EECS 564 INFORMATION FORM OF KALMAN FILTER WINTER 1999

ALTERNATIVE FORMS OF KALMAN FILTER EQNS

BASIC KF EQNS:

$$\hat{\underline{x}}(n|n) = \hat{\underline{x}}(n|n-1) + \underbrace{P(n|n-1)H^T(HP(n|n-1)H^T+R)^{-1}}(\underline{y}(n) - H\hat{\underline{x}}(n|n-1)) \quad (1)$$

$$= \hat{\underline{x}}(n|n-1) + K(n)(\underline{y}(n) - H\hat{\underline{x}}(n|n-1))$$ WHERE $K(n)$ = KALMAN GAIN.

$$P(n|n) = P(n|n-1) - P(n|n-1)H^T(HP(n|n-1)H^T+R)^{-1}HP(n|n-1) \quad (2)$$

MATRIX IDENTITIES:

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} \quad (3)$$

$$A^{-1}B(D-CA^{-1}B)^{-1} = (A-BD^{-1}C)^{-1}BD^{-1} \quad (4)$$

SETTING $A = P(n|n-1)^{-1}$, $B = H^T$, $C = R^{-1}$, $D = H$ IN (3) YIELDS

$$\underbrace{P(n|n) = P(n|n-1) - P(n|n-1)H^T(HP(n|n-1)H^T+R)^{-1}HP(n|n-1)}_{\text{USING (2)}} \underset{\text{USING (3)}}{=} (P(n|n-1)^{-1} + H^TR^{-1}H)^{-1} \quad (5)$$

OR $\underline{P(n|n)^{-1} = P(n|n-1)^{-1} + H^TR^{-1}H}$ **(6)** ALSO NOTE $\underline{P(n|n) = (I - K(n)H)P(n|n-1)}$ **(7)**

SETTING $A = P(n|n-1)^{-1}$, $B = H^T$, $C = -H$, $D = R$ IN (4) YIELDS

$$\underbrace{K(n) = P(n|n-1)H^T(HP(n|n-1)H^T+R)^{-1}}_{\text{USING (1)}} \underset{\text{USING (4)}}{=} (P(n|n-1)^{-1} + H^TR^{-1}H)^{-1}H^TR^{-1} = P(n|n)H^TR^{-1}$$

OR $\underline{K(n) = P(n|n)H^TR^{-1}}$ **(8)** COMPARE TO CONT-TIME KALMAN FILTER GAIN.

INFORMATION FORM OF KF:

PROBLEM: WHAT IF WE HAVE NO IDEA WHAT $\underline{x}(0|-1)$ IS? INITIALIZE KF WITH $P(0|-1) = [\infty]$?

IDEA: INSTEAD OF PROPAGATING $\hat{\underline{x}}(n+1|n)$ AND $P(n+1|n)$, PROPAGATE $\underline{S(n+1|n) = P(n+1|n)^{-1}}$

$$\hat{\underline{n}}(n+1|n) = P(n+1|n)^{-1}\hat{\underline{x}}(n+1|n) = S(n+1|n)\hat{\underline{x}}(n+1|n)$$

THEN INITIALIZE USING $\underline{S(0|-1) = [0]}$ AND $\underline{\hat{\underline{n}}(0|-1) = \underline{0}}$.

NOTE THAT $E[(\underline{n}(n+1) - \hat{\underline{n}}(n+1|n))(\underline{n}(n+1) - \hat{\underline{n}}(n+1|n))^T]$ = ERROR COVARIANCE MATRIX OF $\hat{\underline{n}}$

$$= P(n+1|n)^{-1}P(n+1|n)P(n+1|n)^{-1} = P(n+1|n)^{-1} = S(n+1|n).$$

ORDER UPDATES:

$$\underline{S(n|n) = S(n|n-1) + H^TR^{-1}H}$$ DIRECTLY FROM (6) $S(n|n) = P(n|n)^{-1}$, ETC.

$$\underline{\hat{\underline{n}}(n|n) = \hat{\underline{n}}(n|n-1) + H^TR^{-1}\underline{y}(n)}$$ $\hat{\underline{n}}(n|n) = P(n|n)^{-1}\hat{\underline{x}}(n|n)$, ETC.

PROOF: $\hat{\underline{x}}(n|n) = \hat{\underline{x}}(n|n-1) + K(n)(\underline{y}(n) - H\hat{\underline{x}}(n|n-1)) = (I - K(n)H)\hat{\underline{x}}(n|n-1) + K\underline{y}(n)$. PREMULT BY $P(n|n)^{-1}$:

$$\underbrace{P(n|n)^{-1}\hat{\underline{x}}(n|n)}_{\hat{\underline{n}}(n|n)} = \underbrace{P(n|n)^{-1}(I-K(n)H)P(n|n-1)}_{= I \text{ USING (7)}}\underbrace{P(n|n-1)^{-1}\hat{\underline{x}}(n|n-1)}_{\hat{\underline{n}}(n|n-1)} + \underbrace{P(n|n)^{-1}\overbrace{P(n|n)}^{\text{USING (8)}}}_{I}H^TR^{-1}\underline{y}(n). \quad \text{QED.}$$

TIME UPDATES:

DEFINE $\underline{\Sigma(n) = A^{-T}S(n|n)A^{-1}}$ (ASSUMING A IS INVERTIBLE). THEN:

$$\underline{S(n+1|n) = \Sigma - \Sigma B(Q^{-1} + B^T\Sigma B)^{-1}B^T\Sigma}$$

PROOF: $P(n+1|n) = AP(n|n)A^T + BQB^T = \Sigma^{-1} + BQB^T \rightarrow S(n+1|n) = (\Sigma^{-1} + BQB^T)^{-1}$ [APPLY (3) WITH $A = \Sigma^{-1}$, $B = B$, $C = Q$, $D = B^T$. QED.

$$\underline{\hat{\underline{n}}(n+1|n) = (I - \Sigma B(Q^{-1} + B^T\Sigma B)^{-1}B^T)A^{-T}\hat{\underline{n}}(n|n)}$$

PROOF: $\hat{\underline{x}}(n+1|n) = A\hat{\underline{x}}(n|n) \xrightarrow[\text{MULT. BY } S(n+1|n)]{} \hat{\underline{n}}(n+1|n) = S(n+1|n)AS(n|n)^{-1}\hat{\underline{n}}(n|n) \underset{\text{USING ABOVE FORMULA}}{=} (\Sigma - \Sigma B(Q^{-1}+B^T\Sigma B)^{-1}B^T\Sigma)AS(n|n)^{-1}\hat{\underline{n}}(n)$

$$\underset{\text{POSTFACTOR OUT } \Sigma}{=} (I - \Sigma B(Q^{-1}+B^T\Sigma B)^{-1}B^T)\underbrace{\Sigma AS(n|n)^{-1}}_{= A^{-T} \text{ BY DEF. OF } \Sigma}\hat{\underline{n}}(n|n). \quad \text{QED.}$$

SQUARE ROOT INFO. FILTER — DERIVE AS BEFORE.

ASSUME GIVEN A^{-T}, $Q^{-1/2}$, $R^{-1/2}$, SO DON'T NEED TO COMPUTE THEM FROM A, Q, R.

MEASUREMENT UPDATE:

DEFINE $\hat{\underline{d}}(n|n-1) = P(n|n-1)^{-1/2}\hat{\underline{x}}(n|n-1)$ HAS ERROR COVARIANCE I ALWAYS!

CONSIDER
$$\begin{bmatrix} S(n|n-1)^{1/2} & H^TR^{-1/2} \\ \hat{\underline{d}}(n|n-1)^T & \underline{y}^TR^{-1/2}\end{bmatrix}\begin{bmatrix} S(n|n-1)^{T/2} & \hat{\underline{d}}(n|n-1) \\ R^{-1/2}H & R^{-1/2}\underline{y}\end{bmatrix} = \begin{bmatrix} \underbrace{S(n|n-1)+H^TR^{-1}H}_{=S(n|n)} & \overbrace{S(n|n-1)^{1/2}\hat{\underline{d}}(n|n-1)}^{\hat{\underline{u}}(n|n-1)} + H^TR^{-1}\underline{y}(n) = S(n|n)^{1/2}\hat{\underline{d}}(n|n) \\ (\quad)^T & \text{MESS}\end{bmatrix}$$

ALSO CONSIDER
$$\begin{bmatrix} S(n|n)^{1/2} & 0 \\ \hat{\underline{d}}(n|n)^T & * \end{bmatrix}\begin{bmatrix} S(n|n)^{T/2} & \hat{\underline{d}}(n|n) \\ 0 & * \end{bmatrix} = \begin{bmatrix} S(n|n) & S(n|n)^{1/2}\hat{\underline{d}}(n|n) \\ (S(n|n)^{1/2}\hat{\underline{d}}(n|n))^T & \text{MESS}\end{bmatrix}$$

NOTE WE DON'T NEED TO SPECIFY $*$!

$$\therefore \begin{bmatrix} S(n|n-1)^{1/2} & H^TR^{-1/2} \\ \hat{\underline{d}}(n|n-1)^T & \underbrace{\underline{y}^TR^{-1/2}}_{\text{DATA}}\end{bmatrix}\begin{bmatrix}\text{ORTHON.} \\ \text{MATRIX}\end{bmatrix} = \begin{bmatrix} S(n|n)^{1/2} & 0 \\ \hat{\underline{d}}(n|n)^T & * \end{bmatrix}$$

FORCE ZERO HERE → UPDATE BOTH $S(n|n-1) \to S(n|n)$, $\hat{\underline{d}}(n|n-1) \to \hat{\underline{d}}(n|n)$ USING DATA $\underline{y}(n)$

TIME UPDATE

IN THE SAME WAY, DERIVE

$$\begin{bmatrix} Q^{-1/2} & BA^{-T}S(n|n)^{1/2} \\ 0 & A^{-T}S(n|n)^{1/2} \\ \hline 0 & \hat{\underline{d}}^T(n|n)\end{bmatrix}\begin{bmatrix}\text{ORTHON.} \\ \text{MATRIX}\end{bmatrix} = \begin{bmatrix} * & 0 \\ * & S(n+1|n)^{1/2} \\ \hline * & \hat{\underline{d}}(n+1|n)^T\end{bmatrix}$$

← ADD EXTRA ROW

FORCE ZERO HERE → UPDATE BOTH $S(n|n) \to S(n+1|n)$, $\hat{\underline{d}}(n|n) \to \hat{\underline{d}}(n+1|n)$

EXTENDED KALMAN FILTER

CONSIDER THE NONLINEAR PROBLEM:

$\underline{x}(n+1) = \underline{f}(\underline{x}(n), n) + B(n)\underline{u}(n)$ NONLINEAR STATE EQN

$\underline{y}(n) = \underline{h}(\underline{x}(n), n) + \underline{v}(n)$ NONLINEAR OBSERV. EQN

$\underline{x}(0), \underline{u}(i), \underline{v}(j)$ HAVE USUAL PROPERTIES. f AND h ARE NONLINEAR, TIME-VARYING.

IDEA: LINEARIZE EQN. AT EACH n AROUND $\hat{\underline{x}}(n|n-1)$. ALSO: DO TIME AND MEASUREMENT UPDATES SEPARATELY.

TIME UPDATE: $\hat{\underline{x}}(n+1|n) = \underline{f}(\hat{\underline{x}}(n|n), n)$ DEAD RECKONING NOW NON-LINEAR

ERROR COVARIANCE:

(1) $\hat{\underline{x}}(n+1|n) = \underline{f}(\hat{\underline{x}}(n|n), n)$

(2) $\underline{x}(n+1) = \underline{f}(\underline{x}(n), n) + B\underline{u}(n)$
$\doteq \underline{f}(\hat{\underline{x}}(n|n), n) + \frac{\partial f}{\partial x}\Big|_{\hat{\underline{x}}(n|n)} \cdot (\underline{x}(n) - \hat{\underline{x}}(n|n)) + B\underline{u}(n)$

$\underline{x}(n+1) \doteq \left(\underline{f}(\hat{\underline{x}}(n), n) - \frac{df}{dx}\Big|_{\hat{\underline{x}}} \cdot \hat{\underline{x}}(n)\right) + \frac{df}{dx}\Big|_{\hat{\underline{x}}} \cdot \underline{x} + B\underline{u}(n)$ COMPARE TO $\underline{y}(n)$ EQN.

★ CAN DERIVE (1) BY PROJECTING (2) ONTO SPAN $\{\underline{y}(0) \cdots \underline{y}(n)\}$.

SUBTRACT: (2)−(1)=(3) (3) $\underline{\varepsilon}(n+1|n) = A\underline{\varepsilon}(n|n) + B\underline{u}(n)$ WHERE $A = \frac{\partial f}{\partial x}\Big|_{\hat{\underline{x}}(n|n)}$

→ $P(n+1|n) = AP(n|n)A^T + BQB^T$

MEAS. UPDATE: NOW LINEARIZE OBSERV. EQN.:

$\underline{y}(n) = \underline{h}(\underline{x}(n), n) + \underline{v}(n) \doteq \underline{h}(\hat{\underline{x}}(n|n-1), n) + \frac{\partial h}{\partial x}\Big|_{\hat{\underline{x}}(n|n-1)} \cdot (\underline{x}(n) - \hat{\underline{x}}(n|n-1)) + \underline{v}(n)$ →

$\rightarrow \underline{y}(n) - \underbrace{\left(\underline{h}(\hat{\underline{x}}(n|n-1), n) - \frac{\partial h}{\partial x}\Big|_{\hat{\underline{x}}(n|n-1)} \cdot \hat{\underline{x}}(n|n-1)\right)}_{\text{CONSTANT, SO NO EFFECT ON COVARIANCE}} = H\underline{x}(n) + \underline{v}(n)$ WHERE $H = \frac{\partial h}{\partial x}\Big|_{\hat{\underline{x}}(n|n-1)}$ (COMPARE TO TIME UPDATE)

$\underline{w}(n) \triangleq \underline{y}(n) - \hat{\underline{y}}(n|n-1) \triangleq \underline{y}(n) - \underline{h}(\hat{\underline{x}}(n|n-1), n)$
$= H\underline{\varepsilon}(n) + \underline{v}(n)$ AS BEFORE IF USE $\hat{\underline{y}}(n|n-1) = \underline{h}(\hat{\underline{x}}(n|n-1), n)$

NOW CAN APPLY USUAL MEAS. UPDATE EQNS:

$\hat{\underline{x}}(n|n) = \hat{\underline{x}}(n|n-1) + K(n)\underbrace{(\underline{y}(n) - \underline{h}(\hat{\underline{x}}(n|n-1), n))}_{\text{INNOVATION (SEE WHY?)}}$

ERROR COVAR.: $P(n|n) = P(n|n-1) - PH^T(HPH^T+R)^{-1}HP$

HINT: $w(n) \neq v(n)$

KALMAN GAIN: $K(n) = P(n|n-1)H^T(HP(n|n-1)H^T+R)^{-1} = P(n|n)H^TR^{-1}$ (SOMETIMES EASIER)

EASIEST WAY TO SEE: INFO. FORM OF KF → $P(n|n)^{-1}\hat{\underline{x}}(n|n) = P(n|n-1)^{-1}\hat{\underline{x}}(n|n-1) + H^TR^{-1}\underline{y} \rightarrow \hat{\underline{x}}(n|n) = P(n|n)P(n|n-1)^{-1}\hat{\underline{x}}(n|n-1) + \underbrace{P(n|n)H^TR^{-1}}_{\text{MUST BE KALMAN GAIN!}}\underline{y}(n)$

CORRELATED $\underline{v}(t), \underline{u}(t)$? SEE HANDOUT, AND VAN TREES P. 570-1.

COLORED $v(t)$? AUGMENT STATE, BUT NOW $R(t)=0$! NEED A WHITE PART!

DO BY DIFFERENTIATING OBSERVATIONS OF AUGMENTED STATE (OK — NO NOISE ADDED TO THEM!) (SEE VAN TREES P. 572-3)

REVIEW OF QR FACTORIZATION

THM #1: LET A BE $n\times n$ AND NONSINGULAR (REAL-VALUED THROUGHOUT). THEN $A=QR$, WHERE Q IS ORTHOGONAL ($QQ^T=Q^TQ=I$) AND R IS UPPER TRIANGULAR ($R=$ [upper Δ]).

PROOF: LET $A=[\underline{a}_1|\underline{a}_2|\cdots|\underline{a}_n]$ WHERE $\underline{a}_i$ IS AN n-VECTOR.

GRAM-SCHMIDT ORTHOGONALIZE $\{\underline{a}_1,\underline{a}_2\ldots\underline{a}_n\}$ INTO $\{\underline{q}_1,\underline{q}_2\cdots\underline{q}_n\}$: $\underline{q}_1=c_{11}\underline{a}_1$, $\underline{q}_2=c_{22}\underline{a}_2+c_{12}\underline{a}_1$, $\underline{q}_3=c_{33}\underline{a}_3+c_{23}\underline{a}_2+c_{13}\underline{a}_1,\cdots$ WHERE c_{ij} FOUND FROM GRAM-SCHMIDT. NOTE $c_{ii}\neq 0\to \det C=\prod c_{ii}\neq 0$.

WRITE THIS AS
$$[\underline{q}_1|\underline{q}_2|\cdots|\underline{q}_n]=[\underline{a}_1|\underline{a}_2|\cdots|\underline{a}_n]\begin{bmatrix}c_{11}&c_{12}&\cdots&c_{1n}\\0&c_{22}&\cdots&c_{2n}\\0&0&\ddots&\vdots\\0&\cdots&0&c_{nn}\end{bmatrix}.$$
$Q=$ MATRIX OF ORTHONORMAL VECTORS, $A=$ GIVEN MATRIX, $C=$ UPPER Δ

FACT: IF C IS UPPER Δ, $R=C^{-1}$ IS UPPER Δ.
PROOF: $R=\frac{\text{ADJ}(C)}{\text{DET } C}$. SIMILAR RESULT FOR LOWER Δ.

$Q=AC\to A=QC^{-1}=QR$. QED.

THM #2: LET A BE $n\times p$, $p<n$ AND HAVE RANK p. THEN $A=QR$, WHERE Q IS $n\times n$ ORTHOGONAL AND R IS $n\times p$ UPPER Δ: ${}_n[A]^p={}_n[Q]^n\begin{bmatrix}\nabla\\0\end{bmatrix}$

PROOF: LET A^c BE ANY $n\times(n-p)$ MATRIX SUCH THAT ${}_n[\overset{p}{A}|\overset{n-p}{A^c}]$ IS $n\times n$ AND NONSINGULAR.
USE THM #1 TO WRITE $[A|A^c]=Q\,[\text{upper }\Delta]$. POST MULT. BY $\begin{bmatrix}I\\0\end{bmatrix}\to {}_nA={}_nQ^n\begin{bmatrix}\nabla\\0\end{bmatrix}$ (DROP FINAL $n-p$ COLUMNS). NOTE NOT UNIQUE.

THM #3: LET A BE $p\times n$, $p<n$ AND HAVE RANK p. THEN $A=LQ$, WHERE ${}_p[\overset{n}{A}]={}_p[\overset{p}{L}|\overset{n-p}{0}]\,{}_n[\overset{n}{Q}]$.

PROOF: APPLY THM #2 TO A^T, AND NOTE THAT Q ORTHOGONAL $\to Q^T$ ORTHOGONAL.

IMPORTANT POINT: GIVEN A, $\exists$ FAST, STABLE ALGORITHMS (HOUSEHOLDER TRANSFORMATION, GIVENS ROTATION) TO COMPUTE R WITHOUT EXPLICITLY FINDING THE $n\times n$ (BIG) MATRIX Q. THIS IS WHAT WE WILL NEED.

REVIEW OF MATRIX SQUARE ROOT

LET A BE REAL-VALUED, SYMMETRIC, AND POSITIVE DEFINITE THROUGHOUT.

THM #4: $\exists$ A SYMMETRIC MATRIX S SUCH THAT $SS^T=A$. $S=$ SYMMETRIC SQUARE ROOT OF A.

PROOF: LET A HAVE EIGENVALUES $\lambda_i>0$ AND EIGENVECTORS $\underline{\phi}_i$. LET $\Phi=[\underline{\phi}_1|\underline{\phi}_2|\cdots\underline{\phi}_n]$. $A=\Phi\,\text{DIAG}(\lambda_i)\,\Phi^T$. $S=\Phi\,\text{DIAG}[\sqrt{\lambda_i}]\,\Phi^T$.

THM #5: LET M_1 AND M_2 BOTH BE $n\times n$ SQUARE ROOTS OF A: $M_1M_1^T=M_2M_2^T=A$. THEN $M_1=M_2Q$, Q ORTHOGONAL.
THIS MEANS THAT THE SQUARE ROOT OF A IS UNIQUE TO WITHIN POSTMULT. BY Q, WHERE $QQ^T=I$
(COMPARE TO: SQUARE ROOT OF A POSITIVE REAL NO. IS UNIQUE TO MULT. BY Q WHERE $Q^2=1\to Q=\pm1$: $3^2=9$, $(-3)^2=9$)

PROOF: $(\text{DET } M_1)^2=(\text{DET } M_2)^2=\text{DET } A\neq0$ SO M_1, M_2 NONSINGULAR.
$M_1M_1^T=M_2M_2^T\to M_2^{-1}M_1M_1^TM_2^{-T}=I\to(M_2^{-1}M_1)(M_2^{-1}M_1)^T=I\to M_2^{-1}M_1=Q$, $QQ^T=I\to M_1=M_2Q$. QED.

THM #6: $\exists$ A LOWER TRIANGULAR MATRIX L SUCH THAT $LL^T=A$. $L=$ LOWER Δ SQUARE ROOT OF A.

PROOF: COMPUTE S IN THM #4. COMPUTE QR DECOMP. OF S. LET $L=R^T=$ LOWER Δ. $S=QR\to S^T=R^TQ^T=LQ^T=S$ (S SYMMETRIC)
$A=SS^T=(LQ^T)(LQ^T)^T=LQ^TQL^T=LL^T\to L=$ LOWER Δ SQUARE ROOT.

THM #7: L IS UNIQUE TO POSTMULT. BY DIAG$[\pm1]$.

PROOF: $(\text{DET } L)^2=\text{DET } A\neq0\to L$ NONSINGULAR. LET $L_1L_1^T=L_2L_2^T=A$. THEN $L_2^{-1}L_1=L_2^TL_1^{-T}$. BUT L_2 LOWER $\Delta\to L_2^{-1}$ LOWER $\Delta\to L_2^{-1}L_1$ LOWER Δ; L_1^T UPPER $\Delta\to L_1^{-T}$ UPPER $\Delta\to L_2^TL_1^{-T}$ UPPER Δ.
LOWER $\Delta=$ UPPER $\Delta\to$ BOTH MUST BE DIAGONAL! AND FROM THM #5, $L_2^{-1}L_1$ MUST ALSO BE ORTHOGONAL.
D ORTHOGONAL, DIAGONAL $\to D=\text{DIAG}[\pm1]$. $L_2^{-1}L_1=D\to L_1=L_2D$. QED.

THM #8: NONSQUARE SQUARE ROOTS: IF $\underset{n\times m,\ m>n}{[M_{11}\ M_{12}]}[M_{11}\ M_{12}]^T=\underset{n\times m,\ m>n}{[M_{21}\ M_{22}]}[M_{21}\ M_{22}]^T=A$, THEN $\underset{n\times m}{[M_{11}\ M_{12}]}=\underset{n\times m}{[M_{21}\ M_{22}]}\underset{m\times m}{[Q]}$

PROOF: USE THM #3 TO FIND $m\times m$ ORTHOGONAL Q_1 AND Q_2 SUCH THAT $[M_{11}\ M_{12}]=[L_1|0]Q_1$, $[M_{21}\ M_{22}]=[L_2|0]Q_2$ (L_1,L_2 $n\times n$ LOWER Δ).
THEN $[M_{11}\ M_{12}][M_{11}\ M_{12}]^T=[L_1|0]\underbrace{Q_1Q_1^T}_{I}\begin{bmatrix}L_1^T\\0^T\end{bmatrix}=L_1L_1^T=A$
AND $[M_{21}\ M_{22}][M_{21}\ M_{22}]^T=[L_2|0]\underbrace{Q_2Q_2^T}_{I}\begin{bmatrix}L_2^T\\0^T\end{bmatrix}=L_2L_2^T=A$. BY THM #7, $L_1=L_2D$, $D=\text{DIAG}(\pm1)$
(THIS EXTENDS THM #7 TO NONSQUARE SQUARE ROOTS).
SO $[M_{11}\ M_{12}]Q_1^T=[M_{21}\ M_{22}]Q_2^T\begin{bmatrix}D&0\\0&I\end{bmatrix}\to[M_{11}\ M_{12}]=[M_{21}\ M_{22}]\underbrace{Q_2^T\begin{bmatrix}D&0\\0&I\end{bmatrix}Q_1}_{Q}$. $\underbrace{\left(Q_2^T\begin{bmatrix}D&0\\0&I\end{bmatrix}Q_1\right)}_{Q}\underbrace{\left(Q_1^T\begin{bmatrix}D&0\\0&I\end{bmatrix}Q_2\right)}_{Q^T}=I$ ✓

BOTTOM LINE: GIVEN ${}_p[\overset{n}{A}]$, $\exists\ {}_p[\overset{p}{L}|\overset{n-p}{0}]$ S.T. ${}_p[\overset{n}{A}]=[\overset{p}{L}|\overset{n-p}{0}]\,[\overset{n}{Q}]_n$.
THE LOWER TRIANGULAR L IS EASILY COMPUTED FROM A USING HOUSEHOLDER OR GIVENS TRANSFORMS ("FORCING ZEROS" BY POSTMULT. BY A SERIES OF ORTHOGONAL MATRICES).
L IS UNIQUE TO POSTMULT. BY $D=\text{DIAG}[\pm1]$. ELIMINATE THIS AMBIGUITY BY MAKING ALL DIAGONAL ELEMENTS OF $L>0$.

PROBLEMS WITH PROPAGATING BASIC KF EQNS : $P(n+1|n) = APA^T + Q - APH^T(HPH^T+R)^{-1}HPA^T$

1. A LOT OF COMPUTATION : 8 MATRIX MULTS, 1 MATRIX INVERSE) $\hat{x}(n+1|n) = A\hat{\underline{x}} + APH^T(HPH^T+R)^{-1}(\underline{y} - H\hat{\underline{x}})$
2. COMPUTATION CAN BE UNSTABLE, DUE TO ROUNDOFF ERROR: $(P = P(n|n-1),\ \hat{\underline{x}} = \hat{\underline{x}}(n|n-1),\ B = I$ WLOG$)$
   P OFTEN HAS LARGE ($\sim 10^8$) CONDITION NO., SINCE SOME STATES ESTIMATED BETTER THAN OTHERS.
3. $P(n+1|n)$ MAY NOT BE POSITIVE SEMIDEFINITE (SEE BELOW), DUE TO ROUNDOFF ERROR $\longrightarrow$ NEGATIVE MEAN SQUARE ERRORS!

THE SQUARE ROOT KF (POTTER 1963, SCHMIDT 1970) IS AN ALTERNATIVE TO PROPAGATING THE ABOVE BASIC EQNS.

1. USING HOUSEHOLDER TRANSFORMS OR GIVENS ROTATIONS TO PERFORM UPDATES IS FASTER AND MORE STABLE NUMERICALLY
2. PROPAGATING $P^{1/2}$ REDUCES PROBLEMS CAUSED BY LARGE CONDITION # OF P $\left(\text{COND}\#[P^{1/2}] = \sqrt{\text{COND}\#[P]}\right)$ SINCE P IS SYMMETRIC
3. SINCE P IS (IMPLICITLY) COMPUTED AS $P = (P^{1/2})(P^{1/2})^T$, GUARANTEED $P \geq 0$.
4. AVOID MATRIX INVERSION IN BASIC EQN (DO NEED BACK SUBSTITUTION, BUT THIS IS NO WORSE THAN MATRIX MULT)

DERIVATION OF SQUARE ROOT KF (VERY SIMPLE!)

REWRITE BASIC KF EQNS AS $P_{n+1} = APA^T + Q - \tilde{k}\tilde{k}^T$ AND $\hat{\underline{x}}(n+1|n) = A\hat{\underline{x}} + \underbrace{\tilde{k}}_{\text{NORMALIZED GAIN}}\underbrace{(W^{1/2})^{-1}\cdot(\underline{y} - H\hat{\underline{x}})}_{\text{NORMALIZED INNOVATION}}$

WHERE $k = APH^T$ = KALMAN GAIN AND $\tilde{k} = k(W^{1/2})^{-T}$ = NORMALIZED KALMAN GAIN AND $W = HPH^T + R$ = COVARIANCE MATRIX OF INNOVATION VECTOR.

(NOTE THAT $\tilde{k}\tilde{k}^T = k(W^{1/2})^{-T}(W^{1/2})^{-1}k^T = k(W^{1/2}W^{T/2})^{-1}k^T = kW^{-1}k^T$ AND $\tilde{k}(W^{1/2})^{-1} = k(W^{1/2})^{-T}(W^{1/2})^{-1} = kW^{-1}$.)

$$\begin{bmatrix} R^{1/2} & HP^{1/2} & 0 \\ 0 & AP^{1/2} & Q^{1/2} \end{bmatrix}\begin{bmatrix} R^{T/2} & 0^T \\ P^{T/2}H^T & P^{T/2}A^T \\ 0^T & Q^{T/2} \end{bmatrix} \underset{\text{DEF. OF } k}{=} \begin{bmatrix} HPH^T+R & HPA^T \\ APH^T & APA^T+Q \end{bmatrix} \underset{\substack{\text{DEF. OF } k, W \text{ AND} \\ P \text{ UPDATE EQN}}}{=} \begin{bmatrix} W & k^T \\ k & (P_{n+1}+\tilde{k}\tilde{k}^T) \end{bmatrix} \underset{\text{DEF. OF } \tilde{k}}{=} \begin{bmatrix} W^{1/2} & 0 & 0 \\ \tilde{k} & P_{n+1}^{1/2} & 0 \end{bmatrix}\begin{bmatrix} W^{T/2} & \tilde{k}^T \\ 0^T & P_{n+1}^{T/2} \\ 0^T & 0^T \end{bmatrix}$$

DEF. OF $\tilde{k}$: $\tilde{k}W^{T/2} = k(W^{1/2})^{-T}W^{T/2} = k(W^{T/2})^{-1}W^{T/2} = k$

ALL SQUARE ROOTS LOWER $\triangle$

$$\therefore \begin{bmatrix} R^{1/2} & HP^{1/2} & \\ 0 & AP^{1/2} & \end{bmatrix}\begin{bmatrix} \text{ORTHOGONAL} \\ \text{MATRIX} \end{bmatrix} = \begin{bmatrix} W^{1/2} & 0 & 0 \\ \tilde{k} & P_{n+1}^{1/2} & 0 \end{bmatrix}$$

$\therefore$ USING HOUSEHOLDER XFORM OR GIVENS ROTATION, CAN UPDATE FROM $P^{1/2}$ TO $P_{n+1}^{1/2}$ AND COMPUTE $\tilde{k}$ AND $W^{1/2}$ AS WELL!

TO COMPUTE $\tilde{k}(W^{1/2})^{-1}(\underline{y} - H\hat{\underline{x}})$: SINCE $W^{1/2}$ IS LOWER TRIANGULAR SQUARE ROOT OF W, COMPUTE $(W^{1/2})^{-1}(\underline{y} - H\hat{\underline{x}})$ BY BACK SUBSTITUTION — REQUIRES $\frac{1}{2}n^2$ MULTS. AND ADDS, SAME # OPERATIONS REQUIRED TO COMPUTE $(W^{1/2})(\underline{y} - H\hat{\underline{x}})$ (NO INVERSE)!

$Q^{1/2}, R^{1/2}, P_0^{1/2}$ GIVEN AS PART OF PROBLEM.

CONDITION NUMBER

CONSIDER $\begin{bmatrix} 1 & 10^6 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1{,}000{,}001 \\ 1 \end{bmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. BUT CHANGE TO $\begin{bmatrix} 1 & 10^6 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1{,}000{,}001 \\ 1.1 \end{bmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -99{,}999 \\ 1.1 \end{bmatrix}$

DET = 1
EIGVAL = 1, 1

ONLY CHANGE: $1 \rightarrow 1.1$ HERE. BIG CHANGE!

WHY? $\begin{bmatrix} 1 & 10^6 \\ 0 & 1 \end{bmatrix}^{-1}\left[\begin{array}{c|c} 1{,}000{,}001 & 1{,}000{,}001 \\ 1 & 1.1 \end{array}\right] = \left[\begin{array}{c|c} 1 & -99{,}999 \\ 1 & 1.1 \end{array}\right]$. NUMERICAL PROBS!! E.G., DISCRETE RICATTI EQN. EACH MATRIX MULT!

$\begin{bmatrix} 1 & 0 \\ 10^6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 10^6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 10^6 \\ 10^6 & 10^{12} \end{bmatrix}$

EIGVALS: $10^{12}, 10^{-12}$. DET = 1

$\sqrt{\frac{10^{12}}{10^{-12}}} = 10^{12}$.

REGARD MATRIX AS LINEAR XFORM, VECTOR AS DIRECTION + LENGTH.
DIRECTION CHANGES NOT IMPT -- EACH ELEMENT OF $\underline{x}$ BOUNDED BY $\|\underline{x}\|$.
LENGTH CHANGES IMPT -- IF $\underline{y} = A\underline{x}$, WHAT IS DYNAMIC RANGE OF $\frac{\|y\|}{\|x\|}$?

RAYLEIGH QUOTIENT: $\frac{\underline{x}^T M \underline{x}}{\underline{x}^T\underline{x}}$ HAS $\begin{cases} \text{MAX.} \\ \text{MIN.} \end{cases}$ VALUE OF $\begin{matrix} \lambda_{max}(M) \\ \lambda_{min}(M) \end{matrix}$ AT ASSOC. EIGVEC.

$\frac{\|y\|}{\|x\|} = \sqrt{\frac{y^Ty}{x^Tx}} = \sqrt{\frac{x^T(A^TA)x}{x^Tx}}$ HAS $\begin{cases} \text{MAX.} \\ \text{MIN.} \end{cases}$ VALUE OF $\begin{matrix} \sqrt{\lambda_{max}(A^TA)} = \sigma_{MAX}. \\ \sqrt{\lambda_{min}(A^TA)} = \sigma_{MIN}. \end{matrix}$

DYNAMIC RANGE $= \frac{(\|y\|/\|x\|)_{MAX}}{(\|y\|/\|x\|)_{MIN}} = \frac{\sigma_{MAX}}{\sigma_{MIN}}$ = CONDITION # OF A.

COLORED OBSERVATION NOISE IN CONT-TIME KF : SEE VAN TREES P. 572-3
BRYSON & HO P. 405-6

EECS 564 KALMAN SMOOTHING FILTERS WINTER 1999

BASIC PROBLEM: $\begin{bmatrix} \underline{x}(n+1) = A(n)\underline{x}(n) + B(n)\underline{u}(n) & \text{STATE EQN} \\ \underline{y}(n) = H(n)\underline{x}(n) + \underline{v}(n) & \text{OBSERVATION EQN} \end{bmatrix}$

EVERYTHING 0-MEAN

DRIVING NOISE: $E[\underline{u}(i)\underline{u}(j)^T] = I\,\delta(i-j)$
(CAN INCORPORATE Q IN B MATRIX: $B \to BQ^{1/2}$)
OBSERV. NOISE: $E[\underline{v}(i)\underline{v}(j)^T] = R(i)\,\delta(i-j)$
INITIAL STATE: $E[\underline{x}(0)\underline{x}(0)^T] = \Pi_0$
$\underline{x}(0), \underline{v}(i), \underline{u}(j)$ UNCORRELATED FOR ALL i, j

SMOOTHING: GIVEN $\{\underline{y}(0), \underline{y}(1), \ldots \underline{y}(N)\}$, ESTIMATE $x(n)$ FOR ALL $0 \le n \le N$.

NOTE THIS IS NONCAUSAL. SO WHY NOT USE ∞ SMOOTHING FILTER $\frac{S_x(\omega)}{S_x(\omega) + S_v(\omega)}$? BECAUSE THIS IS IIR, (ALMOST ALWAYS) AND HAVE FINITE LENGTH DATA. ALSO, SINCE NON-STATIONARY.

PLAN: (1) IDEA OF COMPLEMENTARY MODELS (2) HAMILTONIAN SMOOTHING EQNS (2-POINT BOUNDARY VALUE PROB.) (3) DIAGONALIZE TO 2-FILTER FORMULA

COMPLEMENTARY MODELS (WEINERT & DESAI, IEEE TRANS AC, APRIL 1981)

CONSIDER THE PROBLEM OF ESTIMATING $\underline{x}$ FROM NOISY OBSERVATIONS $\underline{y} = H\underline{x} + \underline{v}$, $E[\underline{x}] = E[\underline{v}] = [0]$, $E[\underline{x}\underline{v}^T] = [0]$.

LET T = HILBERT SPACE SPANNED BY ELEMENTS OF RANDOM VECTORS $\underline{x}$ AND $\underline{v}$, AND Y = HILBERT SPACE SPANNED BY ELEMENTS OF RANDOM VECTOR $\underline{y}$.
TYPICAL MEMBER OF T HAS FORM $c_1x_1 + c_2x_2 + \ldots + c_Mx_M + d_1v_1 + \ldots + d_Lv_L$ FOR CONSTANTS c_i, d_i, L, M.
THEN COMPUTING LINEAR-LEAST-SQUARES ESTIMATE $\hat{\underline{x}}(\underline{y}) \leftrightarrow$ PROJECTING $\underline{x} \in T$ ONTO THE SUBSPACE Y.

$T = Y \oplus Y^\perp$
$\underline{x} \in T$ — PROJECT $\underline{x}$ ONTO Y — ERROR AFTER PROJECTION $\in Y^\perp$.

ANOTHER WAY TO LOOK AT THIS: $\underline{y} = H\underline{x} + \underline{v} = [H \;\; I]\begin{bmatrix}\underline{x}\\ \underline{v}\end{bmatrix}$. Q: WHY CAN'T WE GET $\underline{x}$ EXACTLY? A: BECAUSE THIS IS NOT INVERTIBLE: INFORMATION IS LOST.

IDEA: SUPPOSE WE HAD AVAILABLE A 2^{ND} OBSERVATION $\underline{z} = H_1\underline{x} + H_2\underline{v}$? SUPPOSE $\underline{z} \perp \underline{y} \leftrightarrow E[\underline{z}\underline{y}^T] = [0]$?

COMBINE INTO $\begin{bmatrix}\underline{y}\\ \underline{z}\end{bmatrix} = \begin{bmatrix} H & I \\ H_1 & H_2 \end{bmatrix}\begin{bmatrix}\underline{x}\\ \underline{v}\end{bmatrix}$.

(1) IF $\begin{bmatrix} H & I \\ H_1 & H_2 \end{bmatrix}$ NON-SINGULAR, THIS IS INVERTIBLE XFORM: NO INFORMATION LOST!
(2) $E[\underline{z}\underline{y}^T] = [0]$ IF OFF-DIAGONAL ELEMENTS OF $\begin{bmatrix} H & I \\ H_1 & H_2 \end{bmatrix}\begin{bmatrix} K_x & 0 \\ 0 & K_v \end{bmatrix}\begin{bmatrix} H^T & H_1^T \\ I & H_2^T \end{bmatrix}$ ARE $[0]$
$\to H_1K_xH^T + H_2K_v = [0] \to$ CHOOSE $H_1 = K_x^{-1}$, $H_2 = -H^TK_v^{-1}$.

$\boxed{\underline{z} = K_x^{-1}\underline{x} - H^TK_v^{-1}\underline{v}}$

WE KNOW THE FOLLOWING:
(1) KNOW $\begin{bmatrix}\underline{y}\\ \underline{z}\end{bmatrix} \leftrightarrow$ KNOW $\begin{bmatrix}\underline{x}\\ \underline{v}\end{bmatrix}$;
(2) $E[\underline{z}\underline{y}^T] = [0] \leftrightarrow \underline{z} \perp \underline{y}$;
(3) $\therefore Y^\perp$ = HILBERT SPACE SPANNED BY RANDOM VECTOR $\underline{z}$. OF COURSE, $Y^\perp$ ALSO SPANNED BY ERROR. SO WHAT GOOD IS IT? SEE BELOW...

NOTE: IF HILBERT SPACES BOTHER YOU, PUNT THEM. THEY'RE JUST FOR INTERPRETATION; NOT NECESSARY AT ALL!

NOW APPLY THESE IDEAS TO SMOOTHING PROBLEM

RECALL VARIATION OF CONSTANTS FORMULA: $\underline{x}(i) = \underbrace{\Phi(i,0)\underline{x}(0)}_{\text{ZERO INPUT RESPONSE}} + \underbrace{\sum_{j=0}^{i-1}\Phi(i,j+1)B(j)\underline{u}(j)}_{\text{ZERO STATE RESPONSE}}$, $\Phi(i,j) = \prod_{n=j}^{i-1} A(n)$ IF $i > j$

$\to \underline{y}(i) = H(i)\Phi(i,0)\underline{x}(0) + H(i)\sum_{j=0}^{i-1}\Phi(i,j+1)B(j)\underline{u}(j) + \underline{v}(i) \in T$ = HILBERT SPACE SPANNED BY RANDOM VECTORS $\{\underline{x}(0), \{\underline{u}(i)\}, \{\underline{v}(i)\}\}$.

ALSO, Y = " " " " $\{\underline{y}(i)\}$ (ANY MEMBER OF Y IS CLEARLY ALSO IN T, SO Y IS A SUBSPACE OF T).

INSTEAD OF STATES $\{\underline{x}(0), \underline{x}(1), \underline{x}(2), \ldots\}$,
USE INITIAL STATE $\underline{x}(0)$ AND INPUTS $\{\underline{u}(i)\}$ $\xrightarrow{\text{USE VARIATION OF CONSTANTS}}$ $\{\underline{x}(i)\}$
SINCE LINEAR, CAN DO SAME WITH ESTIMATES: $\hat{\underline{x}}(0), \{\hat{\underline{u}}(i)\} \to \{\hat{\underline{x}}(i)\}$.

WHY? COVARIANCE FUNC. OF $\{\underline{x}(i)\}$ = MESS, BUT COVARIANCE FUNC. OF $\{\underline{u}(i)\} = I$ (MUCH EASIER!)

NOW FOLLOW KAILATH & WAX, IEEE TRANS. AC, JUNE 1984 BUT AVOID OPERATOR NOTATION.

WRITE ABOVE EQN IN MATRIX FORM AS

$$\underbrace{\begin{bmatrix}\underline{y}(0)\\ \underline{y}(1)\\ \vdots \\ \underline{y}(N)\end{bmatrix}}_{\mathcal{Y}} = \underbrace{\begin{bmatrix} H(0) & 0 & \cdots & 0 \\ H(1)\Phi(1,0) & H(1)\Phi(1,1)B(0) & & \vdots \\ H(2)\Phi(2,0) & H(2)\Phi(2,1)B(0) & & 0 \\ \vdots & \vdots & & \\ H(N)\Phi(N,0) & H(N)\Phi(N,1)B(0) & \cdots & H(N)\Phi(N,N)B(N-1) \end{bmatrix}}_{\mathcal{H}\text{ (LOWER TRIANGULAR)}} \underbrace{\begin{bmatrix}\underline{x}(0)\\ \underline{u}(0)\\ \underline{u}(1)\\ \vdots \\ \underline{u}(N-1)\end{bmatrix}}_{\mathcal{X}} + \underbrace{\begin{bmatrix}\underline{v}(0)\\ \underline{v}(1)\\ \vdots \\ \underline{v}(N)\end{bmatrix}}_{\mathcal{V}}$$

NOTE INITIAL STATE $\underline{x}(0)$ IS REGARDED AS AN EXTRA INPUT AT TIME -1.
THIS IS WHY THERE'S "$\underline{b}$" IN $\mathcal{Z}$ VECTOR BELOW, SO TIME IN $\mathcal{Z}$ = TIME IN $\mathcal{X}$. $x(0) = u(-1)$, $b = z(-1)$.

THEN COMPLEMENTARY OBSERVATIONS $\mathcal{Z}$ ARE $\mathcal{Z} = K_{\mathcal{X}}^{-1}\mathcal{X} - \mathcal{H}^TK_{\mathcal{V}}^{-1}\mathcal{V}$. $K_{\mathcal{X}} = \begin{bmatrix} \Pi_0 & 0 \\ 0 & I \end{bmatrix}$. $K_{\mathcal{V}} = \text{DIAG}[R(i)]$

WRITE THIS EQN AS

$$\underbrace{\begin{bmatrix}\underline{b}\\ \underline{z}(0)\\ \underline{z}(1)\\ \vdots \\ \underline{z}(N-1)\end{bmatrix}}_{\mathcal{Z}} = \underbrace{\begin{bmatrix}\Pi_0^{-1}\underline{x}(0)\\ \underline{u}(0)\\ \underline{u}(1)\\ \vdots \\ \underline{u}(N-1)\end{bmatrix}}_{K_{\mathcal{X}}^{-1}\mathcal{X}} - \underbrace{\begin{bmatrix} H(0)^T & \Phi^T(1,0)H(1)^T & \cdots & \Phi^T(N,0)H(N)^T \\ 0 & B(0)^T\Phi^T(1,1)H(1)^T & \cdots & B(0)^T\Phi^T(N,1)H(N)^T \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & B(N-1)^T\Phi^T(N,N)H(N)^T \end{bmatrix}}_{\mathcal{H}^T} \underbrace{\begin{bmatrix}R(0)^{-1}\underline{y}(0)\\ R(1)^{-1}\underline{y}(1)\\ \vdots \\ R(N)^{-1}\underline{y}(N)\end{bmatrix}}_{K_{\mathcal{V}}^{-1}\mathcal{V}}$$

JUST A MATRIX TRANSPOSE, BUT COMPARING $\mathcal{H}$ AND $\mathcal{H}^T$, "DUALITY" IS SHRIEKING!

WHAT'S THIS FOR? $E[\mathcal{Z}\mathcal{Y}^T] = 0$. USED BELOW

COMPARING $\mathcal{H}^T$ TO $\mathcal{H}$, IT IS CLEAR THAT ABOVE EQN CAN BE IMPLEMENTED AS THE BACKWARD-PROPAGATING (SINCE $\mathcal{H}^T$ UPPER Δ VS. $\mathcal{H}$ LOWER Δ) **COMPLEMENTARY MODEL**

$$\boxed{\begin{aligned} \underline{\lambda}(n) &= A(n)^T \underline{\lambda}(n+1) + H(n)^T R(n)^{-1} \underline{y}(n) \quad \text{STATE EQN.} \\ \underline{z}(n) &= -B(n)^T \underline{\lambda}(n+1) + \underline{v}(n) \quad \text{OBSERVATION EQN.} \\ &\text{INITIALIZED WITH } \lambda(N+1) = 0 \quad \sim \text{ADJOINT SYSTEM} \end{aligned}}$$

(1) PROPAGATES IN DECREASING TIME n

(2) NOTE OBSERVATION NOISE AND SYSTEM DRIVING NOISE INTER-CHANGED!

(3) 1^{ST} ROW OF ABOVE EQN IS $b = \Pi_0^{-1} \underline{x}(0) - \underline{\lambda}(0)$. NOTE $E[b\, y(i)^T] = 0$.

OR: SET $n=-1$ IN OBSERVATION EQN: $b = \underline{z}(-1) = -B(-1)^T \lambda(0) + \underline{v}(-1) = -\lambda(0) + \Pi_0^{-1}\underline{x}(0)$. $B(-1) \equiv I$ FROM LAST EQN OVER

NOW COMBINE:

$$\begin{bmatrix} \text{ORIGINAL SYSTEM} \\ \underline{x}(n+1) = A(n)\underline{x}(n) + B(n)\underline{u}(n) \\ \underline{y}(n) = H(n)\underline{x}(n) + \underline{v}(n) \end{bmatrix} + \begin{bmatrix} \text{COMPLEMENTARY MODEL} \\ \underline{\lambda}(n) = A(n)^T \underline{\lambda}(n+1) + H(n)^T R(n)^{-1} \underline{y}(n) \\ \underline{z}(n) = -B(n)^T \underline{\lambda}(n+1) + \underline{v}(n) \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} I & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} x(n+1) \\ \lambda(n+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(n) \\ \lambda(n) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -H^T R^{-1} \end{bmatrix} \begin{bmatrix} u(n) \\ y(n) \end{bmatrix}$$

$$\begin{bmatrix} z(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} -B^T \lambda(n+1) \\ H\, x(n) \end{bmatrix} + \begin{bmatrix} u(n) \\ v(n) \end{bmatrix} \quad \text{SUBSTITUTE INTO}$$

$$\rightarrow \begin{bmatrix} I & -BB^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} x(n+1) \\ \lambda(n+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ H^T R^{-1} H & I \end{bmatrix} \begin{bmatrix} x(n) \\ \lambda(n) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -H^T R^{-1} \end{bmatrix} \begin{bmatrix} z(n) \\ y(n) \end{bmatrix}.$$

(DESCRIPTOR SYSTEM)

THIS GIVES THE **HAMILTONIAN SMOOTHING EQNS** (NAME FROM SYMMETRY OF EIGENVALUES)

NOW PROJECT THIS EQN. ON $\mathcal{Y} = \text{SPAN}\{\underline{y}(i)\}$:
$x(n+1) \rightarrow \hat{\underline{x}}(n+1)$; $\underline{y}(n) \in \mathcal{Y}$ SO UNCHANGED;
$\underline{\lambda}(n) \rightarrow \hat{\underline{\lambda}}(n)$; $\underline{z}(n) \in \mathcal{Y}^{\perp}$ SO $\underline{z}(n) \rightarrow \underline{0}$!
$E[z y^T] = 0 \rightarrow \hat{z}(\{\underline{y}(i)\}) = 0$.

ALSO, b UNCORRELATED WITH $\{y(i)\}$, SO $b \rightarrow 0$

$n = N \rightarrow$ 1^{st} ROW: $I \cdot \hat{\underline{x}}(N+1) - BB^T \hat{\underline{\lambda}}(N+1) = A(N)\hat{\underline{x}}(N) - \underline{0}$.
SO "$A(N)$" (UNDEFINED) CAN BE ANYTHING. WLOG $A(N) = I$.

$$\boxed{\begin{bmatrix} I & -B(n)B(n)^T \\ 0 & A(n)^T \end{bmatrix} \begin{bmatrix} \hat{\underline{x}}(n+1) \\ \hat{\underline{\lambda}}(n+1) \end{bmatrix} = \begin{bmatrix} A(n) & 0 \\ H(n)^T R(n)^{-1} H(n) & I \end{bmatrix} \begin{bmatrix} \hat{\underline{x}}(n) \\ \hat{\underline{\lambda}}(n) \end{bmatrix} - \begin{bmatrix} 0 \\ H(n)^T R(n)^{-1} \underline{y}(n) \end{bmatrix}}$$

DATA USED HERE

2-POINT BOUNDARY CONDITIONS:
(1) $\hat{\underline{\lambda}}(N+1) = \underline{0}$; (2) $\underline{0} = \Pi_0^{-1} \hat{\underline{x}}(0) - \hat{\underline{\lambda}}(0)$ (REMEMBER, NOT CAUSAL KF ANYMORE: $\hat{x}(0) = \hat{x}(0|\{\underline{y}(0) \ldots \underline{y}(N)\})$)

HOW TO SOLVE THIS? ONE WAY: (1) RUN USUAL KALMAN FILTER TO GET $\hat{\underline{x}}(N|\{\underline{y}(0) \ldots \underline{y}(N)\}) = \hat{\underline{x}}(N)$; (2) $\hat{\underline{x}}(N+1) = A(N)\hat{\underline{x}}(N)$ DEAD RECKONING;
(3) USE $\hat{\underline{x}}(N+1)$ AND $\hat{\underline{\lambda}}(N+1) = \underline{0}$ TO INITIALIZE AT $n = N+1$, AND RUN HAMILTONIAN SYSTEM BACKWARDS

NOTE HAMILTONIAN EQNS. IN DESCRIPTOR FORM ABOVE: PREMULT. BY $\begin{bmatrix} I & -BB^T \\ 0 & A^T \end{bmatrix}^{-1}$ TO PUT INTO USUAL SYSTEM FORM.

CAN PERFORM A SIMILARITY XFORM ON HAMILTONIAN EQNS TO DIAGONALIZE THEM INTO 2 DECOUPLED SYSTEMS, ONE PROPAGATING FORWARD IN TIME, ONE PROPAGATING BACKWARD IN TIME. BOTH SYSTEMS ARE KALMAN FILTERS! THIS IS MAYNE-FRASER 2-FILTER FORMULA. BUT ALGEBRA IS STRAIGHTFORWARD BUT TEDIOUS (KAILATH + LJUNG, INT. J. CONTROL, 1982 CONT. TIME CASE). A MUCH EASIER OF FOLLOWS BELOW. SCATTERING FRAMEWORK MAKES EASIER.

MAYNE-FRASER 2-FILTER FORMULA

DERIVATION: WAX-KAILATH, INT. J. CONT., 1984

LEMMA: SUPPOSE HAVE 2 NOISY OBSERVATIONS $\underline{y}_1, \underline{y}_2$ OF RANDOM VECTOR $\underline{x}$: $\begin{bmatrix} \underline{y}_1 = H_1 \underline{x} + \underline{v}_1 \\ \underline{y}_2 = H_2 \underline{x} + \underline{v}_2 \end{bmatrix}$ $\underline{v}_1 \sim N(\underline{0}, R_1)$, $\underline{v}_2 \sim N(\underline{0}, R_2)$, $\underline{x} \sim N(\underline{0}, \Pi)$; $\underline{x}, \underline{v}_1, \underline{v}_2$ UNCORRELATED

THEN $\hat{\underline{x}}(\underline{y}_1, \underline{y}_2) = P(P_1^{-1} \hat{\underline{x}}(\underline{y}_1) + P_2^{-1} \hat{\underline{x}}(\underline{y}_2))$ WITH ERROR COVAR. MATRIX $P = (P_1^{-1} + P_2^{-1} - \Pi^{-1})^{-1}$

WHERE $\hat{\underline{x}}(\underline{y}_i)$ IS ESTIMATE OF $\underline{x}$ GIVEN $\underline{y}_i$ AND P_i IS ITS ERROR COVAR. MATRIX, $i = 1, 2$. THINK OF RESISTORS IN PARALLEL.

PROOF: NOTE MATRIX IDENTITY $P - PH^T(HPH^T + R)^{-1}HP = (P^{-1} + H^T R^{-1} H)^{-1}$ ASSUMING ALL INVERSES EXIST.

NOW JUST USE KALMAN FILTER! 1^{ST}, REGARD $\underline{y}_1$ AS OBSERVATION AT $n=1$, $\underline{y}_2$ AS OBSERVATION AT $n=2$: $\boxed{A = I, B = 0}$ $\boxed{x(n+1) = \underline{x}(n)}$

(AT TIME 2 USING y_1 AND y_2)
$P = P_1 - P_1 H_2^T (H_2 P_1 H_2^T + R_2)^{-1} H_2 P_1 = (P_1^{-1} + H_2^T R_2^{-1} H_2)^{-1} \rightarrow P^{-1} = P_1^{-1} + H_2^T R_2^{-1} H_2$.

$\hat{\underline{x}}(y_1, y_2) = \hat{x}(y_1) + P_1 H_2^T (H_2 P_1 H_2^T + R_2)^{-1} (y(2) - H_2 \hat{x}(y_1)) = (I - P_1 H_2^T (H_2 P_1 H_2^T + R_2)^{-1} H_2) \hat{x}(y_1) + (\cdot) y(2)$
$= (P_1 - P_1 H_2^T (H_2 P_1 H_2^T + R_2)^{-1} H_2 P_1) P_1^{-1} \hat{x}(y_1) + (\cdot) y(2) = \underline{P \cdot P_1^{-1} \hat{x}(y_1) + (\cdot) y(2)}$.

NOW REGARD y_2 AS OBS. AT $n=1$ AND $\underline{y}_1$ AS OBS. AT $n=2$. BY SYMMETRY, $\hat{\underline{x}}(y_1, y_2) = P \cdot P_2^{-1} \hat{x}(y_2) + (\cdot) y(1)$.

NOW $\hat{x}(y_1 y_2) = PP_1^{-1}\hat{x}(y_1) + (\cdot) y(2) = (\cdot) y(1) + P \cdot P_2^{-1} \hat{x}(y_2) \stackrel{\circ}{=} P(P_1^{-1}\hat{x}(y_1) + P_2^{-1}\hat{x}(y_2))$ QED. (MUST COMBINE $\underline{y}_1$ AND $\underline{y}_2$ THE SAME WAY)

ALSO, $P_2 = \Pi - \Pi H_2^T (H_2 \Pi H_2^T + R_2)^{-1} H_2 \Pi = (\Pi^{-1} + H_2^T R_2^{-1} H_2)^{-1} \rightarrow P_2^{-1} = \Pi^{-1} + H_2^T R_2^{-1} H_2 \rightarrow \underline{P^{-1} = P_1^{-1} + P_2^{-1} - \Pi^{-1}}$ QED.

NOW REGARD $\underline{y}_1$ AS $[\underline{y}(0) \ldots \underline{y}(n)]^T$ AND $\underline{y}_2$ AS $[\underline{y}(n+1) \ldots \underline{y}(N)]^T$ AND $\underline{x}$ AS $\underline{x}(n)$. ALL NOISES UNCORRELATED, SO OK THERE.

THEN $\hat{\underline{x}}(\underline{y}_1) = \hat{\underline{x}}(n|n)$ CAN BE COMPUTED USING A KALMAN FILTER RUNNING FORWARD IN TIME, AND
$\hat{\underline{x}}(\underline{y}_2) = \hat{\underline{x}}(n|\underline{y}(n+1), \underline{y}(n+2), \ldots \underline{y}(N))$ CAN BE COMPUTED USING A KALMAN FILTER RUNNING BACKWARD IN TIME.

THEN $\boxed{\hat{\underline{x}}(n) = P(n) \Big(\underbrace{P_F^{-1}(n|n) \hat{\underline{x}}_F(n|n)}_{\text{FROM FORWARD KF}} + \underbrace{P_B^{-1}(n|n+1) \hat{\underline{x}}_B(n|n+1)}_{\text{FROM BACKWARD KF}} \Big)}$ WHERE $\boxed{P(n) = (P_F^{-1}(n|n) + P_B^{-1}(n|n+1) - \Pi_0^{-1})^{-1}}$

SMOOTHED ESTIM.

THIS IS THE MAYNE-FRASER 2-FILTER FORMULA

NOTE: IF USE INFORMATION FORM OF FORWARD AND BACKWARDS KFS, AUTOMATICALLY GET $S(n|n) = P(n|n)^{-1}$, $\hat{\lambda}(n|n) = P(n|n)^{-1} \hat{x}(n|n)$ EXACTLY WHAT IS NEEDED IN MAYNE-FRASER FORMULA!

SO WHY NOT JUST USE MAYNE-FRASER INSTEAD OF HAMILTONIAN EQNS?

BECAUSE MAYNE-FRASER REQUIRES 2 KALMAN FILTERS, WHILE HAMILTONIAN REQUIRES 1 KALMAN FILTER AND 1 SYSTEM RUNNING BACKWARDS IN TIME ($\sim 1/2$ COMPUTATION TO GENERATE SAME $\hat{x}(n)$).