WAVELET SCALIING FUNCTION DESIGN BY ITERATION OF 2-SCALE EQUATION

Basic Definitions

2-scale eqn: $\phi(t) = \sum_{n=-\infty}^{\infty} g_n 2^{1/2} \phi(2t-n)$
Take Fourier transform: $\Phi(\omega) = \frac{1}{\sqrt{2}} G(e^{j\omega/2}) \Phi(\frac{\omega}{2})$

$M_0(e^{j\omega}) = \frac{1}{\sqrt{2}} G(e^{j\omega}); \quad M_0(e^{j0}) = \frac{1}{\sqrt{2}} G(e^{j0}) = \frac{\sqrt{2}}{\sqrt{2}} = 1$
if $G(e^{j\omega})$ has a zero at $\omega = \pi$. To see this, set $\omega = 0$ in $|G(e^{j\omega})|^2 + |G(e^{j(\omega+\pi)})|^2 = 2$.

Replacing ω with $\omega/2$, $\omega/4$, etc. and substituting:
$\Phi(\omega) = M_0(e^{j\omega/2}) M_0(e^{j\omega/4}) \ldots M_0(e^{j\omega/2^k}) \Phi(\frac{\omega}{2^k})$
Issue: Does $\lim_{N\to\infty} \prod_{k=1}^{N} M_0(e^{j\omega/2^k})$ exist?
Answer: Limit exists if g_n is a *regular* filter.

Some Results on Regularity

Necessary cond.: $G(e^{j\omega})$ has zero at $\omega = \pi$
PROOF: Recall from discrete-time wavelet series that
$g_0^{m+1}(n) = \sum_{i=-\infty}^{\infty} g_0(i) g_0^m(n - 2^m i)$ ((4.4.16),p.245)
z-xform: $G_0^m(z) = G_0(z) G_0(z^2) G_0(z^4) \ldots G_0(z^{2^{m-1}})$
Note $G_0^{m+1}(z) = G_0(z^{2^m}) G_0^m(z) = G_0(z) G_0^m(z^2)$.
Looking at even and odd powers of z separately
$g_0^{m+1}(2n) = \sum_{i=-\infty}^{\infty} g_0(2i) g_0^m(n-i)$
$g_0^{m+1}(2n+1) = \sum_{i=-\infty}^{\infty} g_0(2i+1) g_0^m(n-i)$
Let $g_0(n)$ be regular. Then $\lim_{m\to\infty} g_0^m(2n), g_0^m(2n+1) = \phi(2t)$
Take $\lim_{m\to\infty}$ of above 2 eqns:
$[\sum_{i=-\infty}^{\infty} g_0(2i)] \phi(2t) = [\sum_{i=-\infty}^{\infty} g_0(2i+1)] \phi(2t)$
$\rightarrow \sum_{i=-\infty}^{\infty} g_0(2i) = \sum_{i=-\infty}^{\infty} g_0(2i+1) \rightarrow G_0(e^{j\pi}) = 0.$
See p.251-252 and Problem 4.3 in V& K.

SUFFICIENT COND.: (Daubechies) $M_0(z) = (z+1)^N R(z)$

That is, let $M_0(e^{j\omega})$ have N zeros at $\omega = \pi$.

Then g_n is regular if $|R(e^{j\omega})| < \frac{1}{2}$.

PROOF: See p.253-4 and Problem 4.11 in V& K.

EXAMPLE: Haar: $G_0(e^{j\omega}) = \frac{1}{\sqrt{2}}(1 + e^{j\omega})$

$\rightarrow M_0(e^{j\omega}) = \frac{1}{2}(1 + e^{j\omega}) \rightarrow R(e^{j\omega}) = \frac{1}{2}$.

$|R(e^{j\omega})|$ attains the bound (Haar discontinuous).

Initialization of Recursion

After N recursions initialized with $\Phi_0(\omega)$,

$\Phi_0(\omega)$ is reduced to $\Phi_0(\frac{\omega}{2^k})$.

So only $\Phi_0(\omega), \omega \approx 0$ matters.

But for multiresolution analysis, need:

1. $\phi(t)$ satisfies 2-scale equation, AND
2. $\phi(t)$ orthogonal to integer translations

Use $\phi_0(t) = \begin{cases} 1, & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$ $\quad \Phi_0(\omega) = e^{-j\omega/2} \frac{\sin(\omega/2)}{\omega/2}$

At each iteration, convolution with $\phi_0(t)$ converts discrete function into piecewise constant function with pieces of length 2^{-k}.

$\prod_{k=1}^{N} M_0(e^{j\omega/2^k})$ periodic with period $2^k 2\pi$

$\rightarrow$sampled every 2^{-k} in time domain.

Example: Continuous-Time Daubechies Basis Function

Use g_n as derived on p.131 of V&K

and rederived (compare them!) on p.257-8 of V&K

First 3 iterations shown below (Fig.4.16, p.244).

Piecewise constant due to convolution with $\phi_0(t)$.