

Engin. 100: Music Signal Processing Lab #3[1]: Computing Signal Spectra

- How do we define the spectrum of a signal?
- Why can we sample without losing information?
- How do we compute the spectrum of a signal?
- How do we use Matlab to compute the spectrum?

- Why should we bother to compute the spectrum?
(to be answered next week-we need to know how to compute spectra before we can use them!)

What is the spectrum of a signal?

- **In Engin. 100, we will only define spectra of periodic (repeating) signals. How come?**
1. Each musical note played by an instrument is a periodic signal, so this is what we care about.
 2. Definition and computation are much easier.
 3. Actually, a non-periodic signal can be viewed as a periodic signal with a VERY long period.

Fourier Series of Periodic Signals

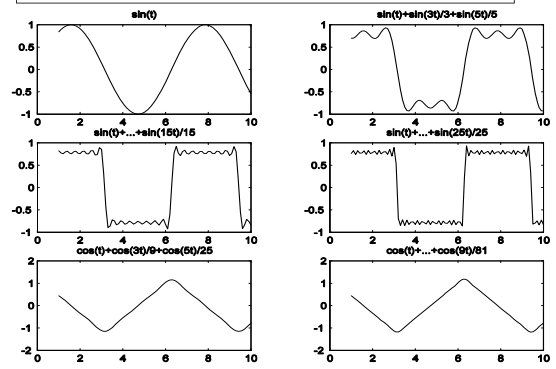
- Any real-world periodic signal with period= T (so $x(t)=x(t+T)$ for all t) can be expanded as:

$$x(t) = c_0 + c_1 \cos(2\pi t/T - \theta_1) + c_2 \cos(4\pi t/T - \theta_2) + \dots$$

constant	DC period= T	fundamental period= T	harmonic period= $T/2$
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The **spectrum** of $x(t)$ is just a stem plot of the amplitudes $\{c_k\}$ vs. the frequencies $\{k/T\}$ Hertz.

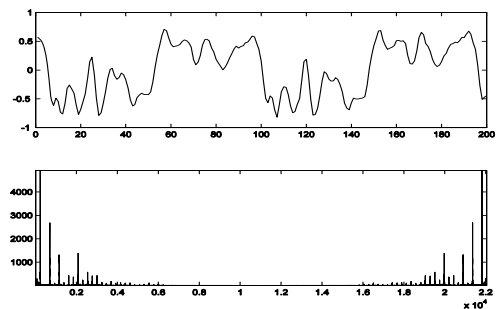
Convergence of Fourier series for square and triangle waves. More terms \rightarrow closer approximation to square or triangle wave.



Fourier Series of Musical Signals

- Phase shift θ cannot be perceived by the ear.
- No DC term in musical signals. So can use:
 $x(t) = a_1 \cos(2\pi t/T) + a_2 \cos(4\pi t/T) + a_3 \cos(6\pi t/T) + \dots$
 fundamental harmonic harmonic
- This model originally proposed by Bernoulli.
- Coefficients $a_n \rightarrow$ timbre (TAM-ber) of sound:
- Is why a clarinet sounds different from a flute.
- Plot of a_k vs. frequency k/T Hertz: sound spectrum.

Clarinet Time Waveform and Spectrum: Sampled at 22100 Hertz; note \approx 220 Hertz



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Fourier Series of Periodic Signals

- Any real-world periodic signal with period= T can be expanded in either of these two forms:
- $x(t) = c_0 + c_1 \cos(2\pi t/T - \theta_1) + c_2 \cos(4\pi t/T - \theta_2) + \dots$
- $x(t) = a_0 + a_1 \cos(2\pi t/T) + a_2 \cos(4\pi t/T) + \dots$
 $+ b_1 \sin(2\pi t/T) + b_2 \sin(4\pi t/T) + \dots$

- Coefficients in these two forms are related by:
- $A \cos(t) + B \sin(t) = C \cos(t - \theta)$ (cf. Lab #1) where $C = (A^2 + B^2)^{1/2}$; $\tan \theta = B/A$; $A = C \cos \theta$; $B = C \sin \theta$.

Example: Period=0.001 seconds

- $x(t)$ has period= $T=0.001$ sec. $x(t)$ has expansion:
- $x(t) = a_0 + a_1 \cos(2\pi 1000t) + a_2 \cos(2\pi 2000t) + \dots$
 $+ b_1 \sin(2\pi 1000t) + b_2 \sin(2\pi 2000t) + \dots$
- 0 1000 Hz (fund.) 2000 Hz (harmonic)

- In general, this expansion is infinite and requires infinitely high frequencies at integer multiples of the fundamental frequency= $1/\text{period}=1000$ Hz.
- But what if the signal is bandlimited to 8000 Hz?

**Example: Period=0.001 seconds;
Bandlimited to 8000 Hertz**

- $x(t) = a_0 + a_1 \cos(2\pi 1000t) + \dots + a_8 \cos(2\pi 8000t)$
 $+ b_1 \sin(2\pi 1000t) + \dots + b_8 \sin(2\pi 8000t)$

- Now the expansion has only finite length. In fact:
- If $x(t)$ is periodic with period=0.001 seconds and bandlimited (maximum frequency) to 8000 Hertz:
- Then it's completely characterized by 17 numbers: $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\}$.

**Sampling Signals with Period= T
seconds and Bandlimited to B Hertz**

- IDEA: If a signal is completely characterized by $2BT+1$ numbers, we can compute those $2BT+1$ numbers by sampling $2BT+1$ times per period!

- THAT IS: Sample *faster* than $2BT$ times/period.

- Sampling rate $> (2BT \text{ times}) / (T \text{ second}) = 2B$ Hertz.
So need: sampling rate $> 2(\text{maximum frequency})$.

**Sampling Signals with Period= T
seconds and Bandlimited to B Hertz**

- Sampling rate $> (2BT \text{ times}) / (T \text{ second}) = 2B$ Hertz.
Need: sampling rate $> 2(\text{maximum frequency})$.

- Note: Period= T does not matter to sampling rate!
We can make T arbitrarily large; this still holds.

- Sampling theorem: discovered by Claude Shannon, U-M CoE Class of 1936 (EE). Bust outside EECS.

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Easiest way to illustrate how to do this:

Use a small numerical example

- $x(t)$ is periodic with period= $T=0.001$ second.
- $x(t)$ has maximum frequency= $B=1000$ Hertz.
- $x(t)$ is then sampled at 4000 samples/second.
- Need: $2BT+1=2(1000)(0.001)+1=3$ numbers.

- $x(t)=a_0+a_1\cos(2\pi t/0.001)+b_1\sin(2\pi t/0.001)$.
- Sampling: Have $x(t)=x(n/4000)$ for $n=0,1,2,3$ since $x(4/4000)=x(0.001)=x(0)$ again (periodic).
- Sub in: $x(n/4000)=a_0+a_1\cos(\pi n/2)+b_1\sin(\pi n/2)$.

Small Numerical Example, Continued

- We observe
- | | | | | |
|-------------|---|---|---|---|
| $n=4000t$ | 1 | 2 | 3 | 4 |
| $x(n/4000)$ | 4 | 1 | 2 | 5 |
- these samples:
 - $x(1/4000)=a_0+a_1\cos(\pi/2)+b_1\sin(\pi/2)=a_0+b_1=4$.
 - $x(2/4000)=a_0+a_1\cos(\pi)+b_1\sin(\pi)=a_0-a_1=1$.
 - $x(3/4000)=a_0+a_1\cos(3\pi/2)+b_1\sin(3\pi/2)=a_0-b_1=2$.
 - $x(4/4000)=a_0+a_1\cos(2\pi)+b_1\sin(2\pi)=a_0+a_1=5$.

 - Collecting: $a_0+b_1=4$; $a_0-a_1=1$; $a_0-b_1=2$; $a_0+a_1=5$.
 - Solving: $a_0=3$; $a_1=2$; $b_1=1$. [4 equations in 3 unknowns]

Closed-Form Solution to the
Linear System of Equations

- General: Signal is sampled at S sample/seconds; has period= T seconds; bandlimited to B Hertz.
- Data: $\{x(1/S), x(2/S), x(3/S), \dots, x(ST/S)=x(T)\}$ (sampling every $1/S$ seconds, up to T seconds).
- Solve: $ST=N$ equations in $2BT+1$ unknowns:
- Set $t=\{1/S, 2/S, \dots, (ST)/S=T\}$ in the equation:
- $x(t)=a_0+a_1\cos(2\pi t/T)+\dots+a_{BT}\cos(2\pi BTt/T)$
- $+b_1\sin(2\pi t/T)+\dots+b_{BT}\sin(2\pi BTt/T)$

Closed-Form Solution to the
Linear System of Equations

- There is a closed-form solution (formula):
- $a_k=(2/N)\sum x(n/S)\cos(2\pi nk/N)$ where $N=ST$.
- $b_k=(2/N)\sum x(n/S)\sin(2\pi nk/N)$ where $N=ST$.
- $a_0=(1/N)\sum x(n/S)$ =average value of $x(n/S)$.
- Computed for $k=0$ to BT ($2BT+1$ numbers).
- All sums are taken from $n=1$ to $n=N=ST$.
- Where does this formula come from? See the Appendix of Lab #3 (MUCH algebra!).

Small Numerical Example, Continued

- We observe
- | | | | | |
|-------------|---|---|---|---|
| $n=4000t$ | 1 | 2 | 3 | 4 |
| $x(n/4000)$ | 4 | 1 | 2 | 5 |
- these samples:
 - $a_0=(1/4)[4+1+2+5]$ =average value of $x(t)=3$.
 - $a_1=(2/4)[4\cos(\pi/2)+1\cos(\pi)+2\cos(3\pi/2)+5\cos(2\pi)]=2$.
 - $b_1=(2/4)[4\sin(\pi/2)+1\sin(\pi)+2\sin(3\pi/2)+5\sin(2\pi)]=1$.

 - Answer: $x(t)=3+2\cos(2\pi 1000t)+1\sin(2\pi 1000t)$.
 - MUCH easier than solving ST simultaneous linear equations in $2BT+1$ unknowns if ST & $2BT$ large!

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Using Matlab to Compute Spectra

- Matlab's "F=fft(X)" computes the following:
- $fft(X) = \sum_{n=1}^N x(n-1)e^{-i2\pi(n-1)(k-1)/N}$ for $k=1, \dots, N$ where sum is from $n=1$ to N , and $e^{-ix} = \cos(x) - i \sin(x)$.
- $a_{k-1} = k^{th}$ element of " $2/N * real(fft(X))$ ", $k=1, 2, \dots$
- $b_{k-1} = k^{th}$ element of " $-2/N * imag(fft(X))$ ", $k=1, 2, \dots$
- $c_{k-1} = k^{th}$ element of " $2/N * abs(fft(X))$ ", $k=1, 2, \dots$
- X=vector of sampled signal; N=ST (integer); we need to ensure $S > 2$ (maximum frequency of x(t)).

Using Matlab's "fft" to compute spectra

- "fft" output has mirror symmetry ($e^{-i2\pi(n-1)(k-1)/N}$).
- **DELETE THE 2nd HALF OF ITS OUTPUT!**
- Watch indexing: Matlab: 1 to N. Math: 0 to N-1.
- $(2/N) * real(fft([x(0) \dots x(N-1)], N))$ gives
[$2a_0 \ a_1 \ a_2 \ \dots \ a_{N/2-2} \ a_{N/2-1} \ a_{N/2} \ a_{N/2-1} \ a_{N/2-2} \ \dots \ a_2 \ a_1$]
- $-(2/N) * imag(fft([x(0) \dots x(N-1)], N))$ gives
[$0 \ b_1 \ b_2 \ \dots \ b_{N/2-2} \ b_{N/2-1} \ 0 \ -b_{N/2-1} \ -b_{N/2-2} \ \dots \ -b_2 \ -b_1$]
- $(2/N) * abs(fft([x(0) \dots x(N-1)], N))$ gives
[$2c_0 \ c_1 \ c_2 \ \dots \ c_{N/2-2} \ c_{N/2-1} \ c_{N/2} \ c_{N/2-1} \ c_{N/2-2} \ \dots \ c_2 \ c_1$]

Small Numerical Example, Continued

x(t) observed to have the samples:

n=4000t	1	2	3	4
x(n/4000)	4	1	2	5

Reorder samples of x(t) to start with x(0)=x(4/4000)=5. Then use:

`>>(2/4)*real(fft([5 4 1 2],4))` outputs [6 2 0 2] so $a_0=6/2=3$ and $a_1=2$.

`>>-(2/4)*imag(fft([5 4 1 2],4))` outputs [0 1 0 -1] so $b_0=0$ and $b_1=1$.

`>>(2/4)*abs(fft([5 4 1 2],4))` outputs [6 2.24 0 2.24] so $c_0=6/2=3$ and $c_1=2.24=(2^2+1^2)^{1/2}$. Also note $a_2=b_2=c_2=0$ agrees with above.

Another Simple Numerical Example

- GIVEN: $x(t) = \cos(2\pi 440t)$ sampled at 8192 Hertz.
- `>>X=cos(2*pi*440*[0:8191]/8192);%T=1 second.`
- `>>F=(2/8192)*abs(fft(X,8192))` outputs all zeros except one at $F(441)=1$ and $F(7753=8193-440)=1$.
- So how do we interpret this Matlab output?
- Remember to delete the mirror image output.
- There is a single sinusoid, but what frequency?

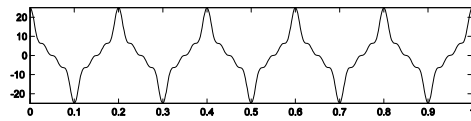
Another Numerical Example, Continued

- S=sampling rate; T=period; N=ST=length of X.
- NOTE: X is assumed by "fft" to be a single period.
- k^{th} index is frequency $(k-1)/T=S(k-1)/N$ Hertz for $k=1, 2, \dots, N/2$ (require maximum frequency $< S/2$).
- Frequency resolution (separation) is $1/T$ Hertz.
- HERE: X is samples of one sinusoid at frequency $(441-1)(8192 \text{ sample/sec.})/(8192 \text{ samples})=440 \text{ Hz}$.
- Plot only first half of F to get the spectrum of X!

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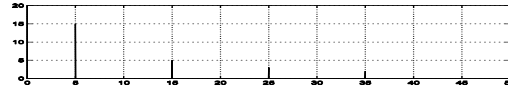
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Example: Spectrum of periodic signal



Period=0.2 second (5 complete periods occur in 1 second).
Other than that, all you can say is that this looks weird. But:

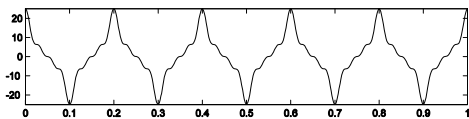
$x(t)=15\cos(10\pi t)+5\cos(30\pi t)+3\cos(50\pi t)+2\cos(70\pi t)$, $0 < t < 1$.
So the spectrum of this signal is actually quite simple:



Great! But, given this waveform, how do I compute that formula?

Periodic and Bandlimited Signals

- General: Signal periodic with period= T seconds, & bandlimited to maximum frequency= B Hertz, is completely characterized by $2BT+1$ numbers.



- If we know (somehow) that this is bandlimited to 35 Hertz, then $2(35)(0.2)+1=15$ numbers suffice. (Every other harmonic was zero in its spectrum.)

Example: Spectrum of periodic signal

- The Matlab commands I used to make this example:

```
T=0:0.01:99.99; %Sampling rate=1/0.01=100 samples/second.
Duration=99.99 seconds. Length=100/0.01=10000 samples.
```

```
X=15*cos(10*pi*T)+5*cos(30*pi*T)+3*cos(50*pi*T)+2*cos(70*pi*T);
%Generates sampled signal. Of course, we don't "know" this formula.
We want to compute the spectrum of this signal from its samples X.
```

```
FX=2/10000*abs(fft(X)); F=[0:4999]/100; plot(F,FX(1:5000))
%Computes the spectrum FX and plots it vs. actual frequency F Hz.
```

Example: Spectrum of periodic signal

- Sampling rate=100 samples/second $>2(\text{max. frequency})=2(35 \text{ Hertz})$.
- Frequency resolution (discretization)= $1/\text{duration}=1/100=0.01$ Hertz. The longer the duration T , the higher the frequency resolution $1/T$.
- If you want to be lazy, just plot($\text{abs}(\text{fft}(X))$) where X =sampled signal.
- Peak at index K is at frequency $f=(K-1)(\text{sampling rate})/(\text{length}(X))$. Here, peak at index $K=501$ is at frequency $(501-1)(100)/10000=5$ Hz.
- Peak at index $K=1501$ is at frequency $(1501-1)(100)/10000=15$ Hertz.
- Peaks at indices $K=2501$ and 3501 are at frequencies 25 and 35 Hertz.

Conclusion

- Periodic signals (including musical notes) can be expanded in a Fourier series.
- The signal is a sum of sinusoids at frequencies that are integer multiples of $1/(\text{period of signal})$.
- If signal is both bandlimited and periodic, then the Fourier series has only a finite #terms, and we can compute coefficients from its samples.
- There is a closed-form solution to the resulting linear system; Matlab can compute it easily.
- Use $f=(K-1)S/N$ to get frequencies from index.