# DISCRETE WAVELETS AND FILTER BANKS AND MULTIRATE DIGITAL SIGNAL PROCESSING

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(Preliminary version)

## OUTLINE OF PRESENTATION

1. Review of Digital Signal Processing
   1.1 z-Transform
   1.2 Discrete-Time Fourier Transform (DTFT)
   1.3 Discrete-Time Frequency Response
2. Multirate Digital Signal Processing
   2.1 Upsampling
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   2.4 Applications of Multirate SP
   2.5 Subband Coding
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3. Perfect-Reconstruction Filter Banks
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   3.2 Orthogonal 2-Channel Filter Banks
   3.3 Filter Design
   3.4 Subband Coding=Discrete Wavelet Xform

# 1. REVIEW OF DIGITAL SIGNAL PROCESSING

## 1.1 z-Transform

Digital or discrete-time signal or sequence:
$x(n) = \{3, 1, 4, 1, 5, 9, 2, 6, 5 \ldots\} (x(0) = 3)$

$$z\text{−}xform\, of\, x(n) = \mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= 3 + 1z^{-1} + 4z^{-2} + 1z^{-3} + 5z^{-4} + 9z^{-5} + \ldots$$

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TO COMPUTE $\mathcal{Z}^{-1}$: 1. Use $DTFT^{-1}$ below.

2. Read off coeff. of power series of $X(z)$.
3. Partial fractions and $\mathcal{Z}\{a^n 1(n)\} = \frac{z}{z-a}$.

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### Properties of z-Transform:

1. Maps convolution to multiplication:
   $w(n) = x(n) * y(n) \rightarrow W(z) = X(z)Y(z)$.
2. Delay: $\mathcal{Z}\{x(n - D)\} = z^{-D} X(z)$.
3. Compare to Laplace transform: $e^s \rightarrow z$.

## 1.2 Discrete-Time Fourier Transform (DTFT)

Digital or discrete-time signal or sequence:
$x(n) = \{3, 1, 4, 1, 5, 9, 2, 6, 5 \ldots\}(x(0) = 3)$

$$DTFT\{x(n)\} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(z)|_{z=e^{j\omega}}$$

$$= 3 + 1e^{-j\omega} + 4e^{-2j\omega} + 1e^{-3j\omega} + 5e^{-4j\omega} + \ldots$$

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$$DTFT^{-1}\{X(e^{j\omega})\} = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

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**Properties of DTFT:**

1. Also maps convolution to multiplication:
   $w(n) = x(n) * y(n) \rightarrow W(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$
2. Delay: $DTFT\{x(n-D)\} = e^{-j\omega D}X(e^{j\omega})$.
3. Compare to discretized Fourier transform:
   $\mathcal{F}\{x(t)\sum_{n=-\infty}^{\infty}\delta(t-n)\} = DTFT\{x(n)\}$
   DTFT is $\mathcal{F}$ of sampled cont.-time signal.
4. PERIODIC in $\omega$ with period $2\pi$.
5. DUAL of Fourier SERIES: $x(n)$ are the Fourier series coeffs of the periodic function $X(e^{j\omega})$.
6. DTFT is $\mathcal{Z}$ evaluated on unit circle $|z| = 1$.

## 1.3 Discrete-Time Frequency Response

NOTE: *discrete* in one Fourier domain
$\leftrightarrow$*periodic* in other Fourier domain.
EXAMPLES: Fourier series, DTFT.

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SYSTEM: Linear (discrete)time-invariant
with discrete-time impulse response $h(n)$.
STEADY-STATE INPUT: $\cos(\omega_o n)$
Note this is a SAMPLED sinusoid.
STEADY-STATE OUTPUT: $M\cos(\omega_o n+\theta)$
for some amplitude $M$ and phase shift $\theta$.

$$\cos(\omega_o n) \rightarrow h(n) \rightarrow |H(e^{j\omega_o})|\cos(\omega_o n + ARG[H(e^{j\omega_o})])$$

where $H(e^{j\omega}) = DTFT\{h(n)\}$=freq. response.
Just like phasors and cont. freq. response,
except now everything PERIODIC in $\omega$.

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**Lowpass and Highpass Filters**

$$h(n) = \frac{\sin(an)}{\pi n} \leftrightarrow H(e^{j\omega}) = \begin{cases} 1, & \text{if } |\omega| < a; \\ 0, & \text{if } a < |\omega| < \pi \end{cases}$$

$$h(n)(-1)^n \leftrightarrow H(e^{j(\omega+\pi)})$$

Modulate by $(-1)^n$: lowpass$\rightarrow$highpass

## 1.3 Discrete-Time Frequency Response, continued

### EXAMPLE: Numerical Integration

GOAL: Compute $y(t) = \int_{-\infty}^{t} x(t')dt'$ numerically.
IDEAL INTEGRATOR: $H(\omega) = \frac{1}{j\omega}$.
RUNNING SUM: $y(n) = y(n-1) + x(n) \rightarrow$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-e^{-j\omega}}.$$

SIMPSON'S RULE:

$$y(n) = y(n-2)+[x(n)+4x(n-1)+x(n-2)]/3 \rightarrow$$

$$H(e^{j\omega}) = \frac{1+4e^{-j\omega}+e^{-2j\omega}}{3(1-e^{-2j\omega})}$$

$|H(e^{j\omega})|$ for
⇐ideal integrator (X)⇒
⇐running sum rule
Simpson's rule⇒

**COMMENTS:**

1. Both rules work well for small $\omega$
2. Simpson's rule better for midrange $\omega$
3. Simpson's rule BLOWS UP at $\omega = \frac{\pi}{2}$!

Say what? Try computing $\int_0^{10} \cos(\pi t)dt = 0$:
RUNNING SUM: $y(10) = \sum_1^{10}(-1)^n = 0$.
SIMPSON'S RULE: $y(10) =$
$\frac{1}{3}(1-4+2-4+2-4+2-4+2-4+1) = -3.33$
WARNING: Oversample if using Simpson's rule!

# 2. MULTIRATE DIGITAL SIGNAL PROCESSING

## 2.1 Upsampling

$x(n) = \{3, 1, 4, 1, 5, 9, 2, 6, 5 \ldots\} (x(0) = 3)$
The *upsampled* (by 2) signal $x_u(n)$ is
$x_u(n) = \{3, 0, 1, 0, 4, 0, 1, 0, 5, 0, 9 \ldots\}$

$$x(n) \rightarrow \uparrow 2 \rightarrow x_u(n); \quad x_u(n) = \begin{cases} x(n/2), & \text{if n is even} \\ 0, & \text{if n is odd} \end{cases}$$

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In the frequency domain:
$X_u(e^{j\omega}) = X(e^{2j\omega})$ now has period $\pi$:

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Can recover $x(n)$ from $x_u(n)$:

1. First lowpass filter $|\omega| < \frac{\pi}{2}$
2. Then downsample (see below).

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*Interpolate* $x(n)$: upsample+lowpass filter

Filtering replaces zeros in $x_u(n)$ with values obtained from sampling a bandlimited continuous-time signal.

## 2.2 Downsampling

$x(n) = \{3, 1, 4, 1, 5, 9, 2, 6, 5 \ldots\}(x(0) = 3)$
The *downsampled* (by 2) signal $x_d(n)$ is
$x_u(n) = \{3, 4, 5, 2, 5, 5 \ldots\}$
$x(n) \rightarrow \downarrow 2 \rightarrow x_d(n); \quad x_d(n) = x(2n)$

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In the frequency domain:
$X_d(e^{j\omega}) = [X(e^{j\omega/2}) + X(e^{j(\omega+2\pi)/2})]/2$ aliased

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Can recover $x(n)$ from $x_d(n)$
ONLY IF $X(e^{j\omega}) = 0$ for $|\omega| > \frac{\pi}{2}$
$\leftrightarrow X(e^{j\omega})$ *bandlimited* $\leftrightarrow x(t)$ *oversampled* $\times 2$

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Can lowpass $x(n)$, then downsample.
*Decimate* $x(n)$: lowpass filter+downsample

Since the lowpass-filtered signal is now oversampled, we can subsample without losing information.

## 2.3 Multirate Signal Processing

Can change sample rate by any *rational* factor $L/M$:

1. Upsample by $L(\uparrow L)$; lowpass filter(cutoff=$\frac{\pi}{L}$)
2. Lowpass filter(cutoff=$\frac{\pi}{M}$); Downsample by $M$.
3. Must upsample first to ensure no aliasing!

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**Comments:**

1. Combine (1b),(2a)→single filter
2. $X(e^{j\omega})$ bandlimited to $|\omega| < \frac{\pi L}{M}$ if $L < M$.
3. The following commute:

   a. $\uparrow L$ AND $\downarrow M$ IF $L, M$ relatively prime
   If $L = U$, require no aliasing occur

   b. Downsampling AND filtering:
   $U(z) \rightarrow \downarrow 2 \rightarrow H(z) \rightarrow Y(z)$ equivalent to
   $U(z) \rightarrow H(z^2) \rightarrow \downarrow 2 \rightarrow Y(z)$.

   c. Upsampling AND filtering:
   $U(z) \rightarrow H(z) \rightarrow \uparrow 2 \rightarrow Y(z)$ equivalent to
   $U(z) \rightarrow \uparrow 2 \rightarrow H(z^2) \rightarrow Y(z)$.

4. Multirate systems: linear, time-varying periodic.
5. Multirate SP equivalent to reconstructing continuous-time signal and then resampling, but faster.

## 2.4 Applications of Multirate SP

1. Interfacing systems with different clock rates
2. Implement fractional time delays
   $H(e^{j\omega}) = e^{j\omega D}$, $D$ rational
   All-pass, linear phase filter
3. Narrowband lowpass filters:
   a. Have signal sampled at 8 kHz
      Want lowpass filter with cutoff=80 Hz
   b. Decimate signal by $\frac{8kHz}{2\times 80Hz} = 50$
      All but lowest 80 Hz of the signal aliased, but DON'T CARE!
   c. Implement lowpass filter at decimated rate
      $\rightarrow$FIR filter much shorter
4. Can implement multirate filtering in several smaller stages of decimation/interpolation, instead of doing it all in one big stage.

## 2.5 Subband Coding

IDEA: Divide up speech signal into frequency bands, each band an *octave* (factor of 2) wide.

1. Code each band separately.
2. Each stage splits lowpass signal from previous band into low(er)pass and highpass signals.
3. Each stage is the first half (analysis part) of a quadrature mirror filter (QMF) (see below)

## 2.6 Polyphase Transforms

*Periodically time-varying system:*
Input $u(n)$ →output $y(n)$ implies
Input $u(n + Nm)$ →output $y(n + Nm)$.

EXAMPLE: Downsample, then upsample.

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**Polyphase Transform:**

Map signal $x(n)$ into *collection* of signals $\{x_i(n), i = 0, 1 \ldots N - 1\}$ where $x_i(n) = x(nN + i)$

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1. $x_i(n)$ shifted and downsampled $x(n)$
   Each has different phase→"polyphase"
2. Recover $x(n)$ by interleaving $x_i(n)$
3. Write using z-transform as
   $X(z) = \sum_{i=0}^{N-1} z^i X_i(z^N)$
   where $X_i(z) = \sum_n x(nN - i)z^{-n}$.

   For example, $X(z) = X_0(z^2) + zX_1(z^2)$.

# 3. PERFECT-RECONSTRUCTION FILTER BANKS

Split up signal into different frequency bands.
Then reassemble bands into original signal.

## 3.1 Quadrature-Mirror Filters

1. $H_0(e^{j\omega})$=lowpass filter
2. $H_1(e^{j\omega})$=*mirror*-image highpass filter:
   $H_1(e^{j\omega}) = H_0(e^{j(\omega+\pi)}); h_1(n) = (-1)^n h_0(n)$
   $H_1(e^{j\omega})$ is reflection of $H_0(e^{j\omega})$ about $\omega = \frac{\pi}{2}$.
3. Why do we need these mirror conditions?
4. In sequel use $H(\omega)$ instead of $H(e^{j\omega})$.

## 3.1 Quadrature-Mirror Filters, continued

3. Condition for perfect reconstruction ($\hat{x}(n) = x(n)$):

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a. Signal at midpoint of top rail is

$X_{uppermid}(\omega) = [H_0(\frac{\omega}{2})X(\frac{\omega}{2}) + H_0(\frac{\omega+2\pi}{2})X(\frac{\omega+2\pi}{2})]/2$

using decimation formula above.

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b. Signal at midpoint of lower rail is

$X_{lowermid}(\omega) = [H_1(\frac{\omega}{2})X(\frac{\omega}{2}) + H_1(\frac{\omega+2\pi}{2})X(\frac{\omega+2\pi}{2})]/2$

using decimation formula above.

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c. Output signal is

$\hat{X}(\omega) = G_0(\omega)X_{uppermid}(2\omega) + G_1(\omega)X_{lowermid}(2\omega)$

using interpolation formula above.

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d. Condition for perfect reconstruction is

$X(\omega) = \hat{X}(\omega) = [G_0(\omega)H_0(\omega) + G_1(\omega)H_1(\omega)]X(\omega)/2$

$+[G_0(\omega)H_0(\omega+\pi) + G_1(\omega)H_1(\omega+\pi)]X(\omega+\pi)/2$

by substituting (b) in (c).

## 3.1 BIorthogonal QMFs, continued

4. Condition for perfect reconstruction is
$X(\omega) = \hat{X}(\omega) = [G_0(\omega)H_0(\omega){+}G_1(\omega)H_1(\omega)]X(\omega)/2$
$+[G_0(\omega)H_0(\omega{+}\pi){+}G_1(\omega)H_1(\omega{+}\pi)]X(\omega{+}\pi)/2$

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a. Want second term(aliasing)=0
for perfect reconstruction:

$G_0(\omega)H_0(\omega+\pi) + G_1(\omega)H_1(\omega+\pi) = 0$
$\rightarrow G_0(\omega) = H_1(\omega+\pi), G_1(\omega) = -H_0(\omega+\pi).$

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Now impose BIorthogonality:
synthesis filters $g(n) = \pm$analysis filters $h(n)$:
$G_0(\omega) = H_0(\omega); G_1(\omega) = -H_1(\omega) = -H_0(\omega+\pi)$
$g_0(n) = h_0(n); \quad g_1(n) = -(-1)^n h_0(n).$

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b. Want first term=1 after cancel $X(\omega)$:
$G_0(\omega)H_0(\omega) + G_1(\omega)H_1(\omega) = 2.$

Substituting expressions from (a):
$H_0^2(\omega) - H_1^2(\omega) = H_0^2(\omega) - H_0^2(\omega+\pi) = 2.$

**Problem:** Impossible! (try $\omega \rightarrow \omega + \pi$). Instead,

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$H_0^2(\omega){-}H_1^2(\omega) = H_0^2(\omega){-}H_0^2(\omega{+}\pi) = 2e^{j\omega(2N-1)}$
$\hat{X}(\omega) = X(\omega)e^{j\omega(2N-1)} \rightarrow \hat{x}(n) = x(n-2N+1)$
(perfect reconstruction except for a delay).
Why $2N-1$? Need consistency for $\omega \rightarrow \omega+\pi$.

**Only FIR solution:** Haar $H(\omega) = \frac{1}{\sqrt{2}}(1+e^{-j\omega})$
(see V&K, bottom of p.121)

## 3.2 Orthogonal 2-Channel Filter Banks

Now make a slight but significant change:
$g(n) = \pm time - reversed\, filters\, h(-n)$

$G_0(\omega) = H_0(-\omega); G_1(\omega) = H_1(-\omega)$

$g_0(n) = h_0(2N-1-n); g_1(n) = h_1(2N-1-n)$

Also $g_1(n) = (-1)^n g_0(2N-1-n)$ (V&K, p.125)

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Repeating above→Smith-Barnwell condition

$|H_0(\omega)|^2 + |H_0(\omega+\pi)|^2 = 2$
$|G_0(\omega)|^2 + |G_0(\omega+\pi)|^2 = 2$ instead of
$H_0^2(\omega) - H_1^2(\omega) = H_0^2(\omega) - H_0^2(\omega+\pi) = 2e^{j\omega(2N-1)}$
**Interpretation:** Power complementary.

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Also get
$G_0(\omega)G_1(-\omega) + G_0(\omega+\pi)G_1(\pi-\omega) = 0$

instead of the aliasing=0 condition
$G_0(\omega)H_0(\omega+\pi) + G_1(\omega)H_1(\omega+\pi) = 0.$

## 3.2 Orthogonal 2-Channel Filter Banks, cont.

Using V&K *modulation* matrix notation:

### BIorthogonal QMFs

$$G_0(\omega)H_0(\omega) + G_1(\omega)H_1(\omega) = 2$$
$$G_0(\omega)H_0(\omega+\pi) + G_1(\omega)H_1(\omega+\pi) = 0$$

can be written as

$$\begin{bmatrix} G_0(\omega) & G_1(\omega) \\ G_0(\omega+\pi) & G_1(\omega+\pi) \end{bmatrix} \begin{bmatrix} H_0(\omega) & H_0(\omega+\pi) \\ H_1(\omega) & H_1(\omega+\pi) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

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### Orthogonal 2-Channel Filter Banks

$$|G_0(\omega)|^2 + |G_0(\omega+\pi)|^2 = 2$$
$$G_0(\omega)G_1(-\omega) + G_0(\omega+\pi)G_1(\pi-\omega) = 0$$

can be written as

$$\begin{bmatrix} G_0(\omega) & G_1(\omega) \\ G_0(\omega+\pi) & G_1(\omega+\pi) \end{bmatrix} \begin{bmatrix} G_0(-\omega) & G_0(\pi-\omega) \\ G_1(-\omega) & G_1(\pi-\omega) \end{bmatrix}$$
$$= \begin{bmatrix} G_{0,0} & G_{0,1} \\ G_{1,0} & G_{1,1} \end{bmatrix} \begin{bmatrix} G_{0,0} & G_{0,1} \\ G_{1,0} & G_{1,1} \end{bmatrix}^H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

consistent with $g(n) = \pm h(-n)$.

## 3.3 Filter Design

## Using Spectral Factorization

Define $P(\omega) = |G_0(\omega)|^2$. Then
$|G_0(\omega)|^2 + |G_0(\omega+\pi)|^2 = 2 \rightarrow P(\omega) + P(\omega+\pi) = 2$.

1. Find half-band lowpass filter $P(\omega)$ where
   $P(\omega) + P(\omega + \pi) = 2$
   Symmetry about $\omega = \pi/2$.
2. Then spectral factorization of
   $P(z) = G_0(z)G_0(1/z)$:
   $G_0(z)$ has all zeros inside $|z| = 1$.
   $P(\omega)$ lowpass $\rightarrow G_0(\omega)$ lowpass.

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## EXAMPLE: Daubechies $Dn$ Basis

Choose $P(z) = (1 + z)^n(1 + z^{-1})^n R(z)$

1. $2n$ zeros at $\omega = \pi$
2. Choose $R(z)$ so $P(z) + P(-z) = 2$
   a. $R(z)$ is an autocorrelation:
      $R(z) = R(1/z)$ and $R(e^{j\omega}) > 0$
      since will spectrally factor.
3. Equate coefficients→matrix eqn.
4. For details see V&K p.131.

## 3.4 Subband Coding=Discrete Wavelet Xform

At each $m^{th}$ stage, signal is split into:

1. A *detail* (highpass) signal $\mathcal{W}_{2^m} x(n)$
2. An *average* (lowpass) signal $x_m(n)$
3. Each stage is the first half (analysis part) of a quadrature mirror filter (QMF)!
4. Can reassemble signal from
   a. its detail signals and
   b. the final average signal

   using second half (synthesis part) of QMF

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**Analysis (Compute Wavelet Xform)**

**average:** $x_m(n) = \sum_i x_{m-1}(i) h_0(2n - i)$
**detail:** $\mathcal{W}_{2^m} x(n) = \sum_i x_{m-1}(i) h_1(2n - i)$

These formulae combine filtering and subsampling.

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**Synthesis (Reconstruct Signal)**

$$x_{m-1}(n) = \sum_i h_0(2i-n) x_m(i) + h_1(2i-n) \mathcal{W}_{2^m} x(i)$$

Note time reversal between analysis and synthesis.

## 3.4 Subband Coding=Discrete Wavelet Xform, cont.

Can also use direct definitions:

**Analysis (Compute Wavelet Xform)**

**average:** $x_m(n) = \sum_i x(i)h_0^m(2^m n - i)$

**detail:** $\mathcal{W}_{2^m}x(i) = \sum_i x(i)h_1^m(2^m n - i)$

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**Synthesis (Reconstruct Signal)**

$x(n) = \sum_{m=1}^{L} \sum_i \mathcal{W}_{2^m}x(i)h_1^m(2^m i - n)$
$+ \sum_i x_L(i)h_0^L(2^L i - n)$

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**Filters at Each Resolution**

$h_0^{m+1}(n) = \sum_i h_0(i)h_0^m(n - 2^m i)$
$h_1^{m+1}(n) = \sum_i h_1(i)h_0^m(n - 2^m i)$

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**Orthogonality of Basis Functions**

$< h_1^m(2^m n - i), h_1^{m'}(2^{m'} n' - i) >= \delta(m - m')\delta(n - n')$
$< h_1^m(2^m n - i), h_0^{m'}(2^{m'} n' - i) >= 0, \; for\, m' > m$
$< h_0^m(2^m n - i), h_0^m(2^m n' - i) >= \delta(n - n')$
where $< f(n), g(n) >= \sum_n f(n)g^*(n)$ if $< \infty$.