DEF: A *discrete* random variable (rv) is a mapping $x : \Omega \to$ *countable* set.
DEF: A *continuous* random variable is a mapping $x : \Omega \to$ *uncountable* set.
WLOG: Countable range of discrete rv is 1-1 with $\mathcal{Z}$=integers.

DEF: The *pmf* (probability mass function) for a *discrete* rv x is
Props: $p_x(X) = Pr[x = X]$. We have: $0 \leq p_x(X) \leq 1$; $\sum_{X=-\infty}^{\infty} p_x(X) = 1$.
For *continuous* rvs, use pdf (probability density function) $f_x(X)$.

EX1: Discrete rv x=#heads in 3 independent flips of a fair coin.
$p_x(0) = p_x(3) = \frac{1}{8}$; $p_x(1) = p_x(2) = \frac{3}{8}$; $p_x(X) = 0$ otherwise.

EX2: *Geometric pmf:* $p_x(X) = (1-a)^{X-1}a, X \geq 1, \quad 0 < a < 1$.
Note: $\sum_{X=1}^{\infty}(1-a)^{X-1}a = a/(1-(1-a)) = 1$ and $0 < (1-a)^{X-1}a < 1$.

Note: PDF (distribution) $F_x(X)$ for discrete rv x is piecewise constant (staircase) function with a countable number of discontinuities.

But: Flip a fair coin. If heads, flip a 2^{nd} coin with Pr[heads]=2/3.
If 2^{nd} coin heads, set rv x=3; if 2^{nd} coin tails, set rv x=1.
If 1^{st} coin tails, spin wheel of fortune and multiply the result by 5.

1. Sample space $\Omega = (\{heads\} \times \{1, 3\}) \cup (\{tails\} \times [0, 5))$.
2. PDF is discontinuous (below left); pdf contains impulses (below right).
3. x is called a *mixture* rv: neither discrete nor continuous.
4. To express using pdf, need impulse to put finite prob. at a single point.
5. Example: $Pr[x \leq 2] = F_x(2) = \frac{1}{2}(1/3) + \frac{1}{2}(1/5)2 = 11/30$.
6. Note $Pr[x < 1] = 0.1 \neq Pr[x \leq 1] = 0.2666...$
   due to the discontinuity of the PDF $F_x(X)$ at $x = 1$.

DEF: Joint PDF $F_{x_1 \ldots x_N}(X_1 \ldots X_N) = Pr[x_1 \leq X_1 \,\&\, x_2 \leq X_2 \ldots]$.

DEF: Joint pdf $f_{x_1 \ldots x_N}(X_1 \ldots X_N) = \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_N} F_{x_1 \ldots x_N}(X_1 \ldots X_N)$.

Note: $Pr[X_1 < x_1 \leq X_1 + \delta X_1 \; AND \ldots X_N < x_N \leq X_N + \delta X_N]$
$= f_{x_1 \ldots x_N}(X_1 \ldots X_N)\delta X_1 \ldots \delta X_N$ (compare to 1-D pdf).

EX: Amy and Bo meet for lunch. 1^{st} to arrive will wait only 15 minutes. Amy and Bo equally likely to arrive anytime between noon and 1 PM.

Q1: Assume arrival times independent. Pr[Amy and Bo meet for lunch]=?

Let: x=Amy's arrival time in fraction of hour after noon; y=Bo's arrival.

Ω: Sample space: $\Omega = [0,1]^2$. $f_{x,y}(X,Y) = 1$ for $(X,Y) \in [0,1]^2$.

Event: A=they meet for lunch=$\{(X,Y) : |X-Y| < \frac{1}{4}\}$. Want Pr[A].

$$Pr[A] = \underbrace{\int\int}_{|X-Y|<1/4} f_{x,y}(X,Y)dX\,dY = Area[|X-Y| < 1/4] = 7/16.$$

Q2: Now suppose Amy and Bo are both more likely to arrive late:

$$f_x(X) = \begin{cases} 2X & \text{if } 0 < X < 1 \\ 0 & \text{otherwise} \end{cases}; \quad f_y(Y) = \begin{cases} 2Y & \text{if } 0 < Y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

$$Pr[A] = \underbrace{\int\int}_{|X-Y|<1/4} f_{x,y}(X,Y)dX\,dY = \underbrace{\int\int}_{|X-Y|<1/4} (2X)(2Y)dX\,dY$$

$$= 1 - 2\int_{\frac{1}{4}}^{1} dX \int_0^{X-\frac{1}{4}} dY\, 4XY = 1 - 2\int_0^{\frac{3}{4}} dX \int_{X+\frac{1}{4}}^{1} dY\, 4XY = 139/256.$$

DEF: The *marginal* pdfs for x and y are computed from joint pdf $f_{x,y}(X,Y)$:
$f_x(X) = \int f_{x,y}(X,Y)dY$; $f_y(Y) = \int f_{x,y}(X,Y)dX$.

DEF: x and y are *independent* rvs if $f_{x,y}(X,Y) = f_x(X)f_y(Y)$.

1. If given that x, y independent (as above), can compute $f_{x,y}(X,Y)$.
2. Given $f_{x,y}(X,Y)$, can compute marginals pdfs $f_x(X), f_y(Y)$ and check whether $f_{x,y}(X,Y) = f_x(X)f_y(Y)$. If so, x, y are independent.
3. If support (nonzero region) of $f_{x,y}(X,Y)$ is not rectangular, we know immediately that x, y are *not* independent rvs.

DEF: $p_{x,y|A}(X,Y|A) = Pr\left[\begin{smallmatrix} x=X \\ y=Y \end{smallmatrix}|A\right] = \begin{cases} p_{x,y}(X,Y)/Pr[A] & \text{if } (X,Y) \in A \\ 0 & \text{if } (X,Y) \notin A \end{cases}.$

EX: Let $p_{x,y}(X,Y) = \frac{1}{4}$ if $(X,Y) \in [1,2]^2$. Let $A = \{(X,Y) : X+Y \geq 3\}$.
$p_{x,y|A}(X,Y|A) = \frac{1}{3}$ if $(X,Y) \in \{(1,2),(2,1),(2,2)\}$; 0 otherwise.

DEF: *Conditional marginal* pmf $p_{x|A}(X|A) = \sum_Y p_{x,y|A}(X,Y|A)$.

EX: Here, $p_{x|A}(1|A) = \frac{1}{3}$; $\quad p_{x|A}(2|A) = \frac{2}{3}$; $\quad p_{x|A}(X|A) = 0$ otherwise.

DEF: *Conditional* pmf $p_{x|y}(X|Y) = Pr[x=X|y=Y] = \frac{p_{x,y}(X,Y)}{\sum_X p_{x,y}(X,Y)}$.

EX: $p_{x|y}(1|2) = p_{x|y}(2|2) = \frac{1}{2}$; $\quad p_{x|y}(X|Y) = 0$ otherwise.

DEF: x, y *conditionally independent* if $p_{x,y|A}(X,Y|A) = p_{x|A}(X|A)p_{y|A}(Y|A)$.

DEF: *Conditional* pdf $f_{x|y} = f_{x,y}(X,Y)/f_y(Y)$.

Note: $Pr[X < x \leq X + \delta X | Y < y \leq Y + \delta Y] = \frac{Pr[X<x\leq X+\delta X \;\&\; Y<y\leq Y+\delta Y]}{Pr[Y<y\leq Y+\delta Y]}$

$\rightarrow f_{x|y}(X|Y)\delta X = f_{x,y}(X,Y)\delta X\, \delta Y/(f_y(Y)\delta Y) = f_{x,y}(X,Y)/f_y(Y)\delta X$

EX: $f_{x,y}(X,Y) = c$ if $0 < X < 1$ & $X < Y < 4X$; otherwise $f_{x,y}(X,Y) = 0$.

Let: $A = \{(X,Y) : X+Y \leq 2\}$. Compute conditional marginal $f_{x|A}(X|A)$.

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c: $1 = \int\int f_{x,y}(X,Y)dX\,dY = c\frac{1}{2}(4\cdot 1 - 1\cdot 1) = c\frac{3}{2} \rightarrow c = \frac{2}{3}$.

Pr[A]: $Pr[A] = c\frac{1}{2}(2\cdot 2 - 2\cdot 1 - 2\cdot 0.4) = 0.4$. See figure below.

1. $f_{x,y|A}(X,Y|A) = \frac{2}{3}/0.4$ for $(X,Y) \in A$; 0 otherwise.
   Always compute $f_{x,y|A}(X,Y|A)$ *first, then* $f_{x|A}(X|A)$.

2. $f_{x|A}(X|A) = \int f_{x,y|A}(X,Y|A)dY = \int(\frac{2}{3}/0.4)dY$ for $(X,Y) \in A$.

3. $f_{x|A}(X|A) = \begin{cases} \frac{1}{0.4}\frac{2}{3}(4X - X) = 5X & \text{if } 0 < X < 0.4 \\ \frac{1}{0.4}\frac{2}{3}(2 - X - X) = \frac{10}{3}(1-X) & \text{if } 0.4 < X < 1 \end{cases}$