EECS 564 THE DISCRETE-TIME KALMAN FILTER WINTER 1999

PROBLEM: $\underline{x}(n+1) = A\underline{x}(n) + B\underline{u}(n)$ STATE EQN. — DRIVING NOISE: $E[\underline{u}(i)\underline{u}(j)^T] = Q\delta(i-j)$

$\underline{y}(n) = H\underline{x}(n) + \underline{v}(n)$ OBSERVATION EQN. — OBSERV. NOISE: $E[\underline{v}(i)\underline{v}(j)^T] = R\delta(i-j)$

[EVERYTHING IS 0-MEAN] MATRICES MAY VARY WITH n (NOT SHOWN TO SAVE SPACE)

INITIAL STATE: $E[\underline{x}(0)\underline{x}(0)^T] = P(0|-1)$.

$\underline{x}(0), \underline{u}(i), \underline{v}(j)$ UNCORRELATED FOR ALL i,j

GOAL: DEFINE $\hat{\underline{x}}(i|j)$ = LINEAR LEAST-SQUARES ESTIMATE OF $\underline{x}(i)$ GIVEN $\{\underline{y}(0), \underline{y}(1), \ldots \underline{y}(j)\}$.

$P(i|j)$ = (PREDICTION) ERROR COVARIANCE MATRIX = $E[(\underline{x}(i)-\hat{\underline{x}}(i|j))(\underline{x}(i)-\hat{\underline{x}}(i|j))^T]$.

GIVEN $\hat{\underline{x}}(n|n-1)$ AND NEW OBSERVATION $\underline{y}(n)$, RECURSIVELY COMPUTE $\hat{\underline{x}}(n+1|n)$.

THE KALMAN FILTER EXISTS DUE TO 3 BASIC IDEAS:

1. THE LINEAR LEAST-SQUARES ESTIMATE (LLSE) $\hat{\underline{x}}(\underline{y}) = K_{xy}K_y^{-1}\underline{y}$, AND THE ERROR COVARIANCE MATRIX IS

$$E[(\underline{x}-\hat{\underline{x}}(\underline{y}))(\underline{x}-\hat{\underline{x}}(\underline{y}))^T] = K_x - K_{xy}K_y^{-1}K_{xy}^T = \underbrace{K_x^{1/2}(I-\rho_{xy}\rho_{xy}^T)K_x^{T/2}}_{\text{MORE ELEGANT FORM}} \text{ WHERE } \underbrace{\rho_{xy} = K_x^{-1/2}K_{xy}K_y^{-1/2}}_{\text{CORRELATION COEFF. MATRIX}}$$

FOR ANY 2 ZERO-MEAN VECTORS $\underline{x}$ AND $\underline{y}$.

PROOFS: LET $\hat{\underline{x}}(\underline{y}) = A\underline{y}$. ORTHOG. PRINCIPLE $\to E[(\underline{x}-\hat{\underline{x}}(\underline{y}))\underline{y}^T] = \underline{0} = E[\underline{x}\underline{y}^T] - AE[\underline{y}\underline{y}^T] \to A = K_{xy}K_y^{-1}$.

LET $\underline{e} = \underline{x}-\hat{\underline{x}}(\underline{y})$. $E[\underline{e}\underline{e}^T] = E[\underline{e}\underline{x}^T] - \underbrace{E[\underline{e}\hat{\underline{x}}(\underline{y})^T]}_{0 \text{ BY ORTHOG. PRIN.}} = E[\underline{x}\underline{x}^T] - E[\hat{\underline{x}}(\underline{y})\underline{x}^T] = K_x - \underbrace{K_{xy}K_y^{-1}}_{\text{SUBSTITUTE FROM ABOVE; CONSTANTS PULL OUT}}\underbrace{E[\underline{y}\underline{x}^T]}_{K_{xy}^T}$.

2. IF $\{\underline{y}(0)\ldots\underline{y}(N)\}$ ARE UNCORRELATED, THEN $\hat{\underline{x}}(\{\underline{y}(0)\ldots\underline{y}(N)\}) = \sum_{i=1}^{N}\hat{\underline{x}}(\underline{y}(i))$, i.e. ESTIMATION PROBLEM DECOUPLES.

PROOF: $\hat{x}_i(y_j) = K_{x_iy_j}K_{y_j}^{-1}y_j = K_{x_iy_j}y_j$, $\hat{x}_i(\underline{y}) = \sum_{j=1}^{N}[K_{xy}]_{ij}y_j = \sum_{j=1}^{N}\hat{x}_i(y_j)$. NOTE NEED $K_y = I$. (UNCORRELATED)

3. $\{\underline{y}(0)\ldots\underline{y}(n)\}$ MAY BE CAUSALLY UNCORRELATED INTO $\{\underline{w}(0)\ldots\underline{w}(n)\}$ USING $\underline{w}(i) = \underline{y}(i) - \hat{\underline{y}}(i|i-1)$.

PROOF: SINCE $\hat{\underline{y}}(i|i-1)$ IS A LINEAR COMB. OF $\{\underline{y}(0)\ldots\underline{y}(i-1)\}$, CLEARLY CAUSAL FILTER OF $\{\underline{y}(i)\}$.

CAN WRITE $\begin{bmatrix}w(0)\\ \vdots\\ w(i)\end{bmatrix} = \underbrace{\begin{bmatrix}\Delta & 0\end{bmatrix}}_{\text{LIN. COMB.}}\begin{bmatrix}y(0)\\ \vdots\\ y(i)\end{bmatrix}$ AND $\begin{bmatrix}y(0)\\ \vdots\\ y(i)\end{bmatrix} = \underbrace{\begin{bmatrix}\Delta & 0\end{bmatrix}^{-1}}_{\text{ALSO LOWER }\Delta}\begin{bmatrix}w(0)\\ \vdots\\ w(i)\end{bmatrix}$, SO CAUSALLY INVERTIBLE FILTER.

$E[\underline{w}(i)\underline{w}(j)^T] = E[(\text{ERROR IN ESTIMATING } \underline{y}(i))(\underbrace{\underline{y}(j) - \text{LINEAR COMB. OF } \{\underline{y}(k), k<j\}}_{\text{DATA USED IN ESTIMATING } \underline{y}(i)})^T] = 0$ BY ORTHOG. PRIN.

ASSUME WLOG $i>j$

LEMMA: $E[\hat{\underline{x}}(\underline{y})\underline{e}^T] = \underline{0}$. **PROOF:** $E[\hat{\underline{x}}(\underline{y})\underline{e}^T] = E[\underline{e}\cdot\hat{\underline{x}}(\underline{y})^T]^T = E[\underline{e}\cdot(\text{LINEAR COMB. OF } y_i)^T]^T = \underline{0}^T$.

AND AWAY WE GO...

$$\hat{\underline{x}}(n|n) = \hat{\underline{x}}(n|\{\underline{y}(0)\ldots\underline{y}(n)\}) \underset{\text{USING \#3}}{=} \hat{\underline{x}}(n|\{\underline{w}(0)\ldots\underline{w}(n)\}) \underset{\text{USING \#2}}{=} \hat{\underline{x}}(n|\{\underline{w}(0)\ldots\underline{w}(n-1)\}) + \hat{\underline{x}}(n|\underline{w}(n)) \underset{\text{USING \#3 BACKWARDS}}{=} \hat{\underline{x}}(n|n-1) + \hat{\underline{x}}(n|\underline{w}(n)).$$

NOW WE USE #1 TO COMPUTE $\hat{\underline{x}}(n|\underline{w}(n))$: $\hat{\underline{x}}(n|\underline{w}(n)) = K_{x(n)w(n)}K_{w(n)}^{-1}\underline{w}(n)$. USE IN

SO WE NEED $K_{x(n)w(n)}$ AND $K_{w(n)}$. TO GET THESE, USE THE OBSERVATION EQN. AND $P(n|n-1)$:

$$\underline{w}(n) = \underline{y}(n) - \hat{\underline{y}}(n|n-1) \underset{\text{LINEARITY OF LLSE}}{=} \underline{y}(n) - H\hat{\underline{x}}(n|n-1) \underset{\text{OBSERV. EQN.}}{=} (H\underline{x}(n)+\underline{v}(n)) - H\hat{\underline{x}}(n|n-1) = H(\underline{x}(n)-\hat{\underline{x}}(n|n-1)) + \underline{v}(n) \quad (*).$$

THEN $K_{w(n)} = E[\underline{w}(n)\underline{w}(n)^T] = HE[(\underline{x}(n)-\hat{\underline{x}}(n|n-1))(\underline{x}(n)-\hat{\underline{x}}(n|n-1))^T]H^T + E[\underline{v}(n)\underline{v}(n)^T] = HP(n|n-1)H^T + R$

$$K_{x(n)w(n)} \underset{\text{USE }(*)}{=} E[\underline{x}(n)\underline{w}(n)^T] = E[\underline{x}(n)(H(\underline{x}(n)-\hat{\underline{x}}(n|n-1)))^T] + \underbrace{E[\underline{x}(n)\underline{v}(n)^T]}_{0}$$

$$= E[(\underline{x}(n)-\hat{\underline{x}}(n|n-1))(H(\underline{x}(n)-\hat{\underline{x}}(n|n-1)))^T] + \underbrace{E[\hat{\underline{x}}(n|n-1)(H(\underline{x}(n)-\hat{\underline{x}}(n|n-1)))^T]}_{0 \text{ BY LEMMA}} = P(n|n-1)H^T.$$

$\underline{x}(n) = (\underline{x}(n)-\hat{\underline{x}}(n)) + \hat{\underline{x}}(n)$

PLUG IN:

$$\boxed{\hat{\underline{x}}(n|n) = \hat{\underline{x}}(n|n-1) + \underbrace{P(n|n-1)H^T}_{K_{x(n)w(n)}}\underbrace{(HP(n|n-1)H^T+R)^{-1}}_{K_{w(n)}^{-1}}\underbrace{(\underline{y}(n)-H\hat{\underline{x}}(n|n-1))}_{\underline{w}(n)}}$$

① ORDER UPDATE

THIS UPDATES $\hat{\underline{x}}(n|n-1)$ TO $\hat{\underline{x}}(n|n)$.

TO UPDATE $P(n|n-1)$ TO $P(n|n)$:

SUBTRACT ① FROM $x(n)$: $(x(n)-\hat{x}(n|n)) = (x(n)-\hat{x}(n|n-1)) - P(n|n-1)H^T(HP(n|n-1)H^T+R)^{-1}w(n)$

MULT. THIS BY ITS TRANSPOSE AND TAKE $E[\cdot]$. 4 TERMS RESULT:

$$P(n|n) = P(n|n-1) + PH^T(HPH^T+R)^{-1}K_{w(n)}(HPH^T+R)^{-T}HP^T - PH^T(HPH^T+R)^{-1}E[w(n)(x(n)-\hat{x}(n|n-1))^T] - E[(x(n)-\hat{x}(n|n-1))w(n)^T](HPH^T+R)^{-T}HP^T.$$

NOW WHAT? PLUG IN FROM OTHER SIDE OF PAGE:

$K_{w(n)} = HP(n|n-1)H^T + R$ AND $E[(x(n)-\hat{x}(n|n-1))w(n)^T] = E[x(n)w(n)^T] - E[\hat{x}(n|n-1)w(n)^T]$

$$= K_{x(n)w(n)} - E\left[\left(\text{LINEAR COMB. OF } w(1)\ldots w(n-1)\right)w(n)^T\right]$$

$$= P(n|n-1)H^T - 0 \text{ BY \#3}$$

THE LAST 3 TERMS ARE ALL EQUAL!! $P(n)$ AND $R(n)$ ARE SYMMETRIC.

PLUG IN: $\boxed{P(n|n) = P(n|n-1) - P(n|n-1)H^T(HP(n|n-1)H^T+R)^{-1}HP(n|n-1)}$ ② UPDATES $P(n|n-1)$ TO $P(n|n)$

THE SECOND HALF IS MUCH EASIER: NOW UPDATE $\hat{x}(n|n)$ TO $\hat{x}(n+1|n)$ USING STATE EQN.

$x(n+1) = Ax(n) + Bu(n) \longrightarrow \boxed{\hat{x}(n+1|n) = A\hat{x}(n|n)}$ ③ UPDATES $\hat{x}(n|n)$ TO $\hat{x}(n+1|n)$ USING "DEAD RECKONING"

SUBTRACT ③ FROM STATE EQN: $(x(n+1)-\hat{x}(n+1|n)) = A(x(n)-\hat{x}(n|n)) + Bu(n)$.

MULT. THIS BY ITS TRANSPOSE AND TAKE $E[\cdot]$: $\boxed{P(n+1|n) = AP(n|n)A^T + BQB^T}$ ④ TIME UPDATE

GRAND FINALE:

INSERT ③ INTO ① →

$$\hat{x}(n|n) = \underbrace{A(n-1)\hat{x}(n-1|n-1)}_{\text{A PRIORI INFO "DEAD RECKONING"}} + \underbrace{P(n|n-1)H(n)^T(H(n)P(n|n-1)H(n)^T+R(n))^{-1}}_{\text{KALMAN GAIN}}\underbrace{(y(n)-H(n)A(n-1)\hat{x}(n-1|n-1))}_{\text{NEW DATA}}$$

OR

$$\hat{x}(n+1|n) = A(n)\hat{x}(n|n-1) + A(n)P(n|n-1)H(n)^T(H(n)P(n|n-1)H(n)^T+R(n))^{-1}\overbrace{(y(n)-H(n)\hat{x}(n|n-1))}^{\text{A POSTERIORI INFO. } w(n) = \text{NEW INFORMATION IN } y(n)}$$

WITH $P(n|n-1)$ RECURSIVELY COMPUTED USING (INSERT ② INTO ④)

$$\boxed{P(n+1|n) = \underbrace{A(n)P(n|n-1)A(n)^T + B(n)Q(n)B(n)^T}_{\text{"DEAD RECKONING" ERROR}} - A(n)\underbrace{P(n|n-1)H(n)^T(H(n)P(n|n-1)H(n)^T+R(n))^{-1}}_{\text{KALMAN GAIN}}H(n)P(n|n-1)A(n)^T}$$

DATA $y(n)$ HELPS

KALMAN GAIN: THE MORE $y(n)$ IS WEIGHTED, THE MORE IT HELPS.

STILL ANOTHER WAY OF WRITING IS AS FOLLOWS:

$$\hat{x}(n|n) = A(n-1)\hat{x}(n-1|n-1) + k(n)(y(n) - H(n)A(n-1)\hat{x}(n-1|n-1))$$

$$k(n) = P(n|n-1)H(n)^T(H(n)P(n|n-1)H(n)^T+R(n))^{-1} = \text{KALMAN GAIN}$$

$$P(n+1|n) = A(n)(I-k(n)H(n))P(n|n-1)A(n)^T + B(n)Q(n)B(n)^T$$

CHOICES FOR KALMAN FILTER:

1. ①,②,③,④ SEPARATELY
2. ABOVE 2 EQNS.
3. 3 EQNS AT LEFT

FINAL COMMENT:

1. THE TRANSFORMATION $\{y(i)\} \to \{w(i)\}$ WHERE $w(i) = y(i) - \hat{y}(i|i-1)$ CAN BE INTERPRETED AS A GRAM-SCHMIDT ORTHOGONALIZATION: SUBTRACT FROM $y(i)$ ITS PROJECTION ON SPAN$[y(0)\ldots y(i-1)]$. THEN THE $\{w(i)\}$ ARE ORTHOGONAL IN THAT $E[w(i)w(j)]=0$ IF $i\neq j$.
2. KALMAN FILTER IS MORE GENERAL THAN WIENER FILTER, SINCE IT ALLOWS NONSTATIONARY PROCESSES. BUT IT'S NOTHING MORE THAN A RECURSIVE IMPLEMENTATION OF $\hat{x}(y) = K_{xy}K_y^{-1}y$
3. CAN COMPUTE $P(n|n-1)$ (PERFORMANCE) AND $K(n)$ (GAINS) AHEAD OF TIME, OFF-LINE.
4. HANDY EQN: (DERIVED ABOVE) $\underbrace{w(n)}_{\text{INNOVATION (UNCORRELATED)}} = H(n)\underbrace{(x(n)-\hat{x}(n))}_{\text{ERROR}} + \underbrace{v(n)}_{\text{OBS. NOISE (UNCORRELATED)}}$ = ERROR IN PREDICTING $y(n)$.

EECS 564 STEADY-STATE KALMAN FILTER APRIL 5, 1999

STEADY-STATE KF: IF:

(1) $A(t), B(t), H(t), Q(t), R(t)$ CONSTANT IN TIME (INDPT OF t)
(2) A STABLE ($\mathrm{RE}[\text{EIGVAL}] < 0$)
(3) $Q > 0$ NOT JUST $Q \geq 0$
(4) $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ CONTROLLABLE

RICCATI EQN: $\frac{dP}{dt} = AP + PA^T + BQB^T - PH^TR^{-1}HP$

$P(t) \to P_{SS} \geq 0$ WHICH SOLVES ALGEBRAIC RICCATI EQN
$0 = AP_{SS} + P_{SS}A^T + BQB^T - P_{SS}H^TR^{-1}HP_{SS}$ SIMULT. QUADRATICS
KALMAN GAIN $\to P_{SS}H^TR^{-1}$

$P(t) \to P_{SS} > 0$ NOT JUST $P_{SS} \geq 0$. (SEE OVER)
WHY DOES THIS MATTER? DIVERGENCE OF KF

PROOF OF SPECIAL CASE $H=0$ [ILLUSTRATES SIGNIFICANCE OF REQUIREMENTS]

$H = 0$ IN RICCATI EQN $\to \frac{dP}{dt} = AP + PA^T + BQB^T \to P(t) = e^{At}P(0)e^{A^Tt} + \int_0^t e^{A\tau}BQB^Te^{A^T\tau}d\tau$

(SEE BOTTOM OF CONT-TIME KF HANDOUT) LINEAR SYSTEMS HANDOUT SET $s=0$; $\Phi(t,s) = e^{A(t-s)}$ ΔVARIABLES: $\tau \to t-\tau$

AS $t \to \infty$: 1^{ST} TERM $\to 0$ IF A STABLE
$\to \infty$ IF A UNSTABLE; 2^{ND} TERM $\to \int_0^\infty e^{A\tau}BQB^Te^{A^T\tau}d\tau \geq 0$.
SHOWS NECESSARY AS WELL AS SUFFICIENT

WHEN IS $P_{SS} = \int_0^\infty e^{A\tau}BQB^Te^{A^T\tau}d\tau > 0$, NOT JUST ≥ 0? [6.432 IX.18]

IN GENERAL, $BQB^T > 0$ REQUIRES: (1) $Q > 0$ (2) B HAS FULL RANK. TO SEE THESE:
$Q \geq 0 \to \exists\, \underline{x}$ SO THAT $\underline{x}^TQ\underline{x} = 0 \to \underline{y}^T(BQB^T)\underline{y} = 0 \to BQB^T \geq 0$ WHERE $B^T\underline{y} = \underline{x}$.
B NOT FULL RANK $\to \exists\, \underline{y}$ SO THAT $B^T\underline{y} = 0 \to \underline{y}^T(BQB^T)\underline{y} = (B^T\underline{y})^TQ\underbrace{(B^T\underline{y})}_{=0} = 0 \to BQB^T \geq 0$

SO NECESSARY SUFFICIENT FOR: (1) $Q > 0$ (2) $e^{At}B$ FULL RANK

$$e^{At}B = B + ABt + \frac{t^2}{2!}A^2B + \frac{t^3}{3!}A^3B + \ldots = \underbrace{[B|AB|A^2B|\cdots]}_{\text{NEED FULL RANK}}\begin{bmatrix} I \\ tI \\ (t^2/2)I \\ (t^3/3!)I \\ \vdots \end{bmatrix}$$

DISCRETE-TIME STEADY-STATE KF: IF:

(1) $A(n), B(n), H(n), Q(n), R(n)$ CONSTANT IN TIME (INDPT OF n)
(2) A STABLE ($|\text{EIGVAL}| < 1$) $\boxed{\text{OR}}$
$\underline{x}(n+1) = A\underline{x}(n) + B\underline{u}(n)$, $y(n) = Hx(n)$ IS OBSERVABLE
(3) $Q > 0$ NOT JUST $Q \geq 0$
(4) $\underline{x}(n+1) = A\underline{x}(n) + B\underline{u}(n)$ CONTROLLABLE

$P(n+1) = AP(n)A^T + BQB^T - AP(n)H^T(HP(n)H^T+R)^{-1}HP(n)A^T$
WHERE $P(n)$ HERE MEANS $P(n|n-1)$

$P(n) \to P_{SS}$ WHICH SOLVES DISCRETE ARE
$P_{SS} = AP_{SS}A^T + BQB^T - AP_{SS}H^T(HP_{SS}H^T+R)^{-1}HP_{SS}A^T$
KALMAN GAIN $\to$ CONSTANT ALSO.

$P(n) \to P_{SS} > 0$ NOT JUST $P_{SS} \geq 0$.
WHY DOES THIS MATTER? DIVERGENCE OF KF
SEE EXAMPLE ON OTHER SIDE.

III. IF ASSUMPTIONS NOT MET? AUGMENT THE STATE:

SUPPOSE HAVE $x(n+1) = a(n)x(n) + d(n)x(n-1) + b(n)u(n)$. FORMULATE AS KF.
OBSERVE $y(n) = c(n)x(n) + v(n)$

LET $\underline{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix}$. THEN $\underline{x}(n+1) = \begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix} = \overbrace{\begin{bmatrix} a & d \\ 1 & 0 \end{bmatrix}}^{A(n)} \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} + \overbrace{\begin{bmatrix} b \\ 0 \end{bmatrix}}^{B(n)} u(n)$

$y(n) = \underbrace{[c(n) \;\; 0]}_{C(n)} \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} + v(n)$ NOW CAN USE KF.

NONZERO MEAN? SEE PROBLEM SET.

COLORED NOISE? IF RATIONAL, ADD DYNAMICS TO STATE MODEL: NOW SUPPOSE $u(n+1) = e(n)u(n) + w(n)$ NONSTATIONARY COLORED NOISE!

$$\begin{bmatrix} x(n+1) \\ x(n) \\ u(n+1) \end{bmatrix} = \begin{bmatrix} a(n) & d(n) & b(n) \\ 1 & 0 & 0 \\ 0 & 0 & e(n) \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ u(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(n) \qquad y(n) = [c(n) \;\; 0 \;\; 0] \begin{bmatrix} x(n) \\ x(n-1) \\ u(n) \end{bmatrix} + v(n).$$

COLORED OBSERVATION NOISE?

DIVERGENCE OF KF: [6.432 V. 36-38]

CONSIDER $\underline{x}(n+1) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \underline{x}(n) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(n)$, $E(u(n)^2) = 1$.

$\underline{y}(n) = \underline{x}(n) + \underline{v}(n)$, $E(\underline{v}(n)\underline{v}(n)^T) = I$. $\underline{x}(0) = \underline{0}$.

SOL'N TO $P = APA^T + BQB^T - APH^T(HPH^T+R)^{-1}HPA^T$ IS $P_{ss} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$;

KALMAN GAIN IS $P_{ss}H^T(HP_{ss}H^T+R)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$.

$\hat{\underline{x}}(n+1|n+1) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \hat{\underline{x}}(n|n) + \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}}^{K} \underline{y}(n)$. (NOTE $KAH = 0$)

(1) FILTER UNSTABLE (2) FILTER IGNORES $y_1(n)$!!!

NOTE: $[A,B]$ CC AND NEEDED FOR $P_{ss} \geq 0$ { $[A,B]$ CO OR A STABLE $\rightarrow P_{ss}$ ... $\rightarrow P_{ss} \geq 0$
HERE: $[A,B]$ NOT CC, A NOT STABLE BUT $[A,C]$ CO $\rightarrow P_{ss} \geq 0$.

REASON: NOT CC; x_1 UNAFFECTED BY NOISE, EITHER DIRECTLY OR COUPLED THROUGH x_2.
SINCE KNOW $x_1(0) = 0$, JUST USE $x_1(n+1) = 2x_1(n) \rightarrow x_1(n) = 0$ FOR ALL n.
NO NEED TO USE OBSERVATIONS.

BUT - IS THIS REALISTIC? ALWAYS SOME ERROR (NUMERICAL IF NOTHING ELSE)
$x_1(n) = 2^n x_1(0) = 2^n \epsilon$. YET KALMAN GAIN $\rightarrow 0$ BLITHELY.
EARLY APOLLO NAVIGATION WORK.

$[A,B]$ NOT CC:
1. MATHE. $\rightarrow$ GOOD! $\exists$ MODE W/O NOISE
2. MODEL $\rightarrow$ BAD ... : ... W/ OBS.

EXAMPLE OF DISCRETE KF

PROBLEM: OBSERVE $y(n) = x + v(n)$, $v \sim N(0,1)$, $x \sim N(0,1)$ UNKNOWN AND CONSTANT. RECURSIVELY ESTIMATE x FROM $y(n)$.

KF: x CONSTANT $\rightarrow x(n+1) = x(n)$. $A=1$, $B=0$ (OR $Q=0$), $H=1$, $R=1$. $P(0|-1)=1$, $\hat{x}(0|-1)=0$ (A PRIORI).

$P(n+1|n) = AP(n|n-1)A^T + BQB^T - PH^T(HP(n|n-1)H^T+R)^{-1}HP \rightarrow P(n+1) = P(n) + 0 - P(n)^2/(P(n)+1)$.

TRY IT: $P(0)=1$. $P(1) = 1 - \frac{1}{2} = \frac{1}{2}$. $P(2) = \frac{1}{2} - \frac{1/4}{3/2} = \frac{1}{3}$. $P(3) = \frac{1}{3} - \frac{1/9}{4/3} = \frac{1}{4}$. $P(n) = \frac{1}{n+1}$.

$\hat{x}(n+1|n) = A\hat{x}(n|n-1) + PH^T(HPH^T+R)^{-1}(y(n) - H\hat{x}(n|n-1)) \rightarrow \hat{x}(n+1) = \hat{x}(n) + \frac{1/(n+1)}{1+1/(n+1)}(y(n) - \hat{x}(n)) = \frac{n+1}{n+2}\hat{x}(n) + \frac{1}{n+2}y(n)$

TRY IT: $\hat{x}(0) = 0$. $\hat{x}(1) \underset{n=0}{=} \frac{1}{2}\hat{x}(0) + \frac{1}{2}y(0) = \frac{1}{2}y(0)$. $\hat{x}(2) \underset{n=1}{=} \frac{2}{3}(\frac{1}{2}y(0)) + \frac{1}{3}y(1) = \frac{1}{3}(0 + y(0) + y(1))$. $\hat{x}(3) \underset{n=2}{=} \frac{1}{4}(0 + y(0) + y(1) + y(2))$.

EECS 564 HANDOUT WINTER 1999

REVIEW OF CONTINUOUS-TIME LINEAR SYSTEMS FOR THE KALMAN-BUCY FILTER

REVIEW OF LINEAR SYSTEM TH.

$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t)$. VAN TREES p. 516-538

DEFINE STATE TRANSITION MATRIX $\Phi(t,s)$ BY $\underline{x}(t) = \Phi(t,s)\underline{x}(s)$ FOR $\underline{u}(t)=0$. ZIR.

PROPERTIES: HOW TO FIND
(1) $\Phi(t,t) = I \quad (\underline{x}(t) = \Phi(t,t)\underline{x}(t))$
(2) $\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s) \quad (\dot{x}(t) = A(t)x(t))$

SEMIGROUP PROP. (3) $\Phi(t,s) = \Phi(t,r)\Phi(r,s)$ (IMMEDIATE FROM $\underline{x}(t) = \Phi(t,s)\underline{x}(s)$)

VARIATION OF CONSTANTS: $\underline{x}(t) = \underbrace{\Phi(t,s)\underline{x}(s)}_{\text{ZIR: ZERO INPUT RESPONSE}} + \underbrace{\int_s^t \Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau}_{\text{ZSR: ZERO STATE RESPONSE}}$

IF A CONSTANT: $\Phi(t,s) = e^{A(t-s)} = I + A(t-s) + \frac{1}{2!}A^2(t-s)^2 + \cdots = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$ MAY BE ABLE TO EVALUATE

IF A NILPOTENT (ALL EIGVALS 0): $A^n = 0$, WHERE A IS $n\times n$. PROOF: [CHAR. EQN: $\lambda^n = 0$; USE CAYLEY-HAMILTON]

EX: $A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$. $\Phi(t,s) = e^{A(t-s)} = I + \begin{bmatrix}0&1\\0&0\end{bmatrix}(t-s) + \frac{1}{2!}\underbrace{\begin{bmatrix}0&1\\0&0\end{bmatrix}^2}_{0}(t-s)^2 + \cdots$
$= \begin{bmatrix}1&t-s\\0&1\end{bmatrix}$

RANDOM PROCESSES THROUGH STATE EQN SYSTEMS

$\underline{u}(t) \longrightarrow \boxed{\dot{\underline{x}}(t) = A\underline{x} + B\underline{u}} \longrightarrow \underline{x}(t)$ $\quad \underline{u}(t)$ IS 0-MEAN WGN: $E[\underline{u}(t)\underline{u}^T(s)] = Q(t)\delta(t-s)$.

(1) LET $\underline{m}(t) = E[\underline{x}(t)]$. THEN $\dot{\underline{m}}(t) = A\underline{m}(t)$. $\therefore$ $\underline{m}(t) = \Phi(t,s)\underline{m}(s)$.

(2) LET $P(t) = E[\underline{x}(t)\underline{x}^T(t)] = K_{\underline{x}(t)}$. THEN $\dot{P}(t) = A(t)P(t) + P(t)A(t)^T + B(t)Q(t)B(t)^T$.

(3) $K_x(t,s) = \begin{cases}\Phi(t,s)P(s), & t>s\\ P(t)\Phi(s,t)^T, & t<s \text{ (JUST TAKE TRANSPOSE)}\end{cases}$ (COMPARE TO $P(n+1) = A(n)P(n)A(n)^T + B(n)Q(n)B(n)^T$)

ALSO,
$$\boxed{P(t) = \Phi(t,s)P(s)\Phi(t,s)^T + \int_s^t \Phi(t,\tau)B(\tau)Q(\tau)B(\tau)^T\Phi(t,\tau)^T d\tau}$$

PROOFS: VAN TREES, p. 533-534

CONTROLLABILITY AND OBSERVABILITY

A SYSTEM IS CONTROLLABLE IF $\exists$ A NON-IMPULSIVE INPUT THAT CAN DRIVE THE SYSTEM FROM ANY INITIAL STATE $X(0)$ TO ANY DESIRED FINAL STATE $X(T)$ IN FINITE TIME T.

THM: A LINEAR-TIME-INVARIANT (LTI) SYSTEM IS $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ WHERE A IS $n\times n$ IS CONTROLLABLE IFF THE MATRIX $\rightarrow [B|AB|A^2B|\cdots|A^{n-1}B]$ HAS FULL RANK.

EASY TO SEE IN DISCRETE TIME: $\underline{x}(n+1) = A\underline{x}(n) + B\underline{u}(n) \rightarrow \underline{x}(n) = A^n\underline{x}(0) + \underbrace{[B|AB|\cdots|A^{n-1}B]}_{\text{IF THIS IS INVERTIBLE, SOLVE FOR MATRIX OF INPUTS.}}\begin{bmatrix}\underline{u}(n-1)\\ \vdots\\ \underline{u}(0)\end{bmatrix}$

A SYSTEM IS OBSERVABLE IF GIVEN KNOWLEDGE OF ITS INPUTS $\{\underline{u}(t)\}$ AND OUTPUTS $\{\underline{y}(t)\}$, ITS STATE $\{x(t)\}$ CAN BE RECONSTRUCTED, ALL FOR ALL t.

THM: A LTI SYSTEM IS OBSERVABLE IFF $\begin{bmatrix}C\\ \vdots\\ CA^{n-1}\end{bmatrix}$ HAS FULL RANK (DUAL OF CONTROLLABILITY).

DISCRETE-TIME KALMAN FILTER → CONT.-TIME KALMAN FILTER

TO CONNECT THESE TWO CASES, REPLACE:

DISCRETE	→	CONT.
n	→	t
$n+1$	→	$t+\Delta$ (THEN LET $\Delta\to 0$)
$B(n), H(n)$	→	$B(t)\Delta,\ H(t)$
$Q(n),\ R(n)$	→	$Q(t)/\Delta,\ R(t)/\Delta$
$A(n)$	→	$I+A(t)\Delta$

THE MAJOR DIFFERENCE BETWEEN DISCRETE & CONT. IS

DISCRETE: $\underline{x}(n+1)=\Phi(n+1,n)\underline{x}(n)=A(n)x(n)$; CONT.: $\underline{x}(t+\Delta)=\Phi(t+\Delta,t)\underline{x}(t)$

THIS IS WHY $A(n)\to I+A(t)\Delta$. $\qquad = e^{A\Delta}\underline{x}(t)=(I+A(t)\Delta)\underline{x}(t)$ FOR Δ SMALL.

CONT. TIME WHITE NOISE HAS ∞ VARIANCE. HENCE $Q(n)\to\frac{Q(t)}{\Delta}$, $R(n)\to\frac{R(t)}{\Delta}$ (UNITS! REMEMBER, VARIANCE)

DISCRETE-TIME KF EQUATIONS

$$\hat{\underline{x}}(n+1|n)=A(n)\hat{\underline{x}}(n|n-1)+A(n)P(n|n-1)H(n)^T\left(H(n)P(n)H(n)^T+R(n)\right)^{-1}\left(\underline{y}(n)-H(n)\hat{\underline{x}}(n|n-1)\right).$$

$$P(n+1|n)=AP(n|n-1)A(n)^T+B(n)Q(n)B(n)^T-A(n)P(n|n-1)H(n)^T\left(H(n)P(n)H(n)^T+R(n)\right)^{-1}H(n)P(n|n-1)A(n)^T.$$

MAKE ABOVE SUBSTITUTIONS: $O(1)$ vs. $O(\Delta)$ $\qquad O(1)$ vs. $O(1/\Delta)$

$$\hat{\underline{x}}(t+\Delta|t)=(I+A(t)\Delta)\hat{\underline{x}}(t)+(I+\cancel{A(t)\Delta})P(t)H(t)^T\left(\cancel{H(t)P(t)H(t)^T}+\frac{R(t)}{\Delta}\right)^{-1}\left(\underline{y}(t)-H(t)\hat{\underline{x}}(t)\right)$$

$$\to\frac{\hat{\underline{x}}(t+\Delta|t)-\hat{\underline{x}}(t)}{\Delta}=A(t)\hat{\underline{x}}(t)+P(t)H(t)^TR(t)^{-1}\left(\underline{y}(t)-H(t)\hat{\underline{x}}(t)\right)\quad \text{LET } \Delta\to 0:$$

$$\to\boxed{\dot{\hat{\underline{x}}}(t)=A(t)\hat{\underline{x}}(t)+P(t)H(t)^TR(t)^{-1}\left(\underline{y}(t)-H(t)\hat{\underline{x}}(t)\right)}$$

ESTIMATOR EQN. FOR CONT.-TIME KF

$$P(t+\Delta)=(I+A(t)\Delta)P(t)(I+A(t)^T\Delta)+(B(t)\Delta)\frac{Q(t)}{\Delta}(B(t)^T\Delta)$$

$$-(I+A(t)\Delta)P(t)H(t)^T\left(H(t)P(t)H(t)^T+\frac{R(t)}{\Delta}\right)^{-1}H(t)P(t)(I+A(t)^T\Delta)$$

$O(1)$ vs. $O(\Delta)$ $\qquad O(1)$ vs. $O(1/\Delta)$ $\qquad O(1)$ vs. $O(\Delta)$

$$=P(t)+A(t)P(t)\Delta+P(t)A(t)^T\Delta+A(t)P(t)A(t)^T\Delta^2+B(t)Q(t)B(t)^T\Delta-P(t)H(t)^TR(t)^{-1}H(t)P(t)\Delta$$

SUBTRACT $P(t)$, DIVIDE BY Δ, LET $\Delta\to 0$: $\qquad O(\Delta^2)$ vs. $O(\Delta)$ (ALL OTHERS)

$$\boxed{\dot{P}(t)=A(t)P(t)+P(t)A(t)^T+B(t)Q(t)B(t)^T-P(t)H(t)^TR(t)^{-1}H(t)P(t)}$$

ERROR COVARIANCE PROPAGATION FOR CONT.-TIME KF

SPECIAL CASES:

I. $H(t)=0 \leftrightarrow$ OBSERVE NOISE ONLY. $\quad\dot{\hat{\underline{x}}}=A(t)\hat{\underline{x}}(t)$, $\dot{P}=AP+PA^T+BQB^T$ (LIN. SYS. REVIEW HO)

$R(t)\to\infty \leftrightarrow R^{-1}(t)=0 \to$ SAME.

II. $Q(t)=0$, $K_{X(0)}=0$: KNOW $\underline{x}(t)$ ALL TIME! $\quad\dot{P}=AP+PA^T-PH^TR^{-1}HP$, $P(0)=0$: $P(t)=0$

III. $A(t)=B(t)=0 \leftrightarrow \underline{x}(t)$ UNKNOWN RV $\to$ $\dot{\hat{\underline{x}}}=PH^TR^{-1}(y-H\hat{\underline{x}})$ $\to \dot{\hat{\underline{x}}}=A(t)\hat{\underline{x}}(t)$.

$\dot{P}=-PH^TR^{-1}HP$. ONCE EQUALS ZERO, STOP.