DEF: Ω=*sample space*=set of all distinguishable outcomes of an experiment.
DEF: $\mathcal{A}$=*event space*=set of subsets of Ω such that $\mathcal{A}$ is a $\sigma - algebra$.

DEF: An $Algebra = \mathcal{A}$=a set of subsets of a set Ω such that:

1. $A \in \mathcal{A}$ and $B \in \mathcal{A} \rightarrow A \cup B \in \mathcal{A}$ and $A \cap B \in \mathcal{A}$;
2. $A \in \mathcal{A} \rightarrow A' = \Omega - A \in \mathcal{A}$. Closed under $\cup, \cap$, complement in Ω.

DEF: A $\sigma - algebra$ is an algebra closed under *countable* number of $\cup, \cap$.
NOTE: Empty set=ϕ and Ω are always members of any algebra $\mathcal{A}$,
since $A \in \mathcal{A} \rightarrow A' \in \mathcal{A} \rightarrow \phi = A \cap A' \in \mathcal{A}$ and $\Omega = A \cup A' \in \mathcal{A}$.
NOTE: $A \in \mathcal{A}$ and $B \in \mathcal{A} \rightarrow A \cup B \in \mathcal{A}$ and $A \cap B = (A' \cup B')' \in \mathcal{A}$.
So DeMorgan's law→closure under $\cup$ and $'$ →closure under $\cap$.

EX: **Experiment:** Flip a coin twice. Let H_i=heads on i^{th} flip.
Sample space: $\Omega = \{H_1H_2, H_1T_2, T_1H_2, T_1T_2\}$ (2^2 elements).
Event space: $\mathcal{A}$=power set of Ω=set of all subsets of Ω (2^{2^2} elements).
$\mathcal{A} = \{\phi, \Omega, \{H_1H_2\}, \{H_1T_2\}, \{T_1H_2\}, \{T_1, T_2\}, \{H_1\}, \{H_2\}, \{T_1\}, \{T_2\},$
$\{H_1H_2\} \cup \{T_1T_2\}, \{H_1T_2\} \cup \{H_2T_1\}, \{H_1H_2\}', \{H_1T_2\}', \{T_1H_2\}', \{T_1T_2\}'\}$.

DEF: The σ-algebra *generated* by the sets $A_n, n = 1, 2 \ldots \subset \Omega$ is the
set of all countable unions, intersections, and complements of A_n.
EX: $\Omega = \{a, b, c\}$. σ-algebra generated by set $\{a, b\}$ is $\{\phi, \Omega, \{a, b\}, \{c\}\}$.

DEF: *Probability* is a *mapping* $Pr : \mathcal{A} \rightarrow [0, 1]$ such that:
Domain: $\mathcal{A} = \sigma$-algebra=set of subsets of Ω. $\mathcal{A}$ is called an "event space."
Range: $[0, 1] = \{x : 0 \leq x \leq 1\}$ (a closed interval of the real line).
and such that Pr satisfies the three *Axioms of Probability*:

1. $Pr[A] \geq 0$ for any $A \in \mathcal{A}$; 2. $Pr[\Omega] = 1$ (maximum value is one);
3. If $\{A_n\}$ are *pairwise disjoint* $\Leftrightarrow A_i \cap A_j = \phi$ for $i \neq j$,
then $Pr[\cup_{n=1}^{\infty} A_n] = \sum_{n=1}^{\infty} Pr[A_n]$ (probs. of disjoint sets add).
In particular, $A \cap B = \phi \rightarrow Pr[A \cup B] = Pr[A] + Pr[B]$.

- $1 = Pr[\Omega] = Pr[A \cup A'] = Pr[A] + Pr[A'] \rightarrow Pr[A'] = 1 - Pr[A]$.
- $Pr[A] = Pr[A \cup \phi] = Pr[A] + Pr[\phi] \rightarrow Pr[\phi] = 0$ since $A \cap \phi = \phi$.
- $Pr[\phi] = 0$ BUT $Pr[A] = 0$ does NOT imply $A = \phi$! (see overleaf).

- In general, *assign* probabilities in sample space Ω.
- Then use the three axioms of probability to *compute* probabilites $Pr[A]$ for each $A \in \mathcal{A}$=event space=domain of Pr mapping.

"Thm.": Omitting "countable," the axioms of probability $\rightarrow 0 = 1$!

"Proof": First, we need the following lemma (small intermediate result):

DEF: A *wheel of fortune* is an experiment that generates an $x \in [0,1) = \Omega$ such that $Pr[\{x\}] = Pr[\{y\}]$ for all $x, y \in [0,1)$ ("equally likely choice").

Lemma: Let x be any specific number in $[0,1)$, e.g., $x = 0.5$. Then $Pr[\{x\}] = 0$.

Proof: Suppose $Pr[\{x\}] = \epsilon > 0$. Let $N = [1/\epsilon] + 1$ ($\epsilon = 0.001 \rightarrow N = 1001$). Then $Pr[\cup_{n=0}^{N-1}\{\frac{n}{N}\}] = \sum_{n=0}^{N-1} Pr[\{\frac{n}{N}\}] = \sum_{n=0}^{N-1} \epsilon = N\epsilon > 1$. No way.

"Proof": $1 = Pr[[0,1)] = Pr[\cup_{x\in[0,1)}\{x\}] = \sum_{x\in[0,1)} Pr[\{x\}] = \sum_{x\in[0,1)} 0 = 0!$

What went wrong? The third = above used the third axiom, assuming it held for $\cup_{x\in[0,1)}$ in the same way it holds for $\cup_{n=1}^{\infty}$.

Clearly there is a difference between $\mathcal{Z} = \{integers\}$ and $[0,1)$: The third axiom holds for the first infinite set but not the second. $\mathcal{Z}$ is *countably* infinite, while $[0,1)$ is *uncountably* infinite.

Four reasons to worry about countable vs. uncountable infinity:

1. The third axiom of probability holds only for countable infinities.
2. σ-algebras are closed only under a countable number of $\cup, \cap$.
   Later, we will encounter the following for random processes:
3. Discrete-time random processes are defined on countable times;
   Continuous-time processes are defined on uncountable times.
4. The *Kolmogorov extension theorem* holds only for countable times.

DEF: The *Borel sets*=$\mathcal{B}$ in $\mathcal{R}$=reals are the σ-algebra *generated* by the set of all open intervals $(a,b) = \{x : a < x < b\}$ for all $a, b \in \mathcal{R}$.

i.e.: Each $B \in \mathcal{B}$ can be written as a *countable* $\cup, \cap, '$ of intervals (a,b).

Who cares? For the wheel of fortune experiment, let $Pr[(a,b)] = b - a$. We can compute $Pr[B]$ for *any* $B \in \mathcal{B} \cap [0,1]$, and *only* for such B!

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1. $\{x\} \in \mathcal{B}$ since $\{x\} = \cap_{n=1}^{\infty}(x - \frac{1}{n}, x + \frac{1}{n})$ (singleton sets Borel).
2. $\{Rationals\} \in \mathcal{B}$ since $\{Rationals\} = \cup_{x\in\, countable\, set}\{x\}$.
3. $\{Irrationals\} \in \mathcal{B}$ since $\{Irrationals\} = \{Rationals\}'$ (σ-algebra).
4. BUT: $\mathcal{B}$ is NOT the power set (set of all subsets) of $\mathcal{R}$!
   There exist "unmeasurable sets" that are subsets of $\mathcal{R}$ but not of $\mathcal{B}$.
   Cannot compute $Pr[unmeasurable]$ using axioms of probability.