EECS 564 DETECTION IN WGN: $s_1(t) \perp s_2(t)$ F FEB. 12, 1999

PS #5 3: $E_{DB} = \|x_{DB}(t)\|^2 = \int x_{DB}^2(t)dt = \int \left(\int_0^T x(u)\,\mathrm{sinc}(t-u)du\right)^2 dt$ NOW WHAT?

$E_{DB} = \int\left(\int_0^T x(u)\,\mathrm{sinc}(t-u)du \int_0^T x(v)\,\mathrm{sinc}(t-v)dv\right)dt = \int_0^T\int_0^T x(u)x(v)\left(\int_{-\infty}^{\infty} \mathrm{sinc}(t-u)\,\mathrm{sinc}(t-v)dt\right)du\,dv$ SINC * SINC = SINC

$x(t)$ = EIGENFUNCTION OF $K(t,s) = \mathrm{sinc}(t-s)$ OVER INTERVAL $[0,T]$ ASSOCIATED WITH LARGEST λ

OBSERVE $r(t) = \begin{cases} s_1(t)+n(t) & H_1 \\ s_2(t)+n(t) & H_2 \end{cases}$ OVER $0 \le t \le T$ $\left[\begin{matrix} n(t) \text{ WGN} \\ S_n(\omega)=\sigma^2 \end{matrix}\right]$ $\left[\begin{matrix} s_1(t) \perp s_2(t) \\ \int_0^T s_1(t)s_2(t)dt = 0 \end{matrix}\right]$ $\left[\begin{matrix} \int_0^T s_1(t)^2 dt = E_1 \\ \int_0^T s_2(t)^2 dt = E_2 \end{matrix}\right]$

K-L: $n(t)$ WHITE $\to$ USE ANY C.O.S. $\left[\begin{matrix} \phi_1(t) = s_1(t)/\sqrt{E_1} & \phi_3(t) \perp \{\phi_1(t), \phi_2(t)\} \\ \phi_2(t) = s_2(t)/\sqrt{E_2} & \phi_4(t) \perp \{\phi_1, \phi_2, \phi_3\} \end{matrix}\right]$ [GRAM-SCHMIDT] $E(n_i^2) = \lambda_i = \sigma^2$, $E(n_i n_j) = \sigma^2 \delta_{ij}$

PROJECT: MULT. BY $\phi_i(t)$ THEN $\int_0^T dt$: $\left[r_1 = \begin{cases} \sqrt{E_1}+n_1 & H_1 \\ n_1 & H_2 \end{cases} \quad r_2 = \begin{cases} n_2 & H_1 \\ \sqrt{E_2}+n_2 & H_2 \end{cases} \quad r_3 = \begin{cases} n_3 & H_1 \\ n_3 & H_2 \end{cases} \dots \right]$

$n(t)$ GAUSS $\to$ n_i: JGRV $\to$ n_i: INDPT

MULTICHANNEL GAUSSIAN: $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \sim \begin{cases} N\left(\begin{bmatrix} \sqrt{E_1} \\ 0 \end{bmatrix}, \sigma^2 I\right) & H_1 \\ N\left(\begin{bmatrix} 0 \\ \sqrt{E_2} \end{bmatrix}, \sigma^2 I\right) & H_2 \end{cases}$ SEE HANDOUT ON "VECTOR GAUSSIAN DETECTION" NO HELP EVEN IN ESTIMATING $n(t)$

SUFFICIENT STATISTIC: $\ell = \Delta \underline{m}^T Q \underline{R} = \left(\begin{bmatrix} \sqrt{E_1} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \sqrt{E_2} \end{bmatrix}\right)^T \frac{1}{\sigma^2} I \begin{bmatrix} \int_0^T R(t)s_1(t)/\sqrt{E_1}\,dt \\ \int_0^T R(t)s_2(t)/\sqrt{E_2}\,dt \end{bmatrix} = \frac{\int_0^T R(t)(s_1(t)-s_2(t))dt}{\sigma^2}$ MATCHED FILTER

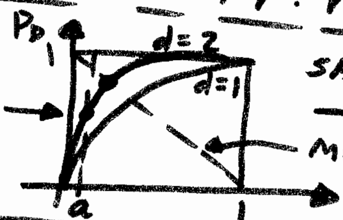
FISHER DISCRIMINANT: $d^2 = \Delta \underline{m}^T Q \Delta \underline{m} = \left(\begin{bmatrix} \sqrt{E_1} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \sqrt{E_2} \end{bmatrix}\right)^T \frac{1}{\sigma^2} I \left(\begin{bmatrix} \sqrt{E_1} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \sqrt{E_2} \end{bmatrix}\right) = \frac{E_1+E_2}{\sigma^2}$. DIMENSIONLESS SIGNAL ENERGIES ADD SINCE ORTHOGONAL

NEYMAN-PEARSON TEST: (H_2 = NULL) $f_{\ell|H_2}(L|H_2) \sim N(\Delta \underline{m}^T Q \underline{m}_2, \Delta \underline{m}^T Q \Delta \underline{m}) \sim N\left(-\frac{E_2}{\sigma^2}, \frac{E_1+E_2}{\sigma^2}\right)$ NOTE $\sigma_\ell^2 = d^2$; SEE HANDOUT

$\alpha = P_F = \Pr[\text{CHOOSE } H_1 | H_2 \text{ TRUE}] = \Pr[\ell > \gamma | H_2] = \mathrm{erfc}\left[\frac{\gamma + E_2/\sigma^2}{\sqrt{(E_1+E_2)/\sigma^2}}\right] = \alpha.$

SOLVE FOR γ: $\gamma = -\frac{E_2}{\sigma^2} + \sqrt{\frac{E_1+E_2}{\sigma^2}}\,\mathrm{erfc}^{-1}[\alpha]$.

ROC:



NEYMAN-PEARSON TEST

SAY $d^2 = \frac{E_1+E_2}{\sigma^2} = 4$ $\to d = 2$

MINIMAX TEST WITH MEP CRITERION

$\int_0^T R(t)(s_1(t)-s_2(t))dt \underset{H_2}{\overset{H_1}{\gtrless}} -E_2 + \sigma\sqrt{E_1+E_2}\,\mathrm{erfc}$

BAYESIAN: $\int_0^T R(t)(s_1(t)-s_2(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{E_1-E_2}{2} + \sigma^2 \log \eta$.

CAN ALSO DERIVE THIS WITHOUT USING RESULTS FROM "VECTOR GAUSSIAN DETECTION":

UNDER H_1: $\ell = \int_0^T R(t)(s_1(t)-s_2(t))dt = \int_0^T (s_1(t)+n(t))(s_1(t)-s_2(t))dt = \underbrace{\int_0^T s_1^2(t)dt}_{E_1} - \underbrace{\int_0^T s_1(t)s_2(t)dt}_{0} + \int_0^T (s_1(t)-s_2(t))n(t)dt$

$E[\ell] = E\left[E_1 - 0 + \int_0^T (s_1-s_2)n(t)dt\right] = E_1 + \int_0^T (s_1(t)-s_2(t))E[n(t)]dt = E_1.$

$\sigma_\ell^2 = \sigma^2\left[E_1 + \int_0^T (s_1-s_2)n(t)dt\right] = E\left[\left(\int_0^T (s_1(t)-s_2(t))n(t)dt\right)^2\right]$ NOW WHAT?

$= E\left[\int_0^T\int_0^T (s_1(u)-s_2(u))n(u)(s_1(v)-s_2(v))n(v)\,du\,dv\right]$

$= \int_0^T\int_0^T (s_1(u)-s_2(u))(s_1(v)-s_2(v))\underbrace{E[n(u)n(v)]}_{\sigma^2\delta(u-v)}du\,dv = \sigma^2\int_0^T (s_1(u)-s_2(u))^2 du = \sigma^2(E_1+E_2)$ SINCE $\perp$

$\therefore f_{\ell|H_1}(L|H_1) \sim N(E_1, \sigma^2(E_1+E_2))$.

$f_{\ell|H_2}(L|H_2) \sim N(-E_2, \sigma^2(E_1+E_2))$.

SIMILARLY, UNDER H_2: $\ell = \int_0^T (s_2(t)+n(t))(s_1(t)-s_2(t))dt = \underbrace{\int_0^T s_2 s_1}_{0} - \underbrace{\int_0^T s_2^2}_{-E_2} + \underbrace{\int_0^T (s_1-s_2)n(t)dt}_{\text{AS BEFORE}}$

NOTE: ℓ DEFINED DIFFERENTLY: NO σ^2.

EECS 564 WINTER 1999

GENERAL BINARY DETECTION IN WHITE NOISE

PROBLEM: OBSERVE $r(t)=\begin{cases} s_1(t)+w(t) & \text{UNDER } H_1\\ s_2(t)+w(t) & \text{UNDER } H_2\end{cases}$ $\int_0^T s_1^2(t)\,dt=E_1$, $\int_0^T s_2^2(t)\,dt=E_2$ FOR $0\le t\le T$. $w(t)$ WGN, STRENGTH σ^2.

CAUCHY-SCHWARZ $\neq$: $\left(\int_0^T s_1(t)s_2(t)\,dt\right)^2 \le \left(\int_0^T s_1^2(t)\,dt\right)\left(\int_0^T s_2^2(t)\,dt\right)=E_1E_2$. DEFINE $\boxed{\cos\theta = \frac{\int_0^T s_1(t)s_2(t)\,dt}{\sqrt{E_1}\sqrt{E_2}}}$ (KNOWN) WHERE $0\le\theta\le 180^\circ$

BASIS FUNCTIONS: $\phi_1(t)=\frac{s_1(t)}{\sqrt{E_1}}$. $\phi_2(t)=\dfrac{s_2(t)-\phi_1(t)\left(\int_0^T s_2(t)\phi_1(t)\,dt\right)}{\sqrt{\int_0^T\left[s_2(t)-\phi_1(t)\left(\int_0^T s_2(t)\phi_1(t)\,dt\right)\right]^2dt}}$ (GRAM-SCHMIDT).

NOW, $\int_0^T s_2(t)\phi_1(t) = \int_0^T \frac{s_1(t)s_2(t)}{\sqrt{E_1}}\,dt=\sqrt{E_2}\cos\theta$ (*).

DENOMINATOR OF $\phi_2(t)$ $= \sqrt{\underbrace{\int_0^T s_2^2(t)}_{E_2}+\underbrace{\int_0^T\phi_1^2(t)}_{1}(E_2\cos^2\theta)-2\underbrace{\int_0^T s_2(t)\phi_1(t)}_{\sqrt{E_2}\cos\theta}\sqrt{E_2}\cos\theta} = \sqrt{E_2(1-\cos^2\theta)}$
$=\sqrt{E_2}\sin\theta$. SINCE $0\le\theta\le180^\circ$

$\therefore\ \phi_2(t)=\dfrac{s_2(t)-\phi_1(t)\sqrt{E_2}\cos\theta}{\sqrt{E_2}\sin\theta}=\dfrac{s_2(t)-\sqrt{\frac{E_2}{E_1}}\cos\theta\, s_1(t)}{\sqrt{E_2}\sin\theta}$. IF $\theta=90^\circ$, $\phi_2(t)=\frac{s_2(t)}{\sqrt{E_2}}$ ✓

NOW LET $r(t)=\sum_{i=1}^{\infty} r_i\phi_i(t)$, $w(t)=\sum_{i=1}^{\infty} w_i\phi_i(t)$. $\phi_1(t),\phi_2(t)$ AS ABOVE. $\{\phi_3(t)\ldots\}\perp\phi_1(t),\phi_2(t)$.
NOTE THAT SINCE $r(t)=w(t)+$ DETERMINISTIC SIGNAL, $r(t)$ AND $w(t)$ HAVE SAME BASIS FUNCTIONS.
NOW TAKE PROJECTIONS ON $\phi_i(t)$ BY MULT. $r(t)=\begin{cases}s_1(t)+w(t) & H_1\\ s_2(t)+w(t) & H_2\end{cases}$ BY $\phi_i(t)$ AND $\int_0^T dt$.
$\Leftrightarrow$ LOOK AT i^{TH} COORDINATES

$i=1$: $r_1=\begin{cases}\int_0^T s_1(t)\phi_1(t)\,dt+w_1\\ \int_0^T s_2(t)\phi_1(t)\,dt+w_1\end{cases}=\begin{cases}\sqrt{E_1}+w_1 & H_1\\ \sqrt{E_2}\cos\theta+w_1 & H_2\end{cases}$ $w(t)=\sum_{i=1}^{\infty} w_i\phi_i(t)$, $E(w_i^2)=\sigma^2$.

$i=2$: $r_2=\begin{cases}\int_0^T s_1(t)\phi_2(t)\,dt+w_2\\ \int_0^T s_2(t)\phi_2(t)\,dt+w_2\end{cases}=\begin{cases}0\ \ (\phi_2(t)\perp s_1(t)=\sqrt{E_1}\phi_1(t))+w_2\\ \dfrac{\underbrace{\int_0^T s_2^2(t)}_{E_2}-\underbrace{\int_0^T s_2(t)\phi_1(t)}_{\sqrt{E_2}\cos\theta}\sqrt{E_2}\cos\theta}{\sqrt{E_2}\sin\theta}+w_2\end{cases}=\begin{cases}w_2 & H_1\\ \sqrt{E_2}\sin\theta+w_2 & H_2\end{cases}$

$i>2$: $r_i=\begin{cases}w_i & H_1\\ w_i & H_2\end{cases}$ **USELESS**.

$\therefore\ \begin{bmatrix}r_1\\r_2\end{bmatrix}\sim N\left(\begin{bmatrix}\sqrt{E_1}\\0\end{bmatrix},\begin{bmatrix}\sigma^2&0\\0&\sigma^2\end{bmatrix}\right)$ UNDER H_1; $\begin{bmatrix}r_1\\r_2\end{bmatrix}\sim N\left(\begin{bmatrix}\sqrt{E_2}\cos\theta\\\sqrt{E_2}\sin\theta\end{bmatrix},\begin{bmatrix}\sigma^2&0\\0&\sigma^2\end{bmatrix}\right)$ UNDER H_2.

TEST IS $\underset{\text{SUFFICIENT STATISTIC}}{\ell=\Delta\underline{m}^T Q\,\underline{R}} = \dfrac{(\sqrt{E_1}-\sqrt{E_2}\cos\theta)R_1+(0-\sqrt{E_2}\sin\theta)R_2}{\sigma^2}\ \underset{H_2}{\overset{H_1}{\gtrless}}\ \gamma$. BUT $R_1=\int_0^T R(t)\phi_1(t)\,dt$, $R_2=\int_0^T R(t)\phi_2(t)\,dt$.

SUBSTITUTING, $\ell=\int_0^T R(t)\underbrace{\sqrt{E_1}\phi_1(t)}_{s_1(t)}\,dt-\cancel{\sqrt{\frac{E_2}{E_1}}\cos\theta\int_0^T R(t)s_1(t)\,dt}-\underbrace{\int_0^T R(t)(\sqrt{E_2}\sin\theta\,\phi_2(t))\,dt}_{\text{CANCELS DENOM.}}$

$\phi_1(t)$

$=\int_0^T R(t)\left[s_2(t)-\cancel{\sqrt{\frac{E_2}{E_1}}\cos\theta\, s_1(t)}\right]dt$

$\rightarrow\ \boxed{\ell=\int_0^T R(t)(s_1(t)-s_2(t))\,dt\ \underset{H_2}{\overset{H_1}{\gtrless}}\ \gamma.}$

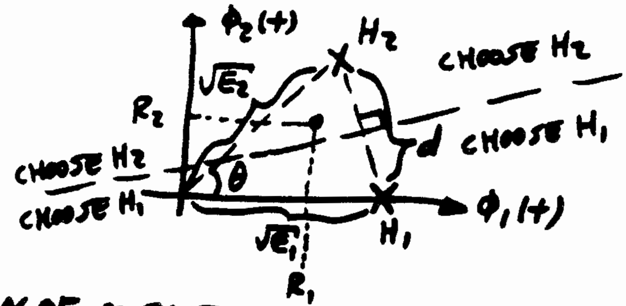
THAT IS, $r(t)=\begin{cases}s_1(t)+w(t) & H_1\\ s_2(t)+w(t) & H_2\end{cases}\ \Leftrightarrow\ r'(t)=r(t)-\underbrace{s_2(t)}_{\text{KNOWN DETERMINISTIC}}=\begin{cases}(s_1(t)-s_2(t))+w(t) & H_1\\ w(t) & H_2\end{cases}$

$\therefore$ MATCHED FILTER TO $s_1(t)-s_2(t)$.

PERFORMANCE OF TEST IS $d^2 = \Delta \underline{m}^T Q \, \Delta \underline{m} = \frac{1}{\sigma^2}\left[\left(\sqrt{E_1} - \sqrt{E_2}\cos\theta\right)^2 + \left(0 - \sqrt{E_2}\sin\theta\right)^2\right]$

$d^2 = \frac{1}{\sigma^2}\left(E_1 + E_2\cos^2\theta - 2\sqrt{E_1}\sqrt{E_2}\cos\theta + E_2\sin^2\theta\right) = \boxed{\frac{1}{\sigma^2}\left(E_1 + E_2 - 2\sqrt{E_1}\sqrt{E_2}\cos\theta\right)}$

SIGNAL SPACE INTERPRETATION:



BY LAW OF COSINES,

$d^2 = E_1 + E_2 - 2\sqrt{E_1}\sqrt{E_2}\cos\theta$ ✓

THEN SCALE BY σ^2.

$R(t) = R_1\phi_1(t) + R_2\phi_2(t) + \ldots \quad R_i = \int_0^T R(t)\phi_i(t)\,dt$

COORDINATES: $[R_1, R_2, R_3, \ldots]$

$s_1(t) = \sqrt{E_1}\,\phi_1(t) + 0\cdot\phi_2(t)$

COORDINATES: $[\sqrt{E_1}, 0, 0, \ldots]$

$s_2(t) = \left(\sqrt{E_2}\cos\theta\right)\phi_1(t) + \left(\sqrt{E_2}\sin\theta\right)\phi_2(t)$

COORDINATES: $[\sqrt{E_2}\cos\theta, \sqrt{E_2}\sin\theta, 0, \ldots]$

COMPUTED THESE USING $\underline{T=1}$, $\underline{T=2}$ ON OTHER SIDE, BUT USING SIGNAL SPACES, CAN SEE WHY THIS IS TRUE!

NOTE THAT ONLY THE 1ST 2 COORDINATES R_1 AND R_2 OF $R(t)$ MATTER:

PROJECT $R(t)$ ONTO SPAN$[\phi_1(t), \phi_2(t)]$

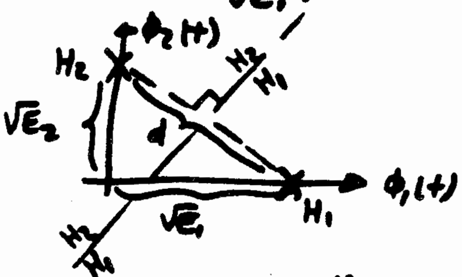
$\longrightarrow (R_1, R_2)$ IN $\phi_1(t)$-$\phi_2(t)$ PLANE.

IF $\theta = 90° \longrightarrow \int_0^T s_1(t)s_2(t)\,dt = 0$

EX: $s_1(t) = \sin 2\pi M_1 t/T$, $s_2(t) = \sin 2\pi M_2 t/T$ SINUSOIDS OF DIFF. FREQS

FREQ-SHIFT KEYING (FSK) CRUDE FORM OF FM.

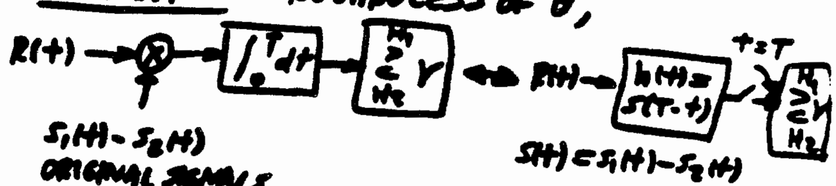
NOW $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$, $\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}}$. $\boxed{d^2 = \frac{E_1 + E_2}{\sigma^2}}$ SET $\theta = 90°$ IN ABOVE EQNS.



BY PYTHAGORAS, $d^2 = E_1 + E_2$ ✓

THEN SCALE BY σ^2.

SUMMARY: REGARDLESS OF θ,



CORRELATION RCVR — MATCHED FILTER

PERFORMANCE: $d^2 = \left(E_1 + E_2 - 2\sqrt{E_1}\sqrt{E_2}\cos\theta\right)/\sigma^2$

USE TO PICK ROC CURVE ON P. 38

(IN GENERAL: $P_D = \text{erfc}[s]$, $P_F = \text{erfc}[s+d]$ AS s VARIES FROM $-\infty$ TO ∞)

EECS 564 GENERAL M-ARY DETECTION IN WGN

WINTER 1999

PROBLEM: OBSERVE $r(t) = \begin{cases} \sqrt{E_1}\, s_1(t) + w(t) & \text{UNDER } H_1 \\ \vdots & \vdots \\ \sqrt{E_M}\, s_M(t) + w(t) & \text{UNDER } H_M \end{cases}$ FOR $0 \le t \le T$

$w(t)$ = 0-MEAN WGN WITH STRENGTH $\frac{N_0}{2}$

$\int_0^T s_i(t) s_j(t)\,dt = \underline{\rho_{ij}}$ $\quad 1 \le i,j \le M$.

NOTE $|\rho_{ij}| \le \rho_{ii} = 1$ WITHOUT LOSS OF GENERALITY.

GOAL: GIVEN $\{R(t), 0 \le t \le T\}$, CHOOSE ONE OF $H_1, H_2, \ldots H_M$.

CRITERION: MIN $Pr[\text{ERROR}]$.

THE FOLLOWING ARE GIVEN: $s_i(t)$, E_i, ρ_{ij}, $N_0/2$, $p_i = Pr[H_i \text{ TRUE}]$ (A PRIORI PROBS)

SOLUTION: (1) PERFORM GRAM-SCHMIDT ON $\{s_1(t) \ldots s_M(t)\} \longrightarrow \{\phi_1(t) \ldots \phi_N(t)\}$

USING
$$\boxed{\phi_n(t) = \frac{s_n(t) - \sum_{i=1}^{n-1} \phi_i(t)\left(\int_0^T s_n(t)\phi_i(t)\,dt\right)}{\sqrt{\int_0^T \left(s_n(t) - \sum_{i=1}^{n-1} \phi_i(t)\left(\int_0^T s_n(t)\phi_i(t)\,dt\right)\right)^2 dt}}}$$
FOR $n = 1, 2, \ldots N$

NOTE: ALMOST SURELY $N = M$. HOWEVER, IF $\{s_1(t) \ldots s_M(t)\}$ IS NOT LINEARLY INDPT, THEN $N < M$.
E.G., $s_M(t) = \sum_{i=1}^{M-1} s_i(t) \longrightarrow$ CAN ONLY GET $N = M-1$ BASIS FUNCS.

(2) USE K-L EXPANSION:

DEFINE $\boxed{m_{ij} = \sqrt{E_j}\int_0^T s_j(t)\phi_i(t)\,dt \quad 1 \le i \le N,\ 1 \le j \le M}$

MULTIPLY "OBSERVE" EQN. BY $\phi_i(t)$ AND $\int_0^T dt$:

$$r_i = \int_0^T r(t)\phi_i(t)\,dt = \begin{cases} m_{i1} + w_i & H_1 \\ m_{i2} + w_i & H_2 \\ \vdots & \vdots \\ m_{iM} + w_i & H_M \end{cases} \quad 1 \le i \le N \qquad = \begin{cases} w_i & H_1 \\ w_i & H_2 \\ \vdots & \vdots \\ w_i & H_M \end{cases} \quad i > N \text{ (NO HELP)}$$

(3) APPLY M-ARY HYPOTHESIS TESTING RESULTS FROM CHAP. 3:

CHOOSE H_i WHERE $\sum_{j \ne i} (C_{ij} - C_{jj}) p_j f_{\underline{r}|H_j}(\underline{R}|H_j)$ SMALLEST [GENERAL RESULT] $\longrightarrow$ CHOOSE H_i WHERE $f_{\underline{r}|H_i}(\underline{R}|H_i) p_i$ LARGEST [MEP: $C_{ij} = 1 - \delta_{ij}$] [MIN Pr(ERROR) RESULT]

FROM ABOVE, (CHANGE i TO j) $f_{r_j|H_i}(R_j|H_i) \sim N(m_{ji}, N_0/2)$ SINCE $w_j \sim N(0, N_0/2)$ AND $f_{\underline{r}|H_i}(\underline{R}|H_i) = \prod_{j=1}^{N} f_{r_j|H_i}(R_j|H_i)$ SINCE $\{w_j\}$ UNCORRELATED GAUSSIANS $\longrightarrow$ INDPT RVS.

$\longrightarrow$ CHOOSE H_i WHERE $\left(\frac{1}{(2\pi N_0/2)^{\frac{N}{2}}} \prod_{j=1}^{N} e^{-(R_j - m_{ji})^2/2\cdot N_0/2}\right) p_i$ LARGEST [PLUG IN]

$\longrightarrow$ CHOOSE H_i WHERE [TAKE LOG] $\boxed{\log p_i - \frac{1}{N_0}\sum_{j=1}^{N} (R_j - m_{ji})^2 \text{ LARGEST}}$ AND $\boxed{R_j = \int_0^T R(t)\phi_j(t)\,dt}$

FINAL PROCEDURE:

(1) $\{s_1(t) \ldots s_M(t)\} \xrightarrow[\text{SCHMIDT}]{\text{GRAM}} \{\phi_1(t) \ldots \phi_N(t)\}$ (2) COMPUTE $m_{ij} = \sqrt{E_j}\int_0^T s_j(t)\phi_i(t)\,dt$ $\quad 1 \le i \le N,\ 1 \le j \le M$
NOTE THESE ARE COMPUTED DURING GRAM-SCHMIDT

NOTE $m_{ij} = 0$ IF $i > j$

(3) READ IN DATA $\{R(t), 0 \le t \le T\}$. COMPUTE $R_j = \int_0^T R(t)\phi_j(t)\,dt$ $\quad 1 \le j \le N$.

(4) COMPUTE $\log p_i - \frac{1}{N_0}\sum_{j=1}^{N} (R_j - m_{ji})^2$ FOR $1 \le i \le M$. (5) CHOOSE H_k WHERE $i = k$ MAXIMIZES (4).

SPECIAL CASE #1: EACH HYPOTHESIS EQUALLY LIKELY A PRIORI: $p_i = \frac{1}{M}$ $\quad 1 \le i \le M$. EQUAL ENERGIES: $E_j = E$.
ALSO, SIGNALS UNCORRELATED: $\int_0^T s_i(t)s_j(t)\,dt = \rho_{ij} = \delta_{ij}$. THEN $\boxed{\phi_i(t) = s_i(t)}$

$\longrightarrow \log p_i - \frac{1}{N_0}\sum_{j=1}^{N}(R_j - m_{ji})^2 = \underbrace{\log \frac{1}{M}}_{\text{[INDPT OF } i]} - \frac{1}{N_0}\left(\underbrace{\sum_{j=1}^{N} R_j^2}_{\text{[INDPT. OF } i]} + E - 2\sum_{j=1}^{N} R_j \sqrt{E}\,\delta_{ij}\right)$

ALSO: SEE OVERLEAF $\quad m_{ij} = \sqrt{E}\,\delta_{ij}$

$\longrightarrow$ $\boxed{\text{CHOOSE } H_i \text{ WHERE } R_i' \text{ LARGEST}}$ AND $\boxed{R_i' = \int_0^T R(t)s_i(t)\,dt} = R_i$ HERE.

NOTE $\frac{1}{N_0}$ = TRADEOFF FACTOR BETWEEN A PRIORI $\log p_i$ AND A POSTERIORI $\Sigma(R_j - m_{ji})$

REMARKABLY, THE ABOVE RESULT HOLDS EVEN IF THE SIGNALS ARE CORRELATED:

SPECIAL CASE #2: IF EACH HYPOTHESIS EQUALLY LIKELY A PRIORI ($P_i = \frac{1}{M}$) AND IF EACH SIGNAL HAS THE SAME ENERGY E ($E_i = E$ FOR $1 \le i \le M$)

THEN THE OPTIMAL RECEIVER IS: $\boxed{\text{CHOOSE } H_i \text{ WHERE } R_i' = \int_0^T R(t) s_i(t)\,dt \text{ IS LARGEST}}$

EVEN IF THE SIGNALS ARE CORRELATED!!

PROOF: RECALL PARSEVAL'S THM FOR REAL ORTHONORMAL EXPANSIONS:

IF $R(t) = \sum_{i=1}^{\infty} R_i \phi_i(t)$ AND $S(t) = \sum_{i=1}^{\infty} S_i \phi_i(t)$, THEN $\int_0^T R(t) S(t)\,dt = \sum_{i=1}^{\infty} R_i S_i$.

THEN $\log P_i - \frac{1}{N_0} \sum_{j=1}^{N} (R_j - m_{ji})^2 = \underbrace{\log \frac{1}{M} - \frac{1}{N_0} \sum_{j=1}^{N} R_j^2}_{\text{INDPT. OF } i} - \frac{1}{N_0} \sum_{j=1}^{N} m_{ji}^2 + \frac{2}{N_0} \sum_{j=1}^{N} R_j m_{ji}$.

BUT $m_{ji} = \int_0^T s_i(t) \phi_j(t) \to s_i(t) = \sum_{j=1}^{N} m_{ji} \phi_j(t)$ AND $R_j = \int_0^T R(t) \phi_j(t)\,dt \to R(t) = \sum_{j=1}^{\infty} R_j \phi_j(t)$.

SO $-\frac{1}{N_0} \sum_{j=1}^{N} m_{ji}^2 = -\frac{1}{N_0} \int_0^T s_i(t)^2\,dt = \underbrace{-\frac{1}{N_0} E}_{\text{INDPT. OF } i}$, LEAVING $\frac{2}{N_0} \sum_{j=1}^{N} R_j m_{ji} = \frac{2}{N_0} \int_0^T R(t) s_i(t)\,dt$.

$\therefore$ CHOOSE H_i WHERE $\log P_i - \frac{1}{N_0} \sum_{j=1}^{N} (R_j - m_{ji})^2$ LARGEST $\to$ CHOOSE H_i WHERE $\int_0^T R(t) s_i(t)\,dt$ LARGEST.

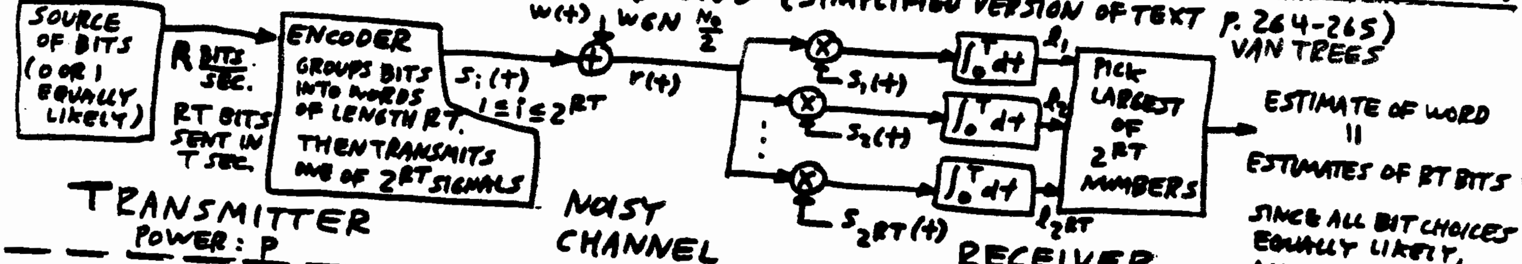
NOTE: THIS ISN'T ALL THAT SURPRISING -- WE SAW IT BEFORE IN THE CASE OF GENERAL BINARY DETECTION. EVEN WHEN $s_1(t)$ AND $s_2(t)$ WERE CORRELATED, THE OPTIMAL TEST WAS $\int_0^T R(t)(s_1(t) - s_2(t))\,dt \underset{H_2}{\overset{H_1}{\gtrless}} \gamma = 0$ BY SYMMETRY IF SAME ENERGIES AND EQUAL A PRIORI PROBS.

$\to \int_0^T R(t) s_1(t) \underset{H_2}{\overset{H_1}{\gtrless}} \int_0^T R(t) s_2(t)\,dt$ WHICH AGREES WITH ABOVE.

SEE TEXT P. 257

WHILE FORM OF OPTIMAL TEST IS UNAFFECTED ABOVE BY CORRELATION BETWEEN SIGNALS, PERFORMANCE OF OPTIMAL TEST IS AFFECTED. BEST PERFORMANCE: USE ORTHOGONAL SIGNALS.

EXAMPLE: CHANNEL CAPACITY BOUND (SIMPLIFIED VERSION OF TEXT P. 264-265) VAN TREES



SINCE ALL BIT CHOICES EQUALLY LIKELY, ALL WORD CHOICES EQUALLY LIKELY.

EACH OF $M = 2^{RT}$ HYPOTHESES EQUALLY LIKELY. EACH $s_i(t)$ HAS EQUAL ENERGY PT.

PERFORMANCE: LET'S COMPUTE A BOUND ON Pr[ERROR].

$Pr[\text{ERROR}] = Pr[\ell_2 > \ell_1 \text{ OR } \ell_3 > \ell_1 \text{ OR} \ldots \text{OR } \ell_{2^{RT}} > \ell_1] \le (2^{RT}-1) Pr[\ell_i > \ell_1] = (2^{RT}-1)\,\text{erfc}\left[\sqrt{PT/N_0}\right]$. BY SYMMETRY, WLOG WE CAN LET H_1 BE TRUE.

[WAYS CHOICE OF ℓ_1 CAN BE WRONG] [UNION BOUND] [EQUAL BY SYMMETRY] [(70) ON P. 38 VAN TREES; $d/2$]

FROM (72) ON P. 39, VAN TREES, WE HAVE $\text{erfc}(x) < \frac{1}{2} e^{-x^2/2}$ FOR $x > 0$ (A VERY CRUDE BOUND)

$\to Pr(\text{ERROR}) \underset{\text{[UNION BOUND]}}{\le} 2^{RT}\,\text{erfc}\left[\sqrt{PT/N_0}\right] \underset{\text{[erfc BOUND]}}{<} 2^{RT} \frac{1}{2} e^{-PT/2N_0} \underset{[2 = e^{\log 2}]}{=} \frac{1}{2} e^{(RT \log 2 - PT/2N_0)} = \frac{1}{2} 2^{T(R - P/2N_0 \log 2)}$.

NOW LET $T \to \infty$ (NOTE THAT $M = 2^{RT}$ = BANDWIDTH $\to \infty$ ALSO). WHAT HAPPENS TO THIS BOUND?

IF $R < P/2N_0 \log 2$, THEN $T(R - P/2N_0 \log 2) < 0$ AND $\frac{1}{2} 2^{T(R - P/2N_0 \log 2)} \to 0$ SO $Pr(\text{ERROR}) \to 0$.

IF $R > P/2N_0 \log 2$, THEN $\frac{1}{2} 2^{T(R - P/2N_0 \log 2)} \to \infty$! ALTHOUGH THIS IS ONLY AN UPPER BOUND ON Pr(ERROR) THRESHOLD! THIS IS NOT A GOOD SIGN!

ACTUAL CHANNEL CAPACITY (SHANNON): $R < P/N_0 \log 2$.

WHY OFF BY 2? TOO MANY BOUNDS.