# SPLINES AND BATTLE-LEMARIE WAVELET

**Splines:** Piecewise polynomial approximation to functions:

1. Divide up real line into intervals;
2. Within interval, use polynomial of degree $k$;
3. At boundaries (called *knots*) have continuous $\frac{d^{k-1}}{dt^{k-1}}$.

**Basis for spline spaces:** B-splines (B-s)

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V&K notation is awful. Summary of notation:

| *source* | $B-s$ | $\mathcal{F}\{B-s\}$ | $\mathcal{F}\{sampled\,B-s\}$ |
|---|---|---|---|
| $V\&K$ | $\beta^{(N)}(t)$ | $\mathcal{B}^{(N)}(\omega)$ | $B^{(N)}(\omega)$ |
| *aey* | $s_N(t)$ | $S_N(\omega)$ | $\tilde{S}_N(\omega)$ |

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## Definitions

**B-spline order** $2N-1$: $S_{2N-1}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{2N}$

**B-spline order** $2N$: $S_{2N}(\omega) = e^{-j\omega/2}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{2N+1}$

Note even-order splines shifted by 1/2 in time so that $s_n(t) \neq 0$ for $0 < t < N+1$.

$s_0(t)$=pulse, $s_1(t)$=triangle, $s_2(t)$=quadratic, etc.

$V_k^N = \{$piecewise polynomials of degree $N$ over intervals $[2^k j, 2^k(j+1)), j \in integers$ having continuous $\frac{d^{k-1}}{dt^{k-1}}$ at $t = 2^k j\}$.

Have chain of subspaces $\ldots V_1^N \subset V_0^N \subset V_{-1}^N \subset \ldots$

$\rightarrow s_N(t)$ satisfies 2-scale equation.

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$x(t)$ orthogonal to its integer translations

$\leftrightarrow \int x(t)x(t-j)dt$=sampled autocorrelation=$\begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } j \neq 0 \end{cases}$

$\leftrightarrow \sum_{k=-\infty}^{\infty} |X(\omega + 2k\pi)|^2 = 1$

(sampling induces periodicity in $\omega$)

$\{s_N(t-j), j \in \mathit{integers}\}$ spans spline space $V_0^N$.
But $\{s_N(t-j)\}$ not *orthonormal* basis for $V_0^N$.

Let $\tilde{S}_{2N+1}(\omega) = \sum_{k=-\infty}^{\infty} |S_N(\omega + 2k\pi)|^2$. Then:

1. $\tilde{S}_{2N+1}(\omega)$ periodic→discrete in time
2. $\tilde{S}_{2N+1}(\omega) = DTFT[\mathit{sampled}\, s_{2N+1}(t)]$
   since $s_N(t) * s_N(t) = s_{2N+1}(t)$ and $s_N(t)$ time-symmetric.

Let $\Phi(\omega) = S_N(\omega)/\sqrt{\tilde{S}_{2N+1}(\omega)}$=scaling function.
Then $\sum_{k=-\infty}^{\infty} |\Phi(\omega + 2k\pi)|^2 = 1$
$\rightarrow \phi(t)$ orthogonal to its translations
$\rightarrow \{\phi(t-j)\}$ *orthonormal* basis for $V_0^N$.

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$s_N(t)$ satisfies 2-scale equation→ $\phi(t)$ does also.
Above→ $\phi(t)$ is a wavelet scaling function.
$\phi(t)$ longer has compact support; decays exponentially.

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**Battle-Lemarie Wavelets:**

1. Algebraic details for $s_1(t)$ on p.233-234 of V&K.
2. $\phi(t)$ longer compact support; decays exponentially.
3. Can also use $\phi(t) = s_N(t)$: lose intrascale orthogonality while keeping interscale orthogonality.
   Advantage: $s_N(t)$ localized in time and frequency.