0-MEAN: NOTE: ALL random variables are assumed to be 0-mean.

$\perp$: **Orthogonality Principle of Least-Squares Estim.:**
Let $\hat{x}(\{y(s)\}) = E[x|\{y(s)\}]$ be LSE of x from $\{y(s)\}$.
Let $e = x - \hat{x}$. Then $E[ey(t)] = 0$ for each value t of s.

PROOF: $E[ey(t)] = E[xy(t)] - E[E[x|\{y(s)\}]y(t)] = E_y E[xy(t)|\{y(s)\}]$
$-E[E[xy(t)|\{y(s)\}]] = 0$ using $E_y[E[xy|y]] = E_y[yE[x|y]]$.

INTERP: span$\{y(s)\}$ = $\{$all linear combs. of $y(s)\}$
= $\{$all possible *linear* estimators of $x\}$.
$\hat{x}$=projection of x on span$\{y(s)\}$.
e=projection error$\perp$span$\{y(s)\}$.

EX#1: Estimate vector x from vector y. $\hat{x} = Ay$ (linear form).
$\perp$: $E[(x - Ay)y^T] = K_{xy} - AK_y = [0] \rightarrow \hat{x} = K_{xy}K_y^{-1}y$.
Much easier than previous derivations!

EX#2: Estimate $x(t)$ from $\{y(s), T_i \leq s \leq T_f\}$.
FORM: $\hat{x}(t) = \int_{T_i}^{T_f} h(t,s)y(s)ds$ (linear; compare to $\hat{x} = Ay$).
$0 =$ $E[(x(t) - \int h(t,u)y(u)du)y(s)] = K_{xy}(t,s) - \int h(t,u)K_y(u,s)du$
Solve this integral equation over $T_i \leq t \leq T_f$ (see below).

∞: **Infinite smoothing filter:** $T_i \rightarrow -\infty, T_f \rightarrow \infty$.
$x(t), y(t)$ jointly WSS$\rightarrow$ $h(t,s) = h(t-s)$ time-invariant.
$\rightarrow$ $K_{xy}(t-s) - \int_{-\infty}^{\infty} h(t-u)K_y(u-s)du = 0$.
$\mathcal{F} \rightarrow$ $S_{xy}(\omega) = H(\omega)S_y(\omega) \rightarrow \bullet H(\omega) = S_{xy}(\omega)/S_y(\omega)\bullet$

Special case: $y(t) = x(t) + v(t)$ and $E[x(t)v(s)] = 0$
$\rightarrow$ $E[x(t)y(s)] = E[x(t)(x(s) + v(s))] = E[x(t)x(s)]$ and
$K_y(t-s) = E[(x(t) + v(t))(x(s) + v(s))] = K_x + K_v$
$\rightarrow$ $S_{xy}(\omega) = S_x(\omega)$ and $S_y(\omega) = S_x(\omega) + S_v(\omega)$. Substitute:
$\rightarrow$ $\bullet H(\omega) = S_x(\omega)/(S_x(\omega) + S_v(\omega))\bullet$ Note *noncausal.*

WANT: $\hat{x}(t|\{y(s), -\infty < s < t\}) \leftrightarrow (T_i \to -\infty, T_f = t)$ (causal).
$x(t), y(t)$ 0-mean, jointly WSS. CAN'T use ∞ smoothing.

LEMMA: $y_1 \dots y_N$ 0-mean with $E[y_i y_j] = \delta_{ij}$. $y = [y_1 \dots y_N]^T$.
THEN: $\hat{x}(\{y_1 \dots y_N\}) = \sum_{i=1}^N \hat{x}(y_i)$. Projections on $\perp$ add.

NOTE: NOT TRUE UNLESS: (1) 0-mean (2) $E[y_i y_j] = \delta_{ij}$!
PROOF: $\hat{x}(y) = K_{xy} K_y^{-1} y = [E[xy_1] \dots E[xy_N]] I y = \sum_{i=1}^N E[xy_i] y_i$.

LEMMA: $\hat{x}(t|\{y(s), -\infty < s < t\}) = \int_{-\infty}^t h(t-s) y(s) ds$ where
$h(t) = \begin{cases} K_{xy}(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$ *provided* $y(t)$ is white.

PROOF: $K_{xy}(t-s) = \int h(t-u) K_y(u-s) du = \int_{-\infty}^t h(t-u) \delta(u-s) du$
$\to$ $K_{xy}(t-s) = h(t-s)$ for $s < t$ and $h(t)$ causal. QED.

What is the transfer function for this $h(t)$?

$\mathcal{L}$: Now use Laplace: $\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt = X(s)$.
$\mathcal{F} \to \mathcal{L}$: $X(s)|_{s=j\omega} = X(j\omega) = \mathcal{F}\{x(t)\}$ (2-sided Laplace).

$X(s) = \frac{\prod(s-z_i)}{\prod(s-p_i)} = \sum \frac{a_i}{s-p_i}$ (partial fraction expansion).

$[\cdot]_+$: $[X(s)]_+ = \sum_{\{\text{RE}\, p_i < 0\}} \frac{a_i}{s-p_i}$ (sum over poles in lhp).

$X(s) = [X(s)]_+ + [X(s)]_-$=realizable+unrealizable parts.
Note $\mathcal{L}\{x(t)\} = X(s) \to \mathcal{L}\{x(t), t > 0; 0, t < 0\} = [X(s)]_+$.

EX: $\frac{-3}{s^2-s-2} = \frac{1}{s+1} - \frac{1}{s-2} \to \left[\frac{-3}{s^2-s-2}\right]_+ = \frac{1}{s+1}$. Use this later.
$\mathcal{L}^{-1}\{\frac{-3}{s^2-s-2}\} = e^{-t}, t > 0; e^{2t}, t < 0$ (2-sided exponential).
NOTE: $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = -e^{at}, t < 0; 0, t > 0$ for $a > 0$ (note signs!).

$y(t)$ white$\to$ $H(s) = [K_{xy}(s)]_+$ where $K_{xy}(s) = \mathcal{L}\{K_{xy}(t)\}$.
Use this after *prewhitening* filter$\to$ *causal* Wiener filter.

EECS 564 RATIONAL WHITENING FILTERS Winter 1999

GOAL: A filter $h(t)$ so that $w(t) = y(t) * h(t)$ is white.
$w(t)$ is *innovations* process associated with $y(t)$.
$h(t)$ must be *causal* and *causally invertible*:
$h(t)$ has *causal inverse* filter $h^{-1}(t)$; $\quad h(t) * h^{-1}(t) = \delta(t)$.

WHY? Know $\{w(s), -\infty < s < t\} \leftrightarrow$know $\{y(s), -\infty < s < t\}$.
So $\hat{x}(t|\{y(s), -\infty < s < t\}) = \hat{x}(t|\{w(s), -\infty < s < t\})$.
$h(t)$ causal+causally invertible$\leftrightarrow$ $h(t)$ *minimum phase*.

HOW? $S_y(j\omega) = \frac{\prod(j\omega - z_i)}{\prod(j\omega - p_i)} \frac{\prod(-j\omega - z_i)}{\prod(-j\omega - p_i)} = S_y(\omega^2)$. Replace $s = j\omega$:

$\mathcal{F} \to \mathcal{L}$: $S_y(s) = \frac{\prod(s - z_i)}{\prod(s - p_i)} \frac{\prod(-s - z_i)}{\prod(-s - p_i)} = S_y(-s^2)$. Watch sign of s^2.

Let $H(s) = \frac{\prod(s-p_i)}{\prod(s-z_i)}$; $\quad \prod$ taken over RE $\{z_i, p_i\} < 0$.
Then $H(s)$=(pre)whitening filter for $y(t)$:

- $h(t) = \mathcal{L}^{-1}\{H(s)\}$ causal since RE $\{z_i < 0\}$.
- $h^{-1}(t) = \mathcal{L}^{-1}\{\frac{1}{H(s)}\}$ causal since RE $\{p_i < 0\}$.

$S_y(s) = S_y^+(s) S_y^-(s)$ spectral factorization. $H(s) = \frac{1}{S_y^+(s)}$.

EX: $S_y(j\omega) = \frac{\omega^4 + 5\omega^2 + 4}{\omega^4 + 25\omega^2 + 144} = S_y(\omega^2)$. Compute the filter $h(t)$.

$$S_y(-s^2) = \frac{s^4 - 5s^2 + 4}{s^4 - 25s^2 + 144} = \frac{(s+1)(s+2)}{(s+3)(s+4)} \frac{(s-1)(s-2)}{(s-3)(s-4)} \quad (s^2 = -\omega^2)$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{(s+3)(s+4)}{(s+1)(s+2)}\right\} = \delta(t) + 6e^{-t} - 2e^{-2t} \text{ for } t \geq 0$$

using $\frac{(s+3)(s+4)}{(s+1)(s+2)} = 1 + \frac{4s+10}{(s+1)(s+2)} = 1 + \frac{6}{s+1} - \frac{2}{s+2}$.

$\sum$: $y(t) \to |h(t)| \to w(t) \to |K_{xw}(t), t > 0| \to \hat{x}(t)$.

501: $S_{xw}(\omega) = S_{xy}(\omega) H^*(\omega) \to S_{xw}(s) = S_{xy}(s) H(-s)$.

$\sum$: $H(s)\mathcal{L}\{K_{xy}(t) * h(-t), t > 0\} = \bullet \frac{1}{S_y^+(s)} \left[\frac{S_{xy}(s)}{S_y^-(s)}\right]_+ \bullet$

EECS 564 EXAMPLES OF WIENER FILTERING Winter 1999

EX#1: Observe $y(t) = x(t)+v(t)$. Want $\hat{x}(t|\{y(s), -\infty < s < t\})$.
SPECS: $E[x(t)v(s)] = 0$; $\quad S_x(j\omega) = \frac{3}{\omega^2+1}$; $\quad S_v(j\omega) = 1$.

$S_y(j\omega) = \frac{3}{\omega^2+1} + 1 = \frac{\omega^2+4}{\omega^2+1} \rightarrow S_y(s) = \frac{s^2-4}{s^2-1}$ $(\omega^2 = -s^2)$

$\rightarrow$ $S_y^+(s) = \frac{s+2}{s+1}$. $\quad S_{xy}(s) = S_x(s) = \frac{3}{1-s^2}$. From before:

$\frac{S_{xy}(s)}{S_y^-(s)} = \frac{3}{1-s^2} / \frac{2-s}{1-s} = \frac{3}{(s+1)(2-s)}$. $\left[\frac{S_{xy}(s)}{S_y^-(s)}\right]_+ = \frac{1}{s+1}$.

$\frac{1}{S_y^+(s)}\left[\frac{S_{xy}(s)}{S_y^-(s)}\right]_+ = \frac{(s+1)}{(s+2)}\frac{1}{(s+1)} = \frac{1}{s+2} \rightarrow h(t) = e^{-2t}, t > 0$.

NOTE: $S_x(j\omega)$ has poles at ± 1; Wiener filter has pole at -2!

EX#2: Find ∞ smoothing filter for EX#1. $H(\omega) = \frac{S_x(\omega)}{S_x(\omega)+S_v(\omega)}$
$= \frac{3/(\omega^2+1)}{3/(\omega^2+1)+1} = \frac{3}{\omega^2+4} \rightarrow h(t) = \frac{3}{4}e^{-2|t|}$ (noncausal).

EX#3: $y(t) = x(t) + v(t); E[x(t)v(s)] = 0; E[v(t)v(s)] = \delta(t-s)$.
Causal Wiener filter$\rightarrow$ •$H(s) = 1 - \frac{1}{S_y^+(s)}$•
if $S_x(s)$ *strictly proper* (#poles>#zeros).

PROOF: $S_{xy} = S_x = S_y - 1 \rightarrow \frac{S_{xy}}{S_y^-} = \frac{S_y-1}{S_y^-} = S_y^+ - \frac{1}{S_y^-}$.

$\frac{1}{S_y^+}\left[\frac{S_{xy}}{S_y^-}\right]_+ = \frac{1}{S_y^+}[S_y^+ - \frac{1}{S_y^-}]_+ = \frac{1}{S_y^+}(S_y^+ - 1) = 1 - \frac{1}{S_y^+}$

since S_x $\substack{\text{strictly}\\ \text{proper}}$ $\rightarrow S_y$ proper$\rightarrow S_y^-$ proper$\rightarrow \frac{1}{S_y^-}$ proper

since #poles=#zeros$\rightarrow$partial fraction=1 + $\substack{\text{strictly}\\ \text{proper}}$.

EX#1: $1 - 1/\frac{s+2}{s+1} = \frac{1}{s+2}$ checks. Recall $y(t) \rightarrow |\frac{1}{S_y^+}| \rightarrow w(t)$.

INNOV-ATIONS: $\hat{x} = y(t) - w(t) \rightarrow w(t) = y(t) - \hat{x}(t) = y(t) - \hat{y}(t)$ (here).
•Innovations=whitened $y(t)$=prediction error for $y(t)$.
Kalman filter uses this to form innovations process.

Pure Prediction of WSS Processes:

GOAL: Compute $\hat{x}(t+d|\{x(s), -\infty < s < t\})$ for "delay" $d > 0$.

SOLN: $K_{xy}(t,s) = E[x(t+d)x(s)] = K_x(t-s+d)$

$\rightarrow$ $S_{xy}(s) = S_x(s)e^{ds}$ (time advance). Substitute:

$$\frac{1}{S_y^+}\left[\frac{S_{xy}}{S_y^-}\right]_+ = \frac{1}{S_x^+}\left[\frac{S_x e^{ds}}{S_x^-}\right]_+ = \frac{1}{S_x^+}[S_x^+ e^{ds}]_+ = \frac{1}{S_x^+}\left[\sum \frac{a_i e^{ds}}{s-p_i}\right]_+$$

• $= \frac{1}{S_x^+}\sum \frac{a_i e^{dp_i}}{s-p_i}$ • using $\left[\frac{ae^{ds}}{s-p}\right]_+ = \frac{ae^{dp}}{s-p}$ for RE $p < 0$.

TIME: Initial part of $e^{pt}, t > 0$ now unrealizable.

ADVANCE Unrealizable part$\rightarrow$even more unrealizable!

EX: $S_x(\omega) = \frac{1}{\omega^2+4} \rightarrow S_y^+(s) = \frac{1}{s+2} \rightarrow (s+2)\frac{e^{-2d}}{s+2} = e^{-2d}$.

$\rightarrow$ $\hat{x}(t+d|\{x(s), -\infty < x < t\}) = e^{-2d}x(t)$ not filtered.

Use only most recent observation: 1^{st}-order Markov!

Performance of Wiener Filters:

GEN: $E[e^2] = E[e(x-\hat{x})] = E[ex] - \int h(t,s)E[ey(s)] = E[ex]$

$= E[(x-\hat{x})x] = E[x^2] - \int h(t,s)K_{xy}(t,s)ds$ (need $h(t)$).

∞: $E[e^2] = \int S_x(\omega)(1 - \frac{|S_{xy}(\omega)|^2}{S_x(\omega)S_y(\omega)})\frac{d\omega}{2\pi}$ (note correlation).

∞: $y = x + v \rightarrow E[e^2] = \int S_v(\omega)\frac{S_x(\omega)}{S_x(\omega)+S_v(\omega)}\frac{d\omega}{2\pi}$ (Parseval).

CAUSAL: $y = x + v, S_v(\omega) = \sigma^2 \rightarrow E[e^2] = \sigma^2 \int \log\left(1 + \frac{S_x(\omega)}{\sigma^2}\right)\frac{d\omega}{2\pi}$.

This is the Yovits-Jackson formula (Van Trees p. 501).

HUH? To make sense of this, try some limiting cases:

$\sigma^2 \rightarrow \infty$: $\log\left(1 + \frac{S_x(\omega)}{\sigma^2}\right) \simeq \frac{S_x(\omega)}{\sigma^2} \rightarrow E[e^2] = \int S_x(\omega)\frac{d\omega}{2\pi}$ (a priori).

$\sigma^2 \rightarrow 0$: Linear factor$\rightarrow 0$ faster than $\log \rightarrow \infty \rightarrow (E[e^2] \rightarrow 0)$.