PRINT YOUR NAME HERE:

HONOR CODE PLEDGE: "I have neither given nor received aid on this exam, nor have I concealed any violations of the honor code." Open book; SHOW ALL OF YOUR WORK!

SIGN YOUR NAME HERE:

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(40) 1. We observe $y(n) = c(-1)^n + v(n)$ where $c \sim N(0,1)$ and $E[c \cdot v(n)] = 0.E[w^2(n)] = 1$. $v(n)$ is a 1^{st}-order AR process: $v(n) = \frac{1}{2}v(n-1) + w(n)$ where $w(n)$ is 0-mean WGN. The problem is to recursively estimate c from observations $\{y(0), y(1)\ldots\}$.

(10) a. *Formulate* this as a Kalman filtering problem (specify matrices A, B, H, Q, R).
(10) b. Let $P(n)$ be the error covariance matrix. EXPLAIN the following:
(i) Why $P(n)$ converges to some matrix P_{ss}; (ii) Why P_{ss} is *not* positive definite;
(iii) Why (ii) *seems* to be a good thing, but is actually a bad thing.
(10) c. Write out the matrix equation for P_{ss} and *solve* it.
HINT: From (b) you know $P_{ss} = \begin{bmatrix} 0 & 0 \\ 0 & p \end{bmatrix}$ for some p.

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(10) d. Now suppose we observe $y(n) = c(-1)^n + w(n)$; recall $w(n)$ is WGN.
Write out *and solve* the Kalman filtering equations. HINT: lecture notes.

WRITE ANSWERS HERE:

(a): A= **B=** **H=** **Q=** **R=**

(b): (i)

(b): (ii)

(b): (iii)

(c): $P_{ss} =$ P_{ss} equation:

(d): A= **B=** **H=** **Q=** **R=**

(d): $\hat{x}(n|n-1) =$ $P(n|n-1) =$

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(20) 2. Continuous-time information Kalman filter:

(10) a. *Derive* the information form of the continuous-time Kalman-Bucy filter. Do *not* use a limiting ($\Delta \to 0$) argument; derive directly from Kalman-Bucy equations. It should propagate $S(t) = P(t)^{-1}$ and $\hat{n}(t) = S(t)\hat{x}(t)$. HINTS: (1) $\frac{d}{dt}P^{-1}(t) = -P^{-1}\frac{dP}{dt}P^{-1}$; (2) Derive Riccati equation first.

We observe $y(t) = x + v(t)$ where $v(t)$ is 0-mean WGN with $S_v(\omega) = 1$ and $E[xv(t)] = 0$. x is an unknown constant. We want to estimate x from $\{y(s), 0 < s < t\}$.

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(5) b. *Formulate* this as a Kalman filtering problem (specify matrices A, B, H, Q, R).

(5) c. Write out and *solve* the *information* Kalman-Bucy filter equations. HINT: This is *much* easier than solving the regular Kalman filter equations.

WRITE ANSWERS HERE:

(a): Information Equations:

(b): A= **B=** **H=** **Q=** **R=**

(c): $s(t) =$ $\hat{n}(t) =$ $\hat{x}(t) =$

#1:

#2:

#3:

$\sum$:

(40) 3. We observe $y(t) = x(t) + v(t)$ where $K_x(t) = e^{-|t|}, K_v(t) = \delta(t) + 3e^{-|t|}, K_{xv}(t) = 0$.

(5) a. Compute the infinite smoothing filter for $\hat{x}(t|\{y(s), -\infty < s < \infty\})$.

(5) b. Compute the minimum-phase spectral factor $S_y^+(s)$ of $S_y(s)$.

(10) c. Compute the causal Wiener filter for $\hat{x}(t|\{y(s), -\infty < s < t\})$.

(5) d. Let $\hat{x}$ be the LLSE of x. Prove $E[\hat{x}^2] = E[x\hat{x}]$.

(5) e. Let $e = x - \hat{x}$. Prove $E[e^2] = E[x^2] - E[\hat{x}^2]$.
Interpret this using the Pythagoras theorem. Be specific.

(5) f. Use these to prove $E[e^2(t)] = E[x^2(t)] - \int h(t)K_{xy}(t)dt$,
where $h(t)$ is the impulse response of the filter in (a) or (c).

(5) g. Use (f) to compute $E[e^2(t)]$ for your answer to (a).

(5) h. Use (f) to compute $E[e^2(t)]$ for your answer to (c).

WRITE ANSWERS HERE:

(a): $H(\omega) =$ **(b):** $S_y^+(s) =$ **(c):** $H(s) =$

(g): $E[e^2(t)] =$ **(h):** $E[e^2(t)] =$
