FALL 2001

EECS 501 HANDOUT: II AND MARKOV RPs

DEF: A CONT.-TIME RP $X(t)$ IS SEPARABLE IF THE KOLMOGOROV EXTENSION THM. HOLDS FOR IT $\leftrightarrow$ KNOWLEDGE OF THE JOINT pdf $f_{X(t_N),X(t_{N-1}),\ldots X(t_1)}(X_N \ldots X_1)$ FOR ANY TIMES $t_N > t_{N-1} > \ldots > t_1$ AND ANY N COMPLETELY SPECIFIES $X(t)$. NOTE THAT ALL DISCRETE-TIME RPs ARE SEPARABLE.

NOTE: IN EECS 501 WE ONLY CONSIDER SEPARABLE RPs. FOR AN EXAMPLE OF A NON-SEPARABLE RP SEE p. 265-6.

DEF: THE INCREMENT OF A RP OVER THE INTERVAL $(s,t]$ IS THE RV $X(t)-X(s)$.

EX: THE INCREMENT OF A POISSON COUNTING PROCESS OVER $(s,t]$ IS THE NUMBER OF ARRIVALS IN $(s,t]$.

DEF: A RP HAS STATIONARY INCREMENTS IF $f_{X(t)-X(s)}(X) = f_{X(t-s)-X(0)}(X) = f_{X(t-s)}(X)$ IF $X(0)=0$ $\leftrightarrow$ pdf FOR INCREMENT OVER $(s,t]$ DEPENDS ONLY ON $t-s$, NOT ON t AND s SEPARATELY (SOUND FAMILIAR?)

DEF: AN INDEPENDENT INCREMENTS (II) RP IS A 1-SIDED RP WITH $X(0)=0$ HAVING INCREMENTS THAT ARE INDEPENDENT OVER NON-OVERLAPPING INTERVALS (2 INTERVALS DO NOT OVERLAP IF THEIR INTERSECTION IS $\emptyset$)

EXS: (1) POISSON COUNTING PROCESS (2) WIENER PROCESS (3) MANY TYPES OF "COUNTING" PROCESSES

NOTE: II RPs ARE USUALLY IMPLICITLY ASSUMED TO HAVE INCREMENTS STATIONARY AS WELL AS INDEPENDENT.

THM: AN II RP IS COMPLETELY SPECIFIED BY ITS MARGINAL pdf $f_{X(t)}(X)$ (ASSUMING: 1. SEPARABLE RP 2. STATIONARY INCREMENTS).

PROOF: RECALL THE FOLLOWING: LET $\underline{Y} = A\underline{X}$ (VECTOR, MATRIX, VECTOR). THEN $f_{\underline{Y}}(\underline{Y}) = \frac{1}{|\det A|} f_{\underline{X}}(A^{-1}\underline{Y})$. $\underline{Y}=A\underline{X} \to f_{A\underline{X}}(A\underline{X}) = \frac{1}{|A|} f_{\underline{X}}(\underline{X})$.

APPLYING THIS TO

$$\underbrace{\begin{bmatrix} X(t_N)-X(t_{N-1}) \\ X(t_{N-1})-X(t_{N-2}) \\ \vdots \\ X(t_1)-\overset{0}{X(0)} \end{bmatrix}}_{\underline{Y}} = \underbrace{\begin{bmatrix} 1 & -1 & & 0 \\ & \ddots & \ddots & \\ & & & -1 \\ 0 & & & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X(t_N) \\ X(t_{N-1}) \\ \vdots \\ X(t_1) \end{bmatrix}}_{\underline{X}}$$

YIELDS $f_{X(t_N),X(t_{N-1})\ldots X(t_1)}(X_N, X_{N-1},\ldots X_1)$

$$= f_{\underbrace{X(t_N)-X(t_{N-1}),\, X(t_{N-1})-X(t_{N-2}),\ldots X(t_1)}_{A\underline{X}}}\underbrace{(X_N - X_{N-1},\ldots X_1)}_{A\underline{X}}$$

$$= \prod_{i=1}^N f_{X(t_i)-X(t_{i-1})}(X_i - X_{i-1}) \text{ USING II}$$

$$= \prod_{i=1}^N \underbrace{f_{X(t_i - t_{i-1})}}_{\text{GIVEN THESE}}(X_i - X_{i-1}) \text{ USING STATIONARY INCREMENTS}$$

$(X(t_0)=X(0)=0)$

$\therefore$ CAN COMPUTE $f_{X(t_N)\ldots X(t_1)}(X_N\ldots X_1)$ FROM $f_{X(t)}(X)$ FOR ANY CHOICE OF $t_N>t_{N-1}>\ldots>t_1$ AND ANY N, USING $\longrightarrow$

$\therefore$ HAVE COMPLETELY SPECIFIED $X(t)$. QED.

THM: ANY II RP (WITH STATIONARY INCREMENTS AND $X(0)=0$) HAS: $\boxed{E[X(t)]=\mu t,\ \sigma^2_{X(t)} = \sigma^2 t, \text{ AND } K_X(t,s)=\sigma^2 \min[t,s]}$ (WHERE $\mu = E[X(1)]$ AND $\sigma^2 = \sigma^2_{X(1)}$).

PROOF: NEED TO ASSUME $E[X(t)]$ AND $\sigma^2_{X(t)}$ ARE CONTINUOUS FUNCTIONS OF t.

LEMMA: IF A FUNCTION $f(x)$ IS CONTINUOUS ON $[0,\infty)$ AND HAS THE PROPERTY $f(x+y)=f(x)+f(y)$ FOR ALL $x,y\geq 0$, THEN $f(x)=f(1)x$ (i.e., $f(x)$ IS LINEAR). (STANDARD ANALYSIS RESULT)

$E[X(t)]$: $E[X(t+s)-X(s)] = \int X f_{X(t+s)-X(s)}(X)\,dX \overset{\uparrow}{=} \int X f_{X(t)}(X)\,dX = E[X(t)]$. THEN $E[X(t+s)]=E[X(t)]+E[X(s)]$ $\to E[X(t)] = E[X(1)]t = \mu t$.

$\sigma^2_{X(t)}$: $E[(X(t+s)-X(s))^2] = \int X^2 f_{X(t+s)-X(s)}(X)\,dX \overset{\text{STATIONARY INCREMENTS}}{=} \int X^2 f_{X(t)}(X)\,dX = E[X(t)^2]$. THEN $\sigma^2_{X(t+s)-X(s)} = \sigma^2_{X(t)}$.

NOW WE USE INDEPENDENT INCREMENTS: $X(t+s) = \underbrace{(X(t+s)-X(s))}_{\text{INCREMENT}} + \underbrace{(X(s)-\overset{0}{X(0)})}_{\text{INCREMENT}}$ (INDEPENDENT SINCE NON-OVERLAPPING) $\to \sigma^2_{X(t+s)} = \sigma^2_{X(t+s)-X(s)} + \sigma^2_{X(s)} = \sigma^2_{X(t)} + \sigma^2_{X(s)}$

$\to \sigma^2_{X(t)} = \sigma^2_{X(1)}t = \sigma^2 t$. QED.

$K_X(t,s)$: ASSUME WLOG $t>s$.

$K_X(t,s) = \lambda_{X(t),X(s)} = \lambda_{(X(t)-X(s)+X(s)),X(s)} = \lambda_{\underbrace{X(t)-X(s)}_{\text{INCREMENT}},\, \underbrace{X(s)-\overset{0}{X(0)}}_{\text{INCREMENT}}} + \underbrace{\lambda_{X(s),X(s)}}_{\sigma^2_{X(s)}} = \sigma^2_{X(s)} = \sigma^2 s$ FROM ABOVE.

λ = "COVARIANCE BETWEEN"; INDEPENDENT SINCE $t>s$; INDPT $\to$ UNCORRELATED.

NOW ASSUME $t<s$. EXCHANGE s AND t IN ABOVE ARGUMENT TO GET $K_X(t,s)=\sigma^2 t$. COMBINE INTO $K_X(t,s)=\sigma^2\min[t,s]$.

THM: $X(n)$ IS A DISCRETE-TIME II RP (WITH STATIONARY INCREMENTS AND $X(0)=0$) IFF $X(n)=\sum_{i=1}^n U(i)$ WHERE $U(i)$ IS iid

PROOF: $\longrightarrow$: LET $U(n)=X(n)-X(n-1)$. THEN $X(n)$ II WITH STATIONARY INCREMENTS $\to U(n)$ iid. QED.

$\longleftarrow$: $X(n_2)-X(n_1) = \sum_{i=n_1+1}^{n_2} U(i)$. SINCE $U(n)$ iid, THE INCREMENTS $X(n_2)-X(n_1)$ ARE INDPT OVER NON-OVERLAP INTERVALS. (iid = INDPT, IDENTICALLY DISTRIBUTED)

NOTE: THIS THM IS NOT TRUE FOR CONT.-TIME RPs, SINCE WOULD NEED $U(t) = \frac{d}{dt}X(t)$ AND $X(t)$ MAY NOT BE DIFFERENTIABLE.

COROLLARY: USING THIS THM, CAN SHOW DIRECTLY THAT IF $X(n)$ IS AN II RP, THEN $E[X(n)]=\mu n$, $\sigma^2_{X(n)}=\sigma^2 n$, $K_X(i,j)=\sigma^2\min[i,j]$ WHERE $\mu=E[U(n)]$ AND $\sigma^2=\sigma^2_{U(n)}$.

PROOF: RECALL $U(n) \to \boxed{h(n)} \to X(n) \to$ $\boxed{E[X(n)] = \sum h(i)E[U(n-i)]}$, $\boxed{K_X(m,n)=\sum h(m-i)\sum h(n-j)K_U(i,j).}$

HERE $X(n) = \sum_{i=1}^n U(i) \to h(n) = \begin{cases} 1 & n\geq 0 \\ 0 & n<0 \end{cases}$.

$E[X(n)] = \sum_{i=1}^n 1\cdot\mu = \boxed{n\mu}$. QED.

$K_X(m,n) = \sum_{i=1}^m\sum_{j=1}^n 1\cdot 1\cdot\sigma^2\delta(i-j) = \boxed{\sigma^2\min[m,n]}$ QED.

DEF: A RP $X(t)$ IS **MARKOV** IF FOR ANY $t_{N+1} > t_N > t_{N-1} > \cdots > t_1$ AND ANY N, THE CONDITIONAL JOINT pdf

$$\underbrace{f_{X(t_{N+1})}}_{\text{FUTURE}} \,|\, \underbrace{X(t_N)}_{\text{PRESENT}}, \underbrace{X(t_{N-1}), \ldots X(t_1)}_{\text{PAST}} (X_{N+1} | X_N, \ldots X_1) = f_{\underbrace{X(t_{N+1})}_{\text{FUTURE}} | \underbrace{X(t_N)}_{\text{PRESENT}}} (X_{N+1} | X_N)$$

$\underbrace{\ldots t_{N-2}\; t_{N-1}}_{\text{PAST}}\quad \underset{\text{PRESENT}}{t_N} \quad \underset{\text{FUTURE}}{t_{N+1}} \rightarrow t$

"THE FUTURE VALUES OF THE RP, GIVEN ITS PRESENT VALUE, IS INDEPENDENT OF ITS PAST VALUES."

NOTE: COMPARE ABOVE DEF. OF A MARKOV RP TO FOLLOWING DEF. OF AN INDPT. RP: $f_{X(t)|X(s)}(X_t | X_s) = f_{X(t)}(X_t)$.
"FOR AN INDPT. RP, PAST DOESN'T MATTER. FOR A MARKOV RP, ONLY THE MOST RECENT PAST MATTERS."

DEF: A RP $X(t)$ IS k^{TH}-ORDER MARKOV IF FOR ANY $t_N > t_{N-1} > \cdots > t_1$ AND ANY N, THE CONDITIONAL JOINT pdf

$$f_{X(t_N)|X(t_{N-1}), X(t_{N-2}), \ldots X(t_1)}(X_N | X_{N-1}, \ldots X_1) = f_{X(t_N)|X(t_{N-1}), X(t_{N-2}), \ldots X(t_{N-k})}(X_N | X_{N-1}, \ldots X_{N-k}).$$

FOR A k^{TH}-ORDER MARKOV RP, ONLY THE k MOST RECENT TIMES MATTER. RELATE TO VECTOR MARKOV RPs (P. 359).

EXS: (1) RANDOM TELEGRAPH WAVE (2) POISSON COUNTING PROCESS (3) WIENER PROCESS (4) DISCRETE-VALUED, FINITE STATE MARKOV RPs = MARKOV CHAINS (P. 278) IMPORTANT IN QUEUEING THEORY (EECS 502)
(5) OUTPUT OF A 1^{ST} ORDER SYSTEM DRIVEN BY AN iid RP:

$v(t)$ iid rp $\rightarrow \boxed{\frac{dx}{dt} = -\lambda x(t) + v(t)} \rightarrow x(t)$ IS: (1) 0-MEAN, WSS. $S_X(w) = \frac{\sigma^2}{\lambda^2 + w^2}$. $R_X(\tau) = \frac{\sigma^2}{2\lambda} e^{-\lambda|\tau|}$.

$S_v(w) = \sigma^2$

(2) MARKOV SINCE THE SOLUTION TO THE 1^{ST}-ORDER DIFF. EQN CAN BE WRIT. AS

$$x(t) = \underbrace{x(s) e^{-\lambda(t-s)}}_{\text{ZERO-INPUT RESPONSE}} + \underbrace{\int_s^t e^{-\lambda(t-r)} v(r)\, dr}_{\text{ZERO-STATE RESPONSE}} \text{ FOR } t > s$$

(BASIC IDEA OF "STATE" OF SYSTEM) DON'T NEED EARLIER VALUES $x(r), 0 < r < s$.

IF $v(t)$ GAUSSIAN, CAN CHANGE "iid" TO "WHITE". $x(t)$ DEPENDS ON $x(s)$ AND $\{v(r), s < r < t\}$ WHICH IS INDEPENDENT OF $\{v(r), 0 < r < s\}$ SINCE $v(t)$ iid rp.

iid $v(t) \rightarrow \boxed{k^{TH} \text{ ORDER SYSTEM}} \rightarrow x(t)$ IS k^{TH}-ORDER MARKOV

THM: A MARKOV RP IS COMPLETELY SPECIFIED (ASSUMING SEPARABLE RP) BY ITS: (1) INITIAL pdf $f_{X(0)}(X)$ (2) TRANSITION pdf $f_{X(t)|X(s)}(X_t|X_s)$

PROOF: ONCE AGAIN WE NEED TO CONSTRUCT $f_{X(t_N), X(t_{N-1}), \ldots X(t_1)}(X_N, \ldots X_1)$ FOR ANY TIMES $t_N > t_{N-1} > \cdots > t_1$ & ANY N.

$N=1$: $f_{X(t)}(X) = \int \underbrace{f_{X(t)|X(0)}(X_t | X_0)}_{\text{TRANSITION pdf}} \underbrace{f_{X(0)}(X_0)}_{\text{INITIAL pdf}} dX_0$. $N=2$: $f_{X(t_2), X(t_1)}(X_2, X_1) = \underbrace{f_{X(t_2)|X(t_1)}(X_2|X_1)}_{\text{TRANSITION pdf}} \underbrace{f_{X(t_1)}(X_1)}_{\text{HAVE FROM } N=1}$

$N=3$: $f_{X(t_3), X(t_2), X(t_1)}(X_3, X_2, X_1) \underset{\text{BASIC DEF. OF COND. pdf}}{=} f_{X(t_3)|X(t_2), X(t_1)}(X_3|X_2, X_1) f_{X(t_2), X(t_1)}(X_2, X_1) \underset{X(t) \text{ IS MARKOV}}{=} \underbrace{f_{X(t_3)|X(t_2)}(X_3|X_2)}_{\text{TRANSITION pdf}} \underbrace{f_{X(t_2), X(t_1)}(X_2, X_1)}_{\text{HAVE FROM } N=2}$

N FROM $N-1$: $f_{X(t_N), X(t_{N-1}) \ldots X(t_1)}(X_N \ldots X_1) \stackrel{*}{=} f_{X(t_N)|X(t_{N-1}), \ldots X(t_1)}(X_N | X_{N-1} \ldots X_1) f_{X(t_{N-1}) \ldots X(t_1)}(X_{N-1}, \ldots X_1)$

(SEE P. 392) SINCE $X(t)$ IS MARKOV, $= \underbrace{f_{X(t_N)|X(t_{N-1})}(X_N|X_{N-1})}_{\text{TRANSITION pdf}} \underbrace{f_{X(t_{N-1}) \ldots X(t_1)}(X_{N-1} \ldots X_1)}_{\text{FROM } N-1}$. INDUCTION COMPLETE.

CHAPMAN-KOLMOGOROV EQN: $f_{X(t)|X(s)}(X_t|X_s)$ IS A VALID TRANSITION pdf IFF $f_{X(t_3)|X(t_1)}(X_3|X_1) = \int f_{X(t_3)|X(t_2)}(X_3|X_2) f_{X(t_2)|X(t_1)}(X_2|X_1)\, dX_2$
THINK ABOUT WHAT THIS IS SAYING. (SEE P. 398)

THM: ANY II RP IS ALSO MARKOV: $\boxed{\text{II} \rightarrow \text{MARKOV}}$

PROOF: LET $X(t)$ BE AN II RP. $f_{X(t_{N+1})|X(t_N) \ldots X(t_1)}(X_{N+1}|X_N \ldots X_1) \stackrel{*}{=} f_{X(t_{N+1}) - X(t_N) | X(t_N) \ldots X(t_1)}(X_{N+1} - X_N | X_N \ldots X_1)$

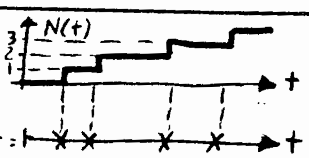
* SINCE WE ARE GIVEN $X(t_N) = X_N$, THIS IS JUST SHIFTING pdf BY A CONSTANT

$= f_{X(t_{N+1}) - X(t_N) | X(t_N) - X(0), X(t_{N-1}) - X(0), \ldots X(t_1) - X(0)}(X_{N+1} - X_N | X_N \ldots X_1) \underset{\text{II}}{=} f_{X(t_{N+1}) - X(t_N)}(X_{N+1} - X_N)$ ($X(0) = 0$)

$\underset{\text{II}}{=} f_{X(t_{N+1}) - X(t_N) | X(t_N) - X(0)}(X_{N+1} - X_N | X_N) = f_{X(t_{N+1})|X(t_N)}(X_{N+1}|X_N) \rightarrow X(t)$ IS MARKOV. QED. (SEE P. 393.)

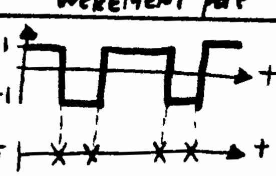
THE CONVERSE OF THIS THM IS NOT TRUE: THE RANDOM TELEGRAPH WAVE IS MARKOV (POISSON PROCESS HAS NO MEMORY) BUT IT IS NOT II SINCE ITS COVARIANCE FUNCTION IS $e^{-2\lambda|t-s|} \neq \sigma^2 \text{MIN}(t,s)$.
IN FACT, AN II RP IS A MARKOV RP SUCH THAT $\underbrace{f_{X(t)|X(s)}(X_t|X_s)}_{\text{TRANSITION pdf}} = \underbrace{f_{X(t) - X(s)}(X_t - X_s)}_{\text{INCREMENT pdf}}$.

POISSON COUNTING PROCESS:



PIECEWISE CONSTANT NONDECREASING II, MARKOV
$E[N(t)] = \lambda t$
$K_N(t,s) = \lambda \text{MIN}[t,s]$

RANDOM TELEGRAPH WAVE:



$X(t) = X(0)(-1)^{N(t)}$
$X(0) = \begin{cases} 1 & \text{PROB. } 1/2 \\ -1 & \text{PROB. } 1/2 \end{cases}$
PIECEWISE CONSTANT MARKOV, NOT II
$E[X(t)] = 0$
$K_X(t,s) = e^{-2\lambda|t-s|}$

$K_X(t,s)$ FOR RANDOM TELEGRAPH WAVE:

$K_X(t,s) = E[X(t)X(s)] = 1 \cdot Pr[X(t) \text{ AND } X(s) \text{ HAVE SAME SIGN}] - 1 \cdot Pr[X(t) \text{ AND } X(s) \text{ HAVE DIFFERENT SIGNS}]$
(0-MEAN SINCE $X(0)$ IS)

$= Pr[\text{EVEN \# ARRIVALS IN } |t-s|] - Pr[\text{ODD \# ARRIVALS IN } |t-s|]$ $\quad (-1)^k = \begin{cases} 1 & k \text{ EVEN} \\ -1 & k \text{ ODD} \end{cases}$

$= \sum_{k=0}^{\infty} (-1)^k (\lambda|t-s|)^k e^{-\lambda|t-s|}/k! = e^{-2\lambda|t-s|}$. QED. (SAME AS GAUSS-MARKOV)