1. $\begin{Bmatrix} z = z(x,y) \\ w = w(x,y) \end{Bmatrix} = \begin{Bmatrix} z = x^2+y^2 \\ w = x \end{Bmatrix} \to$ Inverse $\begin{Bmatrix} x = x(z,w) \\ y = y(z,w) \end{Bmatrix} = \begin{Bmatrix} x = w \\ y = \pm\sqrt{z-w^2} \end{Bmatrix}.$

$|J| = |det \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}| = |det \begin{bmatrix} 2x & 2y \\ 1 & 0 \end{bmatrix}| = |-2y| = 2|\mp\sqrt{z-w^2}| = 2\sqrt{z-w^2}.$

$f_{z,w}(Z,W) = f_{x,y}(x(Z,W), y(Z,W))/|J| = \frac{f_{x,y}(W,+\sqrt{Z-W^2})}{2\sqrt{Z-W^2}} + \frac{f_{x,y}(W,-\sqrt{Z-W^2})}{2\sqrt{Z-W^2}}.$

$f_{z,w}(Z,W) = \frac{1}{2\pi\sigma^2}\frac{1}{\sqrt{Z-W^2}}e^{-\frac{Z}{2\sigma^2}}, Z > 0, |W| < \sqrt{Z}$

$f_z(Z) = \int_{-\infty}^{\infty} f_{z,w}(Z,W)dW = \int_{-\sqrt{Z}}^{\sqrt{Z}} \frac{1}{2\pi\sigma^2}\frac{1}{\sqrt{Z-W^2}}e^{-\frac{Z}{2\sigma^2}}dW = \frac{1}{2\sigma^2}e^{-\frac{Z}{2\sigma^2}}, Z > 0.$

Note: z is an *exponential* pdf.

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2. Try a 45^o rotation: $\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix} \to$

$x^2 - 2\rho xy + y^2 = \frac{1}{4}[(z+w)^2 - 2\rho(z+w)(z-w) + (z-w)^2] = \frac{1}{2}[(1-\rho)z^2 + (1+\rho)w^2] \to$

$f_{z,w}(Z,W) = \frac{1}{2}\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}e^{-\frac{(1-\rho)Z^2}{4\sigma^2(1-\rho^2)} - \frac{(1+\rho)W^2}{4\sigma^2(1-\rho^2)}} = \frac{e^{-Z^2/(4\sigma^2(1+\rho))}}{\sqrt{2\pi 2\sigma^2(1+\rho)}}\frac{e^{-W^2/(4\sigma^2(1-\rho))}}{\sqrt{2\pi 2\sigma^2(1-\rho)}}.$

Can easily see that $f_{z,w}(Z,W) = [N(0, 2\sigma^2(1+\rho))][N(0, 2\sigma^2(1-\rho))] = f_z(Z)f_w(W)$ where $N(0,\sigma^2)$ represents a Gaussian pdf. Note $|J| = 2$; this is the $\frac{1}{2}$ factor above.

Note: This problem will be *much* easier after we study *covariance matrices* later.

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3. Since $d = |x-y|$ is not differentiable, use *method of events*:

$F_d(D) = Pr[d \le D] = Pr[|x-y| \le D] = \iint_{|x-y|\le D} 1\,dX\,dY$

$F_d(D) = \begin{cases} 1-(1-D)^2 & 0 \le D \le 1; \\ 1 & \text{for } D \ge 1 \end{cases} \to f_d(D) = \frac{d}{dD}F_d(D) = \begin{cases} 2(1-D) & 0 \le D \le 1; \\ 0 & \text{otherwise} \end{cases}.$

Check: $F_d(D)$ nondecreasing and continuous;$F_d(-\infty) = 0; F_d(\infty) = 1; \int_{-\infty}^{\infty} f_d(D) = 1.$

4a. $1 = \int_1^2 dY \int_1^Y dX\, AX = \int_1^2 \frac{A}{2}(Y^2-1)dY = \frac{2}{3}A \to A = \frac{3}{2}.$

4b. $f_y(Y) = \int_1^Y \frac{3}{2}X\,dX = \frac{3}{4}(Y^2-1), 1 \le Y \le 2$; 0 otherwise.

4c. $f_{x|y}(X|\frac{3}{2}) = \frac{f_{x,y}(X,3/2)}{f_y(3/2)} = \frac{\frac{3}{2}X}{\frac{3}{4}(\frac{9}{4}-1)} = \frac{8}{5}X, 1 \le X \le \frac{3}{2}$; else 0.

4d. $F_z(Z) = 1 - Pr[(y-x) > Z] = 1 - \int_{Z+1}^2 dY \int_1^{Y-Z} dX(\frac{3}{2}X)$

$F_z(Z) = 1 - \int_{Z+1}^2 [\frac{3}{4}(Y-Z)^2 - 1]dY = \frac{1}{4}Z^3 - \frac{3}{2}Z^2 + \frac{9}{4}Z, 0 \le Z \le 1$; 0 otherwise.

Note: Inner integral $\frac{3}{4}[(Y-Z)^2-1]$ is just $f_y(Y-Z)$ from #4b.

Check: $F_z(0) = 0; F_z(1) = 1.$ $f_z(Z) = \frac{d}{dZ}F_z(Z) = \frac{3}{4}Z^2 - 3Z + \frac{9}{4}, 0 \le Z \le 1$; 0 otherwise.

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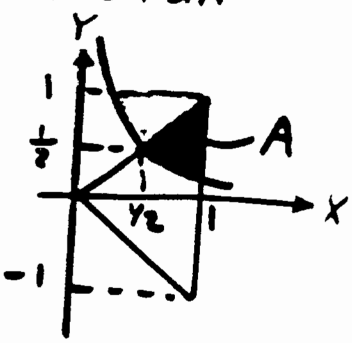
(a) $\iint f_{X,Y}(X,Y)\,dX\,dY = 1$. $\int_0^1 \int_{-X}^{X} cXY^2\,dY\,dX = c\left(\frac{2}{15}\right) = 1$.

$c = \boxed{15/2}$.

(b) $Pr(A) = \iint_A f_{X,Y}(X,Y)\,dX\,dY = \int_{1/2}^{1} \int_{\frac{1}{4X}}^{X} \frac{15}{2} XY^2\,dY\,dX$

$= \int_{1/2}^{1} \frac{5}{2}\left(X^4 - \frac{1}{64X^2}\right) dX = \boxed{57/128}$ (0.445)



(c) $f_{X|A}(X|A) = \int f_{X,Y|A}(X,Y|A)\,dY$

$$= \begin{cases} \frac{15/2}{57/128} \int_{\frac{1}{4X}}^{X} XY^2\,dY, & 1/2 \le X \le 1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$= \begin{cases} \overset{(5.614)}{\frac{320}{57}}\left(X^4 - \frac{1}{64X^2}\right), & 1/2 \le X \le 1 \\ 0 & \text{OTHERWISE} \end{cases}$$ INTEGRATES TO 1. ✓

(d) CDF: $F_W(w) = Pr[w \le W] = Pr\left[\frac{\log|Y|}{\log X} \le W\right]$

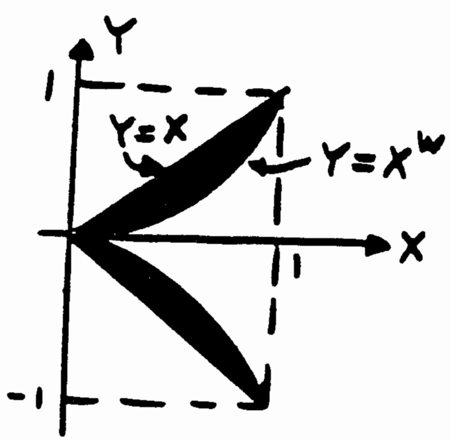
$= Pr[|Y| \ge X^W] = 2\int_0^1 \int_{X^W}^{X} \frac{15}{2} XY^2\,dY\,dX$

WATCH SIGNS! LOG X, LOG |Y| < 0!

$$= \begin{cases} 1 - \frac{5}{3W+2} & W \ge 1 \\ 0 & W \le 1 \end{cases}$$

$F_W(-\infty) = 0$
$F_W(\infty) = 1$
$F_W(W)$ NON-DECREASING.

PDF: $f_W(w) = \frac{d}{dW} F_W(W) = \begin{cases} \frac{15}{(3W+2)^2} & W \ge 1 \\ 0 & W \le 1 \end{cases}$ INTEGRATES TO 1. ✓



RECALL THAT IF $W > 1$, $X^W < X$ ON $[0,1]$.