EECS 564 SOLVING INTEGRAL EQUATIONS Winter 1999

GOAL: Solve $\int_0^T K_x(t,s)\phi(s)ds = \lambda\phi(t)$ for $K_x(t,s) = \sigma^2 \min[t,s]$.

WHY? *Any* 0-mean *independent increments* random process (such as the Wiener process) $x(t)$ can be expanded as: $x(t) = \sum_{n=1}^{\infty} x_n \phi_n(t), 0 \leq t \leq T$, where $E[x_i x_j] = \lambda_i \delta_{i,j}$.

SUB: $\int_0^T \sigma^2 \min[t,s]\phi(s)ds = \int_0^t \sigma^2 s\phi(s)ds + \int_t^T \sigma^2 t\phi(s)ds = \lambda\phi(t)$.

$\frac{\partial}{\partial t}$: $\sigma^2 t\phi(t) - \sigma^2 t\phi(t) + \int_t^T \sigma^2 \frac{\partial t}{\partial t}\phi(s)ds = \lambda\frac{\partial\phi}{\partial t}$ (Leibniz's rule). This is called the "first integral" of the diff. eqn. to follow.

$\frac{\partial}{\partial t}$: $-\sigma^2\phi(t) = \lambda\frac{\partial^2\phi}{\partial t^2}$. A differential equation! (can handle):

SOLN: $\phi(t) = A\cos(\omega t) + B\sin(\omega t)$ where $\omega = \sqrt{\sigma^2/\lambda}$.

FIND A: $K_x(0,0) = \sigma^2\min[0,0] = \sum\lambda_i\phi_i^2(0) = \sum\lambda_i A^2 \rightarrow A = 0$.

FIND B: Substitute $\phi(t) = B\sin(\omega t)$ into "first integral"; set $t = 0$:

$\rightarrow \cos(\omega T) = 0 \rightarrow \omega T = (n+\frac{1}{2})\pi \rightarrow \lambda = \left(\frac{\sigma T}{(n+\frac{1}{2})\pi}\right)^2$.

$\phi(t)$: $\phi(t) = B\sin\left(\frac{(n+\frac{1}{2})\pi}{T}t\right) \rightarrow B = \sqrt{\frac{2}{T}}$ using $\int\phi(t)^2dt = 1$.

K-L: $x(t) = \sum x_n\sqrt{\frac{2}{T}}\sin\left(\frac{(n+\frac{1}{2})\pi}{T}t\right), 0 \leq t \leq T, E[x_n^2] = \left(\frac{\sigma T}{(n+\frac{1}{2})\pi}\right)^2$

Wiener: $x(t)$=Wiener process$\rightarrow x_n \sim N(0,\lambda_n)$.

WIDE-SENSE STATIONARY RANDOM PROCESSES:

PSD: $S_x(\omega) = \frac{\prod(j\omega-z_i)\prod(-j\omega-z_i)}{\prod(j\omega-p_i)\prod(-j\omega-p_i)} = \frac{\prod(\omega^2+z_i^2)}{\prod(\omega^2+p_i^2)} = \frac{N(\omega^2)}{D(\omega^2)}$.

SUB: $\int_0^T K_x(t-s)\phi(s)ds = \int\frac{d\omega}{2\pi}\frac{N(\omega^2)}{D(\omega^2)}e^{j\omega t}\int e^{-j\omega s}\phi(s)ds = \lambda\phi(t)$.

$\mathcal{F}$: $[\lambda D(\omega^2) - N(\omega^2)]\hat{\phi}(\omega) = 0$ (include initial conditions).

$\mathcal{F}^{-1}$: $\lambda\sum d_i(-1)^i\frac{\partial^{2i}\phi}{\partial t^{2i}} - \sum n_i(-1)^i\frac{\partial^{2i}\phi}{\partial t^{2i}} = 0$ (differential eqn.).

For an example: see other handout, Srinath p. 117.