}$ element $(K_x)_{ij} = \lambda_{x_i x_j} = E[x_i x_j] - E[x_i]E[x_j]$.
Note: $K_x = E[(\vec{x} - E[\vec{x}])(\vec{x} - E[\vec{x}])'] = E[\vec{x}\vec{x}'] - E[\vec{x}]E[\vec{x}]'$ (outer products).

Also *Outer* product $\vec{x}\vec{y}' = [x_i y_j] = N \times N$ matrix having rank 1.
Note: *Inner* product $\vec{x}'\vec{y} = \sum x_i y_i$=scalar=Trace of outer product.

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1. K_x is a *symmetric* matrix: $(K_x)_{ij} = \lambda_{x_i x_j} = \lambda_{x_j x_i} = (K_x)_{ji}$.
2. K_x is a *positive semidefinite* matrix: For any vector $\vec{a}$, the *scalar* $\vec{a}' K_x \vec{a} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i (K_x)_{ij} a_j \geq 0$.
3. In particular, the diagonal elements of K_x have $(K_x)_{ii} = \sigma^2_{x_i} \geq 0$. This is necessary but *not* sufficient for K_x to be positive semidefinite.

Thm: Let random vector $\vec{y} = A\vec{x} + \vec{b}$ for any constant matrix A and vector $\vec{b}$. *A need not be square.* Then $E[\vec{y}] = AE[\vec{x}] + \vec{b}$ and $K_y = AK_xA'$.
Proof: $K_y = E[(\vec{y} - E[\vec{y}])(\vec{y} - E[\vec{y}])'] = E[A(\vec{x} - E[\vec{x}])(A(\vec{x} - E[\vec{x}]))']$
$= E[A(\vec{x} - E[\vec{x}])(\vec{x} - E[\vec{x}])'A'] = AK_xA'$ using $(A\vec{x})' = \vec{x}'A'$.
#2: Define rv $y = \vec{a}'\vec{x} = \sum_{i=1}^{N} a_i x_i$. Then $\sigma^2_y = \vec{a}' K_x \vec{a} \geq 0$.

DEF: K_x has N *eigenvalues* λ_i and associated *eigenvectors* v_i which solve $K_x v_i = \lambda_i v_i, i = 1 \dots N$. **Fact:** K_x real & symmetric$\rightarrow \lambda_i$ & v_i real.
Fact: K_x is positive semidefinite iff $\lambda_i \geq 0, i = 1 \dots N$. Matlab: **eig**$\rightarrow \lambda_i, v_i$.

Thm: Let $V = [v_1 | v_2 | \dots | v_N]$ (matrix of eigenvectors) and $\vec{y} = V'\vec{x}$.
Then: $\lambda_{y_i y_j} = E[y_i y_j] - E[y_i]E[y_j] = \lambda_i \delta_{ij} = 0$ if $i \neq j$.
Proof: $K_x v_i = \lambda_i v_i \rightarrow K_x V = V diag[\lambda_i] \rightarrow K_y = V' K_x V = diag[\lambda_i]$ since $\vec{v_i}'\vec{v_j} = 0$ if $i \neq j$ (V is a *unitary* matrix: $V'V = VV' = I$).

Note: This is called *decorrelating* or *(pre)whitening* the vector $\vec{x}$. It is an essential part of communications and signal processing in noise.

DEF: *Cross-correlation* matrix $K_{xy} = E[(\vec{x} - E[\vec{x}])(\vec{y} - E[\vec{y}])']$. $K_{xy} = K'_{yx}$.
Props: $K_{x+y} = K_x + K_y + K_{xy} + K_{yx} = K_x + K_y + K_{xy} + K'_{xy}$ symmetric.
$\vec{z} = A\vec{x} + \vec{b} \rightarrow K_{zy} = AK_{xy}$ and $K_{yz} = K_{yx}A'$. Compare to σ^2_{x+y}.
DEF: $\vec{x}$ and $\vec{y}$ are *uncorrelated* if $K_{xy} = [0] \leftrightarrow E[\vec{x}\vec{y}'] = E[\vec{x}]E[\vec{y}]'$.

EECS 501 APPLICATIONS OF COVARIANCE MATRICES Fall 2001

1. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. $E[\vec{x}] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. $K_x = \begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}$.

Note: K_x symmetric (obvious), positive semidefinite (check: Matlab's "eig").

Q: K_x has $\lambda_3 = 0$ and $v_3 = [1, -2, 1]'$. Significance of 0 eigenvalue?

A: Let $y = v_3'\vec{x} = 1x_1 - 2x_2 + 1x_3$. Then $\sigma_y^2 = v_3' K_x v_3 = v_3' \lambda_3 v_3 = 0$.
$y = x_1 + x_3 - 2x_2 = E[y] = v_3' E[\vec{x}] = 0$ with probability 1.
Not very random vector: $x_1 = 2, x_2 = 3 \rightarrow x_3 = 4$ with probability 1!

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2. Suppose eigenvalues are 100,98,95,2,1,0.1. Significance of grouping?
$\vec{y} = V'\vec{x} \rightarrow \vec{x} = V\vec{y} = \sum_{i=1}^{N} y_i v_i$ where y_i uncorrelated and $\sigma_{y_i}^2 = \lambda_i$.

DEF: This is the finite-dimensional *Karhunen-Loeve expansion* of $\vec{x}$.

Idea: Since $\sigma_{y_i}^2 \approx 0$ for $i = 4, 5, 6$, approximate $y_i \approx E[y_i]$ for $i = 4, 5, 6$.

i.e.: Treat y_1, y_2, y_3 as uncorrelated rvs; y_4, y_5, y_6 as known constants.

Point: Have *compressed* data $[x_1 \ldots x_6]'$ to $[y_1, y_2, y_3]'$; *reduced dimension.*

EECS 501 MULTIDIMENSIONAL GAUSSIAN PDF Fall 2001

DEF: $\{x_1 \ldots x_N\}$ are *jointly Gaussian rvs* (JGRV) if their joint pdf is
$f_{\vec{x}}(\vec{X}) = \frac{1}{(2\pi)^{N/2}\sqrt{|det\, K_x|}} exp[-\frac{1}{2}(\vec{X} - \vec{\mu})' K_x^{-1} (\vec{X} - \vec{\mu})]$. $\vec{x} \sim N(\vec{\mu}, K_x)$.

1. $\int \ldots \int f_{\vec{x}}(\vec{X}) dX_1 \ldots dX_N = 1$: See p.250. V=matrix of eigenvectors.

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2. $\vec{y} = V'\vec{x} \rightarrow f_{\vec{y}}(\vec{Y}) = \frac{1}{|det\, V|} f_{\vec{x}}(\vec{X} = V\vec{Y})$ so integrates to 1. $|det\, V| = 1$.
$= \frac{1}{(2\pi)^{N/2}\sqrt{|det\, V' K_x V|}} exp[-\frac{1}{2}(\vec{Y} - E[\vec{y}])' diag[\frac{1}{\lambda_i}](\vec{Y} - E[\vec{y}])]$
$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\lambda_i}} exp[-\frac{1}{2}(Y_i - E[y_i])^2/\lambda_i] = \prod_{i=1}^{N} f_{y_i}(Y_i)$

Point: For $\{x_1 \ldots x_N\}$ JGRV, uncorrelated$\leftrightarrow$independent. Unusual!
JGRV $\{x_1 \ldots x_n\}$ have diagonal $K_x \rightarrow \{x_1 \ldots x_N\}$ independent rvs.

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3. Any linear combination of JGRV is JGRV. $\{x_i\}$ Gaussian$\neq$ $\{x_i\}$ JGRV.
4. $f_{\vec{x},\vec{y}}(\vec{X}, \vec{Y})$ above form$\rightarrow$ $\vec{x}$ and $\vec{y}$ **each** JGRV; $f_{\vec{x}|\vec{y}}(\vec{X}|\vec{Y})$ above form.
5. 2-D: $f_{x,y}(X, Y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}(\frac{X^2}{\sigma_x^2} + \frac{Y^2}{\sigma_y^2} - \frac{2\rho XY}{\sigma_x\sigma_y})\right]$

where $K_{[x,y]'} = \begin{bmatrix} \sigma_x^2 & \lambda_{xy} \\ \lambda_{xy} & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & \rho_{xy} \\ \rho_{xy} & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix}$.