1a. Note $E[(x(n)-y(n))^2]=E[x(n)^2]+E[y(n)^2]-2E[x(n)y(n)]=R_x(n,n)+R_y(n,n)-2R_{xy}(n,n)=0\rightarrow x(n)=y(n)$ **with probability one** from #6 of Problem Set #5.

1b. Let $y(n)=x(n+T)-x(n)$. Then $E[y(n)]=E[x(n+T)]-E[x(n)]=0-0=0$ and $\sigma^2_{y(n)}=E[(x(n+T)-x(n))^2]=R_x(0)+R_x(0)-2R_x(T)=0\rightarrow x(n+T)=x(n)$ $\rightarrow x(n)$ periodic with probability one, using the result of #1a above.
Also, $R_x(n+T)=E[x(i)x(i+n+T)]=E[x(i)x(i+n)]=R_x(n)\rightarrow R_x(n)$ periodic.

NOTE: Chebyschev$\not\rightarrow$ $Pr[y>\epsilon]\leq\frac{\sigma_y^2}{\epsilon^2}=0\rightarrow Pr[y=0]=1$ ($\Leftrightarrow$#6 of Problem Set #5).

2. $x(n)=\rho x(n-1)+w(n), x(0)=0\Leftrightarrow x(n)=\sum_{i=0}^{n-1}h(i)w(n-i)$ where $h(n)=\rho^n, n\geq 0$
Note that the upper limit is $(n-1)$ since $x(0)=0$ and $w(n)$ is only defined for $n\geq 1$.

2a. $E[x(n)]=\sum_{i=0}^{n-1}\rho^i E[w(n-i)]=0$. Plugging directly into lecture formulae:

2b. $K_x(m,n)=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\rho^{i+j}K_w(m-n-i+j)=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\rho^{i+j}\sigma_w^2\delta(m-n-i+j)$

$K_x(m,n)=\sigma_w^2\sum_{j=0}^{n-1}\rho^{m-n+2j}=\sigma_w^2\rho^{m-n}\frac{1-\rho^{2n}}{1-\rho^2}=\sigma_w^2\frac{\rho^{m-n}-\rho^{m+n}}{1-\rho^2}$ assuming $m\geq n$.

$K_x(m,n)=\sigma_w^2\frac{\rho^{|m-n|}-\rho^{m+n}}{1-\rho^2}$ since $K_x(m,n)$ symmetric (or just redo for $m\leq n$).

2c. For $|\rho|<1$, clearly $\lim_{m,n\to\infty}K_x(m,n)=\frac{\sigma_w^2\rho^{|m-n|}}{1-\rho^2}$. Hence $x(n)$ is *asymptotically* WSS.
This makes sense: the initial condition $x(0)=0$ is now in the infinite past.

2d. As $\rho\to 1\Leftrightarrow e=1-\rho\to 0$, then $K_x(m,n)=\sigma_w^2\frac{\rho^{|m-n|}-\rho^{m+n}}{1-\rho^2}\to\sigma_w^2\frac{(1-e|m-n|)-(1-e(m+n))}{1-(1-e)^2}$ $\to\sigma_w^2\frac{e((m+n)-|m-n|)}{2e}=\sigma_w^2\min[m,n]$, which is the II process result from lecture!

3. **N(n):** $N(n)$ is the sum of a 1-sided iid random process (Bernoulli), so have $N(n)=\sum_{k=1}^n x(k)\to E[N(n)]=nE[x]=np$ and $K_N(i,j)=\sigma_x^2\min[i,j]=p(1-p)\min[i,j]$.

Y(n): $E[Y(n)]=E[(-1)^{N(n)}]=E[\prod_{k=1}^n(-1)^{x(k)}]=\prod_{k=1}^n E[(-1)^{x(k)}]=(1-2p)^n$
since $x(n)$ independent$\to(-1)^{x(n)}$ uncorrelated and $(-1)^{x(n)}=\begin{cases}1 & \text{with prob. } 1-p\\ -1 & \text{with prob. } p\end{cases}$.
$E[Y(n)Y(n+k)]=E[\prod_{i=1}^n(-1)^{x(i)}\prod_{j=1}^{n+k}(-1)^{x(j)}]=E[\prod_{j=n+1}^{n+k}(-1)^{x(j)}]=(1-2p)^k$, for $k>0$, since $(-1)^{2x(j)}=1$. Hence $R_Y(k)=(1-2p)^{|k|}$ (use same idea for $k<0$).

But $K_Y(i,j)=R_Y(|i-j|)-E[Y(i)]E[Y(j)]=(1-2p)^{|i-j|}-(1-2p)^{(i+j)}$.
$Y(n)$ is *asymptotically* WSS since $\lim_{n\to\infty}(1-2p)^n=0$. Note: Given $p\neq\frac{1}{2}$.

4. **Pmf for x(n):** $\{p_{x(n)}(n)=\frac{1}{n};\quad p_{x(n)}(0)=1-\frac{1}{n}\}\to E[x(n)^2]=\frac{1}{n}n^2+(1-\frac{1}{n})0^2=n$.

4a. Convergence in prob.$\Leftrightarrow\lim_{n\to\infty}Pr[|x(n)-0|>\epsilon]=\lim_{n\to\infty}Pr[x(n)=1]=\lim_{n\to\infty}\frac{1}{n}=0$.

4b. Convergence with prob. 1$\Leftrightarrow Pr[\{s:\lim_{n\to\infty}x(n,s)=0\}]=1$. Fix $s\neq 0$ where $s\in\Omega$.
Then $x(n,s)=0$ for $n>\frac{1}{s}\to\lim_{n\to\infty}x(n,s)=0$. $Pr[\{s:s\neq 0\}]=1\to$converges a.s.

4c. Convergence in mean square$\Leftrightarrow\lim_{n\to\infty}E[(x(n)-0)^2]=\lim_{n\to\infty}n\neq 0\to$ doesn't converge.

5. Note: $E[M(n)x(n)] = \frac{1}{n}\sum_{i=1}^{n} E[x(i)x(n)] = \frac{1}{n}\sum_{i=1}^{n} R_x(i,n) = C(n)$ since 0-mean.

Only if: Schwarz≠: $C(n)^2 = E[M(n)x(n)]^2 \leq E[M(n)^2]E[x(n)^2] = E[M(n)^2]R_x(n,n)$.

Then $\underset{n\to\infty}{\text{L.I.M.}} M(n) = 0 \Leftrightarrow \underset{n\to\infty}{\text{LIM}} E[(M(n)-0)^2] = 0 \to \underset{n\to\infty}{\text{LIM}} C(n) \to 0$. QED.

If: $E[M(n)^2] = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E[x(i)x(j)] = \frac{1}{n^2}\sum_{i=1}^{j}\sum_{j=1}^{n} R_x(i,j)$
$= \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{i} R_x(j,i) - \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2_{x(i)} = \frac{2}{n^2}\sum_{i=1}^{n} iC(i) - \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2_{x(i)}$.

Then $\underset{n\to\infty}{\text{LIM}} C(n) = 0 \to \underset{n\to\infty}{\text{LIM}} E[M(n)^2] = 0 \to \underset{n\to\infty}{\text{L.I.M.}} M(n) = 0$ using the hint

and $\frac{1}{n^2}\sum_{i=1}^{n}\sigma^2_{x(n)} \leq \frac{1}{n}\underset{i}{\text{MAX}} R_x(i,i) \to 0$ since finite $\sigma^2_{x(i)}$. QED.