PROBLEM: GIVEN $\underline{x}, \underline{y}$ ARE 2 VECTOR JOINTLY GAUSSIAN RANDOM VARIABLES,

PROVE $\hat{\underline{x}}_{MSE}(\underline{Y}) = E[\underline{x}|\underline{y}=\underline{Y}] = \boxed{E(\underline{x}) + K_{xy}K_y^{-1}(\underline{Y}-E(\underline{y}))}$

WHERE $K_{xy} = E[(\underline{x}-E(\underline{x}))(\underline{y}-E(\underline{y}))^T]$ AND $K_y = K_{yy}$.

PROOF: WE SHOW $p_{\underline{x}|\underline{y}}(\underline{X}|\underline{Y}) = \frac{1}{(2\pi)^{N/2}|K_{x|y}|^{1/2}} e^{-(\underline{X}-E[\underline{x}|\underline{y}])^T K_{x|y}^{-1}(\underline{X}-E[\underline{x}|\underline{y}])/2}$

WHERE $E(\underline{x}|\underline{y}) = E(\underline{x}) + K_{xy}K_y^{-1}(\underline{Y}-E[\underline{y}])$ AND $K_{x|y} = K_x - K_{xy}K_y^{-1}K_{xy}^T$.

THE DESIRED RESULT FOLLOWS IMMEDIATELY. NOTE $E[\underline{e}\,\underline{e}^T] =$ ↑ (TAKE $E_y(\cdot)$)

LET $\underline{z} = \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix}$ SO THAT $E(\underline{z}) = \begin{bmatrix} E(\underline{x}) \\ E(\underline{y}) \end{bmatrix}$ AND $K_z = \begin{bmatrix} K_x & K_{xy} \\ K_{xy}^T & K_y \end{bmatrix}$.

LET $\underline{z}' = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \underline{z}$, WHERE A IS TO BE SPECIFIED. LET $\underline{z}' = \begin{bmatrix} \underline{x}' \\ \underline{y} \end{bmatrix}$ (NOTE $\underline{y}' = \underline{y}$).

THEN $K_{z'} = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} K_x & K_{xy} \\ K_{xy}^T & K_y \end{bmatrix} \begin{bmatrix} I & 0 \\ -A^T & I \end{bmatrix} = \begin{bmatrix} K_x - AK_{xy}^T - K_{xy}A^T + AK_yA^T & K_{xy} - AK_y \\ K_{xy}^T - K_yA^T & K_y \end{bmatrix}$

$\therefore$ CHOOSE $A = K_{xy}K_y^{-1}$ TO UNCORRELATE $\underline{x}'$ AND $\underline{y}$.

THIS CHOICE OF $A \longrightarrow E(\underline{x}') = E(\underline{x}) - AE(\underline{y}) = E(\underline{x}) - K_{xy}K_y^{-1}E(\underline{y})$ AND

$K_{\underline{x}'} = K_x - AK_{xy}^T - K_{xy}A^T + AK_yA^T = K_x - K_{xy}K_y^{-1}K_{xy}^T$.

NOW, $p_{\underline{z}}(\underline{Z}) = p_{\underline{x},\underline{y}}(\underline{X},\underline{Y}) = p_{\underline{x}',\underline{y}}(\underline{X}',\underline{Y}) = p_{\underline{z}'}(\underline{Z}')$ SINCE $\det \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} = 1$.

THUS $p_{\underline{x}|\underline{y}}(\underline{X}|\underline{Y}) = p_{\underline{x},\underline{y}}(\underline{X},\underline{Y})/p_{\underline{y}}(\underline{Y}) \overset{\text{FROM}}{=} p_{\underline{x}',\underline{y}}(\underline{X}',\underline{Y})/p_{\underline{y}}(\underline{Y}) = p_{\underline{x}'|\underline{y}}(\underline{X}'|\underline{Y}) = p_{\underline{x}'}(\underline{X}')$.

$\therefore p_{\underline{x}|\underline{y}}(\underline{X}|\underline{Y}) = p_{\underline{x}'}(\underline{X}')$

$= \frac{1}{(2\pi)^{N/2}|K_{x'}|^{1/2}} e^{-(\underbrace{\underline{X}-A\underline{Y}}_{X'}-E(\underline{x}'))^T K_{x'}^{-1}(\underbrace{\underline{X}-A\underline{Y}}_{X'}-E[\underline{x}'])/2}$

SINCE $\underline{x}'$ AND $\underline{y}$ ARE UNCORRELATED $\rightarrow$ INDEPENDENT SINCE JOINTLY GAUSSIAN.

$= \frac{1}{(2\pi)^{N/2}|K_{x'}|^{1/2}} e^{-(\underline{X}-[\underbrace{E(\underline{x})+K_{xy}K_y^{-1}(\underline{Y}-E(\underline{y}))}_{=E[\underline{x}|\underline{Y}]}])^T K_{x'}^{-1}(\underline{X}-[\underbrace{E(\underline{x})+K_{xy}K_y^{-1}(\underline{Y}-E(\underline{y}))}_{=E[\underline{x}|\underline{Y}]}])/2}$

SINCE $K_{x'} = K_{x|y}$ (DEFINED ABOVE), WE ARE DONE.

NOTE THAT $\hat{\underline{x}}(\underline{Y}) = E(\underline{x}) + K_{xy}K_y^{-1}(\underline{Y}-E(\underline{y}))$ IS BEST LINEAR LEAST-SQUARES ESTIMATE OF $\underline{x}$ GIVEN $\underline{y}=\underline{Y}$. THIS CAN BE DERIVED AS IN SCALAR CASE, ALTHOUGH MESSIER