A New Universal Random-Coding Bound for Average Probability Error Exponent for Multiple-Access Channels

Ali Nazari, Achilleas Anastasopoulos and S. Sandeep Pradhan Electrical Engineering and Computer Science Dept.

University of Michigan, Ann Arbor, MI 48109-2122, USA

E-mail: {anazari,anastas,pradhanv}@umich.edu

Abstract

In this work, a new upper bound for average error probability of a two-user discrete memoryless (DM) multiple-access channel (MAC) is derived. This bound can be universally obtained for all discrete memoryless MACs with given input and output alphabets. This is the first bound of this type that explicitly uses the method of expurgation. It is shown that the exponent of this bound is greater than or equal to those of previously known bounds.

I. INTRODUCTION

A crucial problem in network information theory is determining the average probability of error that can be achieved on a discrete memoryless multiple-access channel. More specifically, a two-user DM-MAC is defined by a stochastic matrix[1] $W : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, where the input alphabets, $\mathcal{X}$, $\mathcal{Y}$, and the output alphabet, $\mathcal{Z}$, are finite sets. The channel transition probability for sequences of length n is given by

$$W^n(\mathbf{z}|\mathbf{x},\mathbf{y}) \triangleq \prod_{i=1}^{n} W(z_i|x_i, y_i) \tag{1}$$

where

$$\mathbf{x} \triangleq (x_1, ..., x_n) \in \mathcal{X}^n, \mathbf{y} \triangleq (y_1, ..., y_n) \in \mathcal{Y}^n$$

and

$$\mathbf{z} \triangleq (z_1, ..., z_n) \in \mathcal{Z}^n.$$

This work was supported by NSF ITR grant CCF-0427385.

[1] We use the following notation throughout this work. Script capitals $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$,... denote finite, nonempty sets. To show the cardinality of a set $\mathcal{X}$, we use $|\mathcal{X}|$. We also use the letters P, Q,... for probability distributions on finite sets, and U, X, Y,... for random variables.

It has been proven, by Ahlswede [1] and Liao's [11] coding theorem, that for any (R_X, R_Y) in the interior of a certain set $\mathcal{C}$, and for all sufficiently large n, there exists a multiuser code with an arbitrary small average probability of error. Conversely, for any (R_X, R_Y) outside of $\mathcal{C}$, the average probability of error is bounded away from 0. The set $\mathcal{C}$, called *capacity region* for W, is the closure of the set of all rate pairs (R_X, R_Y) satisfying [15]

$$0 \leq R_X \leq I(X \wedge Z|Y, U) \tag{2a}$$

$$0 \leq R_Y \leq I(Y \wedge Z|X, U) \tag{2b}$$

$$0 \leq R_X + R_Y \leq I(XY \wedge Z|U), \tag{2c}$$

for all choices of joint distributions over the random variables $U,\ X,\ Y,\ Z$ of the form $p(u)p(x|u)p(y|u)W(z|x,y)$ with $U \in \mathcal{U}$ and $|\mathcal{U}| \leq 4$. As we can see, this theorem was presented in an asymptotic nature, i.e., it was proven that the error probability of the channel code can go to zero as the block length goes to infinity. Yet, it does not tell us how large the block length must be in order to achieve a specific error probability. On the other hand, in practical situations, there are limitations on the delay of the communication. Additionally, the block length of the code cannot go to infinity. Therefore, it is important to study how the probability of error drops as the block length goes to infinity. A partial answer to this question is provided by examining the error exponent of the channel.

Error exponents have been meticulously studied for discrete memoryless channels in point to point data communications. Lower and upper bounds are known on the error exponent of these channels. A lower bound, known as the random coding exponent, was developed by Fano [8]. The random coding bound in information theory provides a well-known upper bound for the probability of decoding error of the best code, of a given rate and block length. This bound is constructed by upper-bounding the average error probability over an ensemble of codes. Gallager [6] demonstrated that the random coding bound is the true error exponent for the random code ensemble. This result illustrates that the weakness of the random coding bound, at low rates, is not due to upper-bounding the ensemble average. Rather, this weakness is due to the fact that the best codes perform much better than the average, especially at low rates. Barg and Forney [2] investigated two different upper bounds on the average probability of error, called the typical random coding bound and the expurgated bound. The typical bound is basically the typical performance of the ensemble. By this, we mean that almost all random codes exhibit this performance. In addition, they have shown that the typical random code performs much better than the average performance over the random coding ensemble, at least, at low rates. The random coding exponent may be improved at low rates by a process called "expurgation" which yields a new bound that exceeds the random coding bound at low rates. It has been shown that the expurgated bound is strictly larger than both the random coding and the typical random coding bounds at low rates. It has also been demonstrated that both the expurgated and the typical random coding bounds are equal at $R = 0$. At this specific rate, the upper bound on the reliability function is also equal to these bounds [4, pg. 189].

In regard to the Multiple-Access Channels, stronger versions of Ahlswede and Liao's coding theorem, giving exponential upper and lower bounds for the error probability, have been derived by numerous other authors. Slepian and Wolf [15], Dyachkov [5], Gallager [7], Pokorny and Wallmeier [14], and Liu and Hughes [12] have all studied upper bounds on the error probability. Haroutunian [10] and Nazari [13] studied lower bounds on the error probability. The random coding bound for MAC was studied by Gallger [9], Pokorny and Wallmeier [14], and Liu and Hughes [12]. In this paper, we mostly concentrate on the result of [14] and [12]. Both of these random coding theorems are universal, i.e., a fixed choice of codewords and decoding sets achieve their upper bounds for all MACs with given input and output alphabets. In deriving both bounds, three crucial steps are observed. The first step is the choice of the ensemble. In [14], each codeword of each code in the ensemble is chosen from T_{P_X} and T_{P_Y}, for some P_X and P_Y. However, in [12] for a fixed distribution, $P_U P_{X|U} P_{Y|U}$, the codewords of each code in the ensemble are chosen from $T_{P_{X|U}}(\mathbf{u})$ and $T_{P_{Y|U}}(\mathbf{u})$ for some sequence $\mathbf{u} \in T_{P_U}$. The second step is the packing lemma, in which the existence of some particular code with certain properties is proven. The way the existence of such a code is proved is through random coding argument over the ensemble. As a side result of this step, it can be shown that most codes in the ensemble of [14] [12] have these properties. In the third step, an appropriate decoding rule is first chosen, and the performance of the code, found in the packing step, is analyzed. It has been shown that the result of Liu and Hughes is tighter than Pokorny's since they used a different ensemble and a differnet decoding rule. In this work, we follow a similar three-step approach. First, we start with an ensemble identical to [12]. Then, we provide a new packing lemma in which the resulting code has more constraints in comparison to the packing lemmas in [14] and [12]. This packing lemma is very similar to Pokorny's packing lemma, in the sense that only channel inputs appear in the packing inequalities. One of the advantages of this packing lemma, in comparison to [14], is that it enables us to partially expurgate some of the codewords and end up with a new code with stronger properties. In general, expurgation has not been studied in MAC, since by eliminating some of the codeword pairs, we may end up with correlated input sequences. In this work, we do not eliminate pairs of codewords. Rather, we expurgate codewords from only one of the codebooks. Finally, we analyze the performance of the expurgated code and end up with a new upper bound on the probability of error.

This paper is organized as follows: section II introduces terminology, and section III summarizes our main results. The proofs of some of these results are given in the Appendix.

II. Preliminaries

For any alphabet $\mathcal{X}$, $\mathcal{P}(\mathcal{X})$ denotes the set of all probability distributions on $\mathcal{X}$. The *type* of a sequence $\mathbf{x} = (x_1, ..., x_n) \in \mathcal{X}^n$ is the distributions $P_{\mathbf{x}}$ on $\mathcal{X}$ defined by

$$P_{\mathbf{x}}(x) \triangleq \frac{1}{n} N(x|\mathbf{x}), \qquad x \in \mathcal{X}, \tag{3}$$

where $N(x|\mathbf{x})$ denotes the number of occurrences of x in $\mathbf{x}$. Let $\mathcal{P}_n(\mathcal{X})$ denote the set of all types in $\mathcal{X}^n$, and define the set of all sequences in $\mathcal{X}^n$ of type P as

$$T_P \triangleq \{\mathbf{x} \in \mathcal{X}^n : P_{\mathbf{x}} = P\}. \tag{4}$$

The joint type of a pair $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$ is the probability distribution $P_{\mathbf{x},\mathbf{y}}$ on $\mathcal{X} \times \mathcal{Y}$ defined by

$$P_{\mathbf{x},\mathbf{y}}(x, y) \triangleq \frac{1}{n} N(x, y|\mathbf{x}, \mathbf{y}), \qquad (x, y) \in \mathcal{X} \times \mathcal{Y}, \tag{5}$$

where $N(x,y|\mathbf{x},\mathbf{y})$ is the number of occurrences of (x,y) in $(\mathbf{x},\mathbf{y})$. The relative entropy or *Kullback-Leibler* distance between two probability distribution $P, Q \in \mathcal{P}(\mathcal{X})$ is defined as

$$D(P||Q) \triangleq \sum_{x\in\mathcal{X}} P(x)\log\frac{P(x)}{Q(x)}. \tag{6}$$

Let $\mathcal{W}(\mathcal{Y}|\mathcal{X})$ denote the set of all stochastic matrices with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. Then, given stochastic matrices $V,\ W \in \mathcal{W}(\mathcal{Y}|\mathcal{X})$, the conditional *I-divergence* is defined by

$$D(V||W|P) \triangleq \sum_{x\in\mathcal{X}} P(x)D(V(\cdot|x)||W(\cdot|x)). \tag{7}$$

Definition 1. *An (n,M,N) multi-user code for a given MAC W, is a set $\{(\mathbf{x}_i,\mathbf{y}_j,D_{ij}) : 1\leq i\leq M, 1\leq j\leq N\}$ with*

- $\mathbf{x}_i \in \mathcal{X}^n,\ \mathbf{y}_j \in \mathcal{Y}^n,\ D_{ij}\subset \mathcal{Z}^n$
- $D_{ij}\cap D_{i'j'} = \varnothing$ *for* $(i,j)\neq(i',j')$.

Definition 2. *When message (i,j) is transmitted, the conditional probability of error of the multiuser code $\mathcal{C}$ is given by*

$$e_{ij}(\mathcal{C},W) \triangleq W^n(D_{ij}^c|\mathbf{x}_i,\mathbf{y}_j).$$

The average probability of error for multiuser code, $\mathcal{C}$, is defined as

$$e(\mathcal{C},W) \triangleq \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N} e_{ij}(\mathcal{C},W). \tag{8}$$

III. MAIN RESULT

In this section, we present a new, universally achievable upper bound on the average error probability of multiple-access channel. We observe that the mutual position of the codewords plays a crucial role in determining the decoding error. Intuitively, we expect that the codewords in a “good” code must be far from each other. In accordance with the ideas of Csiszar and Korner [4], we use conditional types to quantify this statement. Basically, we shall select a prescribed number of sequences in $\mathcal{X}^n$ and $\mathcal{Y}^n$ so that the shells around each pair have small intersections with the shells around other other sequences. In general, we have two types of packing lemmas based on whether the output of the shell belongs to the channel input space or channel output space. The Packing lemma in [14] belongs to the first type, and the one in [12] belongs to the second type. All the inequalities in the first type depend only on the channel input sequences. However, in the second type, the lemma incorporates the channel output into the packing inequalities. In this work, we use the first type. In the following, we prove three packing lemmas. In lemma 1, we show that there exists a good code with some certain properties. The nature of these properties is average, in the sense that they guarantee ,on the average, the codewords in the code are far from each other. One can easily show that by using this packing lemma and an appropriate decoder, all the results of [14] and [12] can be re-derived and unified. In lemma 2, we go one step further, by proving that the code found in lemma 1 has some additional properties that are now guaranteed for all individual pairs of sequences. If we use this packing lemma in bounding the average probability of error, we will get a tighter bound, especially at low rates. One can show that most of the random codes from the ensemble have these properties. Hence, this kind of bound is called the typical random coding bound in accordance to [2]. Finally, In lemma 3, we use one of these typical codes and eliminate some of its codewords. The resulting code has all the previous properties mentioned in lemma 1 and lemma 2. In addition, this code satisfies some additional stronger constraints. In lemma 4, we show that only some of the joint type can be seen in the expurgated code. Finally, we calculate a new upper bound for the average probability of error, depending only on the properties of the set of codewords resulting from expurgation.

Lemma 1. *For every finite set* $\mathcal{U}$*,* $P_{XYU} \in \mathcal{P}_n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})$ *such that* $X - U - Y$*,* $R_X \geq 0$*,* $R_Y \geq 0$ *,* $\delta > 0$*, and* $\boldsymbol{u} \in T^n_{P_U}$*, there exists sets of codewords* $\mathcal{C}_X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{M_X}\}$ *and* $\mathcal{C}_Y = \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_{M_Y}\}$ *with* $\boldsymbol{x}_i \in T^n_{P_{X|U}}(\boldsymbol{u})$*,* $\boldsymbol{y}_j \in T^n_{P_{Y|U}}(\boldsymbol{u})$ *for all i and j,* $M_X \geq 2^{nR_X}$*, and* $M_Y \geq 2^{nR_Y}$*, such that for every joint type* $V_{UXY\tilde{X}\tilde{Y}} \in \mathcal{P}_n(\mathcal{U} \times (\mathcal{X} \times \mathcal{Y})^2)$*, whenever* $n \geq n_0(|\mathcal{U}|, |\mathcal{X}|, |\mathcal{Y}|, \delta)$*,*

$$\frac{1}{M_X M_Y} \sum_{i=1}^{M_X} \sum_{j=1}^{M_Y} 1_{T_{V_{UXY}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j) \leq 2^{-n[F(V) - 2\delta]} \tag{9}$$

$$\frac{1}{M_X M_Y} \sum_{i=1}^{M_X} \sum_{j=1}^{M_Y} \sum_{l \neq j} 1_{T_{V_{UXY\tilde{Y}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{y}_l) \leq 2^{-n[F_Y(V) - 3\delta]} \tag{10}$$

$$\frac{1}{M_X M_Y} \sum_{i=1}^{M_X} \sum_{j=1}^{M_Y} \sum_{k \neq i} 1_{T_{V_{UXY\tilde{X}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{x}_k) \leq 2^{-n[F_X(V) - 3\delta]} \tag{11}$$

$$\frac{1}{M_X M_Y} \sum_{i=1}^{M_X} \sum_{j=1}^{M_Y} \sum_{k \neq i} \sum_{l \neq j} 1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{x}_k, \boldsymbol{y}_l) \leq 2^{-n[F_{XY}(V) - 4\delta]} \tag{12}$$

where

$$F(V) \triangleq I_V(X \wedge Y|U) \tag{13}$$

$$F_X(V) \triangleq I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge Y|U) + I_V(\tilde{X} \wedge X|UY) - R_X \tag{14}$$

$$F_Y(V) \triangleq I_V(X \wedge Y|U) + I_V(X \wedge \tilde{Y}|U) + I_V(\tilde{Y} \wedge Y|UX) - R_Y \tag{15}$$

$$F_{XY}(V) \triangleq I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{X}\tilde{Y} \wedge XY|U) - R_X - R_Y \tag{16}$$

Here U*,* X*,* Y*,* $\tilde{X}$*,* $\tilde{Y}$ *denote random variables with common distribution* $V_{UXY\tilde{X}\tilde{Y}} \in \mathcal{P}_n(\mathcal{U} \times (\mathcal{X} \times \mathcal{Y})^2)$*, and* V_{UXY}*,* $V_{UXY\tilde{X}}$*, and* $V_{UXY\tilde{Y}}$ *are appropriate marginal distributions of* $V_{UXY\tilde{X}\tilde{Y}}$*.*

Proof: In this proof, we use a similar random coding argument that J. Pokorny used in [14]. The main difference is that our lemma uses a different code ensemble which results in a tighter bound. Instead of choosing our sequences from T_{P_X} and T_{P_Y}, we choose our random sequences uniformly from $T^n_{P_{X|U}}(\mathbf{u})$, and $T^n_{P_{Y|U}}(\mathbf{u})$ for a given $\mathbf{u} \in T_{P_U}$. In [12], we see a similar random code ensemble, however, their packing lemma incorporates the channel output **z** into the packing inequalities. One can easily show that, by using this packing lemma and considering the minimum equivocation decoding rule, we would end up with the random coding bound derived in [12]. ■

Lemma 2. *For every finite set $\mathcal{U}$, $P_{XYU} \in \mathcal{P}_n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})$ such that $X - U - Y$, $R_X \geq 0$, $R_Y \geq 0$, $\delta > 0$, and $\boldsymbol{u} \in T^n_{P_U}$, there exists sets of codewords $\mathcal{C}_X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{M_X}\}$ and $\mathcal{C}_Y = \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_{M_Y}\}$ with $\boldsymbol{x}_i \in T^n_{P_{X|U}}(\boldsymbol{u})$, $\boldsymbol{y}_j \in T^n_{P_{Y|U}}(\boldsymbol{u})$ for all i and j, $M_X \geq 2^{nR_X}$, and $M_Y \geq 2^{nR_Y}$, such that for every joint type $V_{UXY\tilde{X}\tilde{Y}} \in \mathcal{P}_n(\mathcal{U} \times (\mathcal{X} \times \mathcal{Y})^2)$,* (9)- (12) *are satisfied provided $n \geq n_0(|\mathcal{U}|, |\mathcal{X}|, |\mathcal{Y}|, \delta)$. Moreover, for any $1 \leq i \leq M_X$, and any $1 \leq j \leq M_Y$*

$$1_{T_{V_{UXY}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j) \leq 2^{-n[F(V) - R_X - R_Y - 2\delta]} \tag{17}$$

$$\sum_{k \neq i} 1_{T_{V_{UXY\tilde{X}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{x}_k) \leq 2^{-n[F_X(V) - R_X - R_Y - 3\delta]} \tag{18}$$

$$\sum_{l \neq j} 1_{T_{V_{UXY\tilde{Y}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{y}_l) \leq 2^{-n[F_Y(V) - R_X - R_Y - 3\delta]} \tag{19}$$

$$\sum_{k \neq i} \sum_{l \neq j} 1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{x}_k, \boldsymbol{y}_l) \leq 2^{-n[F_{XY}(V) - R_X - R_Y - 4\delta]}, \tag{20}$$

Proof: Let us use the result of lemma 1, and multiply both sides of the inequalities (9)- (12) by $M_X M_Y$. ■

Lemma 3. *For every finite set $\mathcal{U}$, $P_{XYU} \in \mathcal{P}_n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})$ such that $X - U - Y$, $R_X \geq 0$, $R_Y \geq 0$, $\delta > 0$, and $\boldsymbol{u} \in T^n_{P_U}$, there exists sets of codewords $\mathcal{C}^*_X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{M^*_X}\}$ and $\mathcal{C}^*_Y = \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_{M^*_Y}\}$ with $\boldsymbol{x}_i \in T^n_{P_{X|U}}$, $\boldsymbol{y}_j \in T^n_{P_{Y|U}}$ for all i and j, $M^*_X \geq 2^{n(R_X - \delta)}$, and $M^*_Y \geq 2^{n(R_Y - \delta)}$, such that for every joint type*

$V_{UXY\tilde{X}\tilde{Y}} \in \mathcal{P}_n(\mathcal{U} \times (\mathcal{X} \times \mathcal{Y})^2)$,

$$\frac{1}{M_X^* M_Y^*} \sum_{i=1}^{M_X^*} \sum_{j=1}^{M_Y^*} 1_{T_{V_{UXY}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j) \leq 2^{-n[F(V)-3\delta]} \tag{21}$$

$$\frac{1}{M_X^* M_Y^*} \sum_{i=1}^{M_X^*} \sum_{j=1}^{M_Y} \sum_{k \neq i} 1_{T_{V_{UXY\tilde{X}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{x}_k) \leq 2^{-n[F_X(V)-4\delta]} \tag{22}$$

$$\frac{1}{M_X^* M_Y^*} \sum_{i=1}^{M_X^*} \sum_{j=1}^{M_Y^*} \sum_{l \neq j} 1_{T_{V_{UXY\tilde{Y}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{y}_l) \leq 2^{-n[F_Y(V)-4\delta]} \tag{23}$$

$$\frac{1}{M_X^* M_Y^*} \sum_{i=1}^{M_X^*} \sum_{j=1}^{M_Y^*} \sum_{k \neq i} \sum_{l \neq j} 1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{x}_k, \boldsymbol{y}_l) \leq 2^{-n[F_{XY}(V)-5\delta]} \tag{24}$$

and for any $1 \leq i \leq M_X^*$, *and any* $1 \leq j \leq M_Y^*$

$$1_{T_{V_{UXY}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j) \leq 2^{-n[F(V)-\min\{R_X,R_Y\}-3\delta]} \tag{25}$$

$$\sum_{k \neq i} 1_{T_{V_{UXY\tilde{X}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{x}_k) \leq 2^{-n[F_X(V)-\min\{R_X,R_Y\}-4\delta]} \tag{26}$$

$$\sum_{l \neq j} 1_{T_{V_{UXY\tilde{Y}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{y}_l) \leq 2^{-n[F_Y(V)-\min\{R_X,R_Y\}-4\delta]} \tag{27}$$

$$\sum_{k \neq i} \sum_{l \neq j} 1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\boldsymbol{u}, \boldsymbol{x}_i, \boldsymbol{y}_j, \boldsymbol{x}_k, \boldsymbol{y}_l) \leq 2^{-n[F_{XY}(V)-\min\{R_X,R_Y\}\min\{R_X,R_Y\}-5\delta]}, \tag{28}$$

whenever

$$n \geq n_0(|\mathcal{U}|, |\mathcal{X}|, |\mathcal{Y}|, \delta)$$

where $F(V), F_X(V), F_Y(V), F_{XY}(V)$ *are defined in* (13)-(16).

Proof: Let $\mathcal{C}_X = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{M_X}\}$ and $\mathcal{C}_Y = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_{M_Y}\}$ be the collections of codewords whose existence is asserted in lemma 1. From lemma 1, the codewords satisfy

$$\frac{1}{M_Y} \sum_{j=1}^{M_Y} \frac{1}{M_X} \sum_{i=1}^{M_X} 1_{T_{V_{UXY}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j) \leq 2^{-n[F(V)-2\delta]}. \tag{29}$$

Therefore, there exist $M_Y^1 \geq \frac{M_Y}{2}$ codewords in $\mathcal{C}_Y$ that satisfy

$$\frac{1}{M_X} \sum_{i=1}^{M_X} 1_{T_{V_{UXY}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j) \leq 2^{-n[F(V)-2\delta]} \times 2. \tag{30}$$

Let us call this set of codewords $\mathcal{C}_Y^1$. By multiplying both sides of (30) with M_X, and considering the fact that all terms in the summation are nonnegative, it can be concluded that for every $\mathbf{x}_i \in \mathcal{C}_X$, $\mathbf{y}_j \in \mathcal{C}_Y^1$,

$$1_{T_{V_{UXY}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j) \leq 2^{-n[F(V)-2\delta-R_X]} \times 2. \tag{31}$$

We can make a similar argument and conclude that there exists a subset of $\mathcal{C}_X$, called $\mathcal{C}_X^1$, with $M_X^1 \geq \frac{M_X}{2}$ codewords such that for any $\mathbf{x}_i \in \mathcal{C}_X^1$, $\mathbf{y}_j \in \mathcal{C}_Y$

$$1_{T_{V_{UXY}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j) \leq 2^{-n[F(V)-2\delta-R_Y]} \times 2. \tag{32}$$

Without loss of generality, let us assume $R_X < R_Y$. In this case, (31) will end up with a tighter result. Using (10), we conclude that

$$\frac{1}{M_X M_Y} \sum_{\substack{i \in \mathcal{C}_X \\ j \in \mathcal{C}_Y^1}} \sum_{l \neq j} 1_{T_{V_{UXY\tilde{Y}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_l) \leq 2^{-n[F_Y(V)-3\delta]}.$$

Since $M_Y^1 \geq \frac{M_Y}{2}$,

$$\frac{1}{M_X M_Y^1} \sum_{\substack{i \in \mathcal{C}_X \\ j \in \mathcal{C}_Y^1}} \sum_{l \neq j} 1_{T_{V_{UXY\tilde{Y}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_l) \leq 2^{-n[F_Y(V)-3\delta]} \times 2,$$

again, by a similar argument, there exists $M_Y^2 \geq \frac{M_Y^1}{2}$ codewords, $\mathbf{y}_j$, in $\mathcal{C}_Y^1$ such that

$$\begin{aligned} \frac{1}{M_X} \sum_{i=1}^{M_X} \sum_{l \neq j} & 1_{T_{V_{UXY\tilde{Y}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_l) \\ & \leq 2^{-n[F_Y(V)-3\delta]} \times 4. \end{aligned} \tag{33}$$

Let us call this subset of $\mathcal{C}_Y^1$ as $\mathcal{C}_Y^2$. Therefore, for any $\mathbf{x}_i \in \mathcal{C}_X$, $\mathbf{y}_j \in \mathcal{C}_Y^2$,

$$\sum_{l \neq j} 1_{T_{V_{UXY\tilde{Y}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_l) \leq 2^{-n[F_Y(V)-3\delta-R_X]} \times 4. \tag{34}$$

By using (11), we can conclude that

$$\frac{1}{M_X M_Y} \sum_{\substack{i \in \mathcal{C}_X \\ j \in \mathcal{C}_Y^2}} \sum_{k \neq i} 1_{T_{V_{UXY\tilde{X}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k) \leq 2^{-n[F_X(V)-3\delta]}$$

Considering the fact that $M_Y^2 \geq \frac{M_Y^1}{2} \geq \frac{M_Y}{4}$, we can conclude that

$$\frac{1}{M_X M_Y^2} \sum_{\substack{i \in \mathcal{C}_X \\ j \in \mathcal{C}_Y^2}} \sum_{k \neq i} 1_{T_{V_{UXY\tilde{X}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k) \leq 2^{-n[F_X(V)-3\delta]} \times 4$$

Hence, there exists $C_Y^3 \subset C_Y^2$, with $M_Y^3 \geq \frac{M_Y^2}{2}$ codewords such that

$$\begin{aligned} \frac{1}{M_X} \sum_{i \in \mathcal{C}_X} \sum_{k \neq i} & 1_{T_{V_{UXY\tilde{X}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k) \\ & \leq 2^{-n[F_X(V)-3\delta]} \times 8 \end{aligned} \tag{35}$$

Therefore, for all $\mathbf{x}_i \in \mathcal{C}_X$, $\mathbf{y}_j \in \mathcal{C}_Y^3$,

$$\sum_{k \neq i} 1_{T_{V_{UXY\tilde{X}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k) \leq 2^{-n[F_X(V)-R_X-3\delta]} \times 8. \tag{36}$$

Similarly, by using (12), we can conclude that

$$\frac{1}{M_X M_Y}\sum_{\substack{i\in\mathcal{C}_X\\ j\in\mathcal{C}_Y^3}}\sum_{k\neq i}\sum_{l\neq j}1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{y}_l)$$

$$\leq 2^{-n[F_{XY}(V)-3\delta]}. \tag{37}$$

By a similar argument and using the fact that $M_Y^3 \geq \frac{M_Y^2}{2} \geq \frac{M_Y^1}{4} \geq \frac{M_Y}{8}$, we conclude that

$$\frac{1}{M_X M_Y^3}\sum_{\substack{i\in\mathcal{C}_X\\ j\in\mathcal{C}_Y^3}}\sum_{k\neq i}\sum_{l\neq j}1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{y}_l)$$

$$\leq 2^{-n[F_{XY}(V)-3\delta]}\times 8. \tag{38}$$

Therefore, there exist $\mathcal{C}_Y^4 \subset \mathcal{C}_Y^3$, with $M_Y^4 \geq \frac{M_Y^3}{2}$ codewords, such that

$$\frac{1}{M_X}\sum_{i\in\mathcal{C}_X}\sum_{k\neq i}\sum_{l\neq j}1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{y}_l)$$

$$\leq 2^{-n[F_{XY}(V)-3\delta]}\times 16. \tag{39}$$

Similarly, since $M_Y^4 \geq \frac{M_Y^3}{2} \geq \frac{M_Y^2}{4} \geq \frac{M_Y^1}{8} \geq \frac{M_Y}{16}$, we conclude that for all $\mathbf{x}_i \in \mathcal{C}_X$, $\mathbf{y}_j \in \mathcal{C}_Y^4$,

$$\sum_{k\neq i}\sum_{l\neq j}1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{y}_l)$$

$$\leq 2^{-n[F_{XY}(V)-R_X-3\delta]}\times 16. \tag{40}$$

Since $\mathcal{C}_Y^4 \subset \mathcal{C}_Y^3 \subset \mathcal{C}_Y^2 \subset \mathcal{C}_Y^1$, any codeword belonging to C_Y^4 has all the properties we derived in (31), (34), (36), (40). Therefore, we have proven that there exists a codebook $\mathcal{C}_Y^4 \subset C_Y$ with $M_Y^4 \geq \frac{M_Y}{16}$ codewords such that for any $\mathbf{x}_i \in \mathcal{C}_X$, $\mathbf{y}_j \in \mathcal{C}_Y^4$, we have the properties (31), (34), (36), (40). As shown, we have eliminated some of the codewords from $\mathcal{C}_Y$. Similarly, we can do the expurgation on $\mathcal{C}_X$. If $R_X < R_Y$, the expurgation on $\mathcal{C}_Y$ results in a tighter result. However, if $R_X > R_Y$, the expurgation on $\mathcal{C}_X$ would end up with a tighter bound. Thus, in general, there exists a pair of codebooks ($\mathcal{C}_X^*$, $\mathcal{C}_Y^*$), with $|\mathcal{C}_X^*||\mathcal{C}_Y^*| \geq \frac{|\mathcal{C}_X||\mathcal{C}_Y|}{16}$, such that for any $\mathbf{x}_i \in \mathcal{C}_X^*$, $\mathbf{y}_j \in \mathcal{C}_Y^*$,

$$1_{T_{V_{UXY}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j) \leq 2^{-n[F(V)-\min\{R_X,R_Y\}-3\delta]} \tag{41}$$

$$\sum_{k\neq i}1_{T_{V_{UXY\tilde{X}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k) \leq 2^{-n[F_X(V)-\min\{R_X,R_Y\}-4\delta]} \tag{42}$$

$$\sum_{l\neq j}1_{T_{V_{UXY\tilde{Y}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{y}_l) \leq 2^{-n[F_Y(V)-\min\{R_X,R_Y\}-4\delta]} \tag{43}$$

$$\sum_{k\neq i}\sum_{l\neq j}1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{y}_l)$$

$$\leq 2^{-n[F_{XY}(V)-\min\{R_X,R_Y\}\min\{R_X,R_Y\}-5\delta]} \tag{44}$$

The only difference between the exponents in (25)-(28) and the ones in (9)-(12) is $\min\{R_X,R_Y\}$. Despite of the (9)-(12) which are upper bounds for some quantities averaged over all pairs of sequences belonging to $(\mathcal{C}_X,\mathcal{C}_Y)$, the results in (25)-(28) are valid for all pairs of codewords in ($\mathcal{C}_X^*$,$\mathcal{C}_Y^*$). Let us define $M_X^* \triangleq |\mathcal{C}_X^*|$, $M_Y^* \triangleq |\mathcal{C}_Y^*|$. In the following, we will show that the new codebook pair, ($\mathcal{C}_X^*$,$\mathcal{C}_Y^*$), still satisfies the same average performance bound we obtained for the original codebook pair, $(\mathcal{C}_X, \mathcal{C}_Y)$. The functions in (9)-(12) for the new codebook pair can be upperbounded as follows,

$$\begin{aligned}
&\frac{1}{M_X^* M_Y^*}\sum_{i=1}^{M_X^*}\sum_{j=1}^{M_Y^*} 1_{T_{V_{UXY}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j)\\
\leq\;& \frac{1}{M_X^* M_Y^*}\sum_{i=1}^{M_X}\sum_{j=1}^{M_Y} 1_{T_{V_{UXY}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j)\\
\leq\;& \frac{16}{M_X M_Y}\sum_{i=1}^{M_X}\sum_{j=1}^{M_Y} 1_{T_{V_{UXY}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j)\\
\leq\;& 16 * 2^{-n[F(V)-3\delta]} \leq 2^{-n[F(V)-2\delta]}.
\end{aligned} \tag{45}$$

We can use a similar argument and show that

$$\frac{1}{M_X^* M_Y^*}\sum_{i=1}^{M_X^*}\sum_{j=1}^{M_Y^*}\sum_{k\neq i}\sum_{l\neq j} 1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{y}_l) \leq 2^{-n[F_{XY}(V)-4\delta]} \tag{46}$$

$$\frac{1}{M_X^* M_Y^*}\sum_{i=1}^{M_X^*}\sum_{j=1}^{M_Y^*}\sum_{l\neq j} 1_{T_{V_{UXY\tilde{Y}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{y}_l) \leq 2^{-n[F_{Y}(V)-4\delta]} \tag{47}$$

$$\frac{1}{M_X^* M_Y^*}\sum_{i=1}^{M_X^*}\sum_{j=1}^{M_Y^*}\sum_{k\neq i} 1_{T_{V_{UXY\tilde{X}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k) \leq 2^{-n[F_{X}(V)-5\delta]} \tag{48}$$

Here, by method of expurgation, we end up with a code with a similar average bound as we had for the original code. However, all pairs of codewords in the new code also satisfy (25)-(28). Therefore, we did not lose anything in terms of average performance, however, as we see in theorem 1 , we would end up with a tighter random coding bound since we have more constraints on any particular pair of codewords in our codebook pair. ■

Lemma 4. *For any type $V_{UXY\tilde{X}\tilde{Y}} \in \mathcal{P}_n(\mathcal{U}\times(\mathcal{X}\times\mathcal{Y})^2)$ such that for some $\boldsymbol{x}_i, \boldsymbol{x}_k \in C_X^*$, and $\boldsymbol{y}_j, \boldsymbol{y}_l \in C_Y^*$,*

$$(\boldsymbol{u},\boldsymbol{x}_i,\boldsymbol{y}_j,\boldsymbol{x}_k,\boldsymbol{y}_l) \in T_{V_{UXY\tilde{X}\tilde{Y}}} \tag{49}$$

the following inequalities must be satisfied

$$\begin{aligned}
&V_{XU} = V_{\tilde{X}U} = P_{XU}, V_{YU} = V_{\tilde{Y}U} = P_{YU} \\
&I_V(X \wedge Y|U), I_V(X \wedge \tilde{Y}|U) \leq \min\{R_X, R_Y\} + 3\delta \\
&I_V(\tilde{X} \wedge Y|U), I_V(\tilde{X} \wedge \tilde{Y}|U) \leq \min\{R_X, R_Y\} + 3\delta \\
&I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge Y|U) + I_V(\tilde{X} \wedge X|UY) \\
&\qquad \leq R_X + \min\{R_X, R_Y\} + 4\delta \\
&I_V(X \wedge \tilde{Y}|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{X} \wedge X|U\tilde{Y}) \\
&\qquad \leq R_X + \min\{R_X, R_Y\} + 4\delta \\
&I_V(X \wedge Y|U) + I_V(X \wedge \tilde{Y}|U) + I_V(\tilde{Y} \wedge Y|UX) \\
&\qquad \leq R_Y + \min\{R_X, R_Y\} + 4\delta \\
&I_V(\tilde{X} \wedge Y|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{Y} \wedge Y|U\tilde{X}) \\
&\qquad \leq R_Y + \min\{R_X, R_Y\} + 4\delta \\
&I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{X}\tilde{Y} \wedge XY|U) \\
&\qquad \leq R_X + R_Y + \min\{R_X, R_Y\} + 5\delta \\
&I_V(\tilde{X} \wedge Y|U) + I_V(X \wedge \tilde{Y}|U) + I_V(X\tilde{Y} \wedge \tilde{X}Y|U) \\
&\qquad \leq R_X + R_Y + \min\{R_X, R_Y\} + 5\delta
\end{aligned} \tag{50}$$

Proof: Let (C_X^*, C_Y^*) be the collections of codewords whose existence is asserted in lemma 3. Consider any $\mathbf{x}_i, \mathbf{x}_k \in C_X^*$, and $\mathbf{y}_j, \mathbf{y}_l \in C_Y^*$. Let us call their joint empirical distribution of $(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_l)$ as $V_{UXY\tilde{X}\tilde{Y}}(u, x, y, \tilde{x}, \tilde{y})$. Using (41), and the fact that $(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j) \in T_{V_{UXY}}$,

$$1 \leq 2^{-n[F(V) - \min\{R_X, R_Y\} - 3\delta]} \tag{51}$$

Therefore,

$$I_V(X \wedge Y|U) \leq \min\{R_X, R_Y\} + 3\delta \tag{52}$$

Similarly, using the empirical distribution of $(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_l)$, $(\mathbf{u}, \mathbf{x}_k, \mathbf{y}_j)$, and $(\mathbf{u}, \mathbf{x}_k, \mathbf{y}_l)$, we conclude that

$$I_V(\tilde{X} \wedge Y|U) \leq \min\{R_X, R_Y\} + 3\delta \tag{53}$$

$$I_V(X \wedge \tilde{Y}|U) \leq \min\{R_X, R_Y\} + 3\delta \tag{54}$$

$$I_V(\tilde{X} \wedge \tilde{Y}|U) \leq \min\{R_X, R_Y\} + 3\delta \tag{55}$$

Since $(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k) \in T_{V_{UXY\tilde{X}}}$,

$$1 \leq \sum_{k \neq i} 1_{T_{V_{UXY\tilde{X}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k) \tag{56}$$

Using (56), and the upper bound we obtained in (42),

$$I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge Y|U) + I_V(\tilde{X} \wedge X|UY)$$
$$\leq R_X + \min\{R_X, R_Y\} + 4\delta \tag{57}$$

Similarly, since $(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_l, \mathbf{x}_k) \in T_{V_{UX\tilde{Y}\tilde{X}}}$, we conclude that

$$I_V(X \wedge \tilde{Y}|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{X} \wedge X|U\tilde{Y})$$
$$\leq R_X + \min\{R_X, R_Y\} + 4\delta \tag{58}$$

By a similar argument for the empirical distribution of $(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{y}_l)$, $(\mathbf{u}, \mathbf{x}_k, \mathbf{y}_j, \mathbf{y}_l)$, and using the upper bound we obtained in (43), the following would respectively be concluded

$$I_V(X \wedge Y|U) + I_V(X \wedge \tilde{Y}|U) + I_V(\tilde{Y} \wedge Y|UX)$$
$$\leq R_Y + \min\{R_X, R_Y\} + 4\delta \tag{59}$$
$$I_V(\tilde{X} \wedge Y|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{Y} \wedge Y|U\tilde{X})$$
$$\leq R_Y + \min\{R_X, R_Y\} + 4\delta \tag{60}$$

Finally, using the empirical distribution of $(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_l)$, $(\mathbf{u}, \mathbf{x}_k, \mathbf{y}_j, \mathbf{x}_i, \mathbf{y}_l)$, and the upper bound in (44),

$$I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{X}\tilde{Y} \wedge XY|U)$$
$$\leq R_X + R_Y + + \min\{R_X, R_Y\} + 5\delta \tag{61}$$
$$I_V(\tilde{X} \wedge Y|U) + I_V(X \wedge \tilde{Y}|U) + I_V(X\tilde{Y} \wedge \tilde{X}Y|U)$$
$$\leq R_X + R_Y + + \min\{R_X, R_Y\} + 5\delta. \tag{62}$$

■

Theorem 1. *For every finite set $\mathcal{U}$, $\mathcal{P}_{XYU} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y} \times \mathcal{U})$ such that $X - U - Y$, $R_X \geq 0$, $R_Y \geq 0$, $\delta > 0$, and $\boldsymbol{u} \in T^n_{P_U}$, there exists a multi-user code*

$$\mathcal{C} = \{(\boldsymbol{x}_i, \boldsymbol{y}_j, D_{ij}) : i = 1, ...M^*_X, j = 1, ...M^*_Y\} \tag{63}$$

with $\boldsymbol{x}_i \in T_{P_{X|U}}(\boldsymbol{u})$, $\boldsymbol{y}_j \in T_{P_{Y|U}}(\boldsymbol{u})$ for all i and j, $M^*_X \geq 2^{n(R_X - \delta)}$, and $M^*_Y \geq 2^{n(R_Y - \delta)}$, such that for every MAC $W : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$

$$e(\mathcal{C}, W) \leq 2^{-n[E_{ex}(R_X, R_Y, W, P_{XYU}) - \delta]} \tag{64}$$

whenever $n \geq n_1(|\mathcal{Z}|, |\mathcal{X}|, |\mathcal{Y}|, |\mathcal{U}|, \delta)$, where

$$E_{ex}(R_X, R_Y, W, P_{XYU})$$
$$\triangleq \min_{\beta = X, Y, XY} \quad E_\beta(R_X, R_Y, W, P_{XYU}) \tag{65}$$

and $E_\beta(R_X, R_Y, W, P_{XYU})$, $\beta = X, Y, XY$ *are defined respectively by*

$$E_X(R_X, R_Y, W, P_{XYU}) \triangleq$$
$$\min_{V_{UXY\tilde{X}Z}\in\mathcal{V}_X} D(V_{Z|XYU}||W|P_{XYU}) + I_V(X \wedge Y|U)$$
$$+ |I(\tilde{X} \wedge XZ|YU) + I_V(\tilde{X} \wedge Y|U) - R_X|^+ \tag{66}$$

$$E_Y(R_X, R_Y, W, P_{XYU}) \triangleq$$
$$\min_{V_{UXY\tilde{Y}Z}\in\mathcal{V}_Y} D(V_{Z|XYU}||W|P_{XYU}) + I_V(X \wedge Y|U)$$
$$+ |I(\tilde{Y} \wedge YZ|XU) + I_V(X \wedge \tilde{Y}|U) - R_Y|^+ \tag{67}$$

$$E_{XY}(R_X, R_Y, W, P_{XYU}) \triangleq$$
$$\min_{V_{UXY\tilde{X}\tilde{Y}Z}\in\mathcal{V}_{XY}} D(V_{Z|XYU}||W|P_{XYU}) + I_V(X \wedge Y|U)$$
$$+ |I(\tilde{X}\tilde{Y} \wedge XYZ|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) - R_X - R_Y|^+ \tag{68}$$

where

$$\mathcal{V}_X \triangleq \{V_{UXY\tilde{X}Z} : \alpha(V_{UXYZ}) \geq \alpha(V_{U\tilde{X}YZ})$$
$$\textit{and } V_{UXY\tilde{X}}\textit{satisfies the relevant conditions in Lemma 4}\}$$
$$\mathcal{V}_Y \triangleq \{V_{UXY\tilde{Y}Z} : \alpha(V_{UXYZ}) \geq \alpha(V_{UX\tilde{Y}Z})$$
$$\textit{and } V_{UXY\tilde{Y}}\textit{satisfies the relevant conditions in Lemma 4}\}$$
$$\mathcal{V}_{XY} \triangleq \{V_{UXY\tilde{X}\tilde{Y}Z} : \alpha(V_{UXYZ}) \geq \alpha(V_{U\tilde{X}\tilde{Y}Z})$$
$$\textit{and } V_{UXY\tilde{X}\tilde{Y}}\textit{satisfies all the conditions in Lemma 4}\} \tag{69}$$

Remark 1. *This exponential error bound can be universally obtained for all MAC's with given input and output alphabets. Note, it is a universal bound since the choice of the codewords does not depend on the channel, and the decoding rule is independent of the channel statistics.*

Proof: Fix $\mathcal{U}$, $\mathcal{P}_{XYU} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y} \times \mathcal{U})$ with $X - U - Y$, $R_X \geq 0$, $R_Y \geq 0$, $\delta > 0$, and $\mathbf{u} \in T^n_{P_U}$. Let $\mathcal{C}^*_X = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{M^*_X}\}$ and $\mathcal{C}^*_Y = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_{M^*_Y}\}$ be the collections of codewords whose existence is asserted in lemma 3. Consider the multiuser code

$$\mathcal{C} = \{(\mathbf{x}_i, \mathbf{y}_j, D_{ij}) : i = 1, ...M^*_X, j = 1, ...M^*_Y\} \tag{70}$$

where the D_{ij} are α-decoding sets for $\mathbf{u}$. Taking into account the given $\mathbf{u}$, the α-decoding yields the decoding sets

$$D_{ij} = \{\mathbf{z} : \alpha(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{z}) \leq \alpha(\mathbf{u}, \mathbf{x}_k, \mathbf{y}_l, \mathbf{z}) \text{ for all}(k, l) \neq (i, j)\}$$

The average probability of this multiuser code can be written as

$$\begin{aligned}
e(C,W) &\triangleq \frac{1}{M_X^* M_Y^*}\sum_{i,j} W^n(D_{ij}^c|\mathbf{x}_i,\mathbf{y}_j)\\
&= \frac{1}{M_X^* M_Y^*}\sum_{i,j} W^n(\bigcup_{k\neq i} D_{kj}|\mathbf{x}_i,\mathbf{y}_j)\\
&+ \frac{1}{M_X^* M_Y^*}\sum_{i,j} W^n(\bigcup_{l\neq j} D_{il}|\mathbf{x}_i,\mathbf{y}_j)\\
&+ \frac{1}{M_X^* M_Y^*}\sum_{i,j} W^n(\bigcup_{\substack{k\neq i\\ l\neq j}} D_{kl}|\mathbf{x}_i,\mathbf{y}_j)
\end{aligned} \tag{71}$$

The first term on the right side can be written as

$$\begin{aligned}
&\frac{1}{M_X^* M_Y^*}\sum_{i,j} W^n\Big(\{\mathbf{z}:\alpha(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{z})>\alpha(\mathbf{u},\mathbf{x}_k,\mathbf{y}_j,\mathbf{z}),\\
&\qquad\qquad \text{for some } k\neq i\}|\mathbf{u},\mathbf{x}_i,\mathbf{y}_j\Big)\\
&= \sum_{V_{UXY\tilde{X}Z}\in\mathcal{V}_X} 2^{-n[D(V_{Z|XYU}||W|V_{XYU})+H_V(Z|XYU)]}\\
&.\Big[\frac{1}{M_X^* M_Y^*}\sum_{i,j} 1_{T_{V_{UXY}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j)\\
&.\big|\{\mathbf{z}:(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{z})\in T_{V_{UXY\tilde{X}Z}} \text{for some } k\neq i\}\big|\Big]
\end{aligned} \tag{72}$$

The second term in (72) can be upper bounded by

$$\begin{aligned}
&\frac{1}{M_X^* M_Y^*}\sum_{i,j} 1_{T_{V_{UXY}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j)\\
&.\big|\{\mathbf{z}:(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k,\mathbf{z})\in T_{V_{UXY\tilde{X}Z}} \text{for some } k\neq i\}\big|\\
&\leq \frac{1}{M_X^* M_Y^*}\sum_{i,j}\sum_{k\neq i} 1_{T_{V_{UXY\tilde{X}}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k)\\
&\qquad .\big|\{\mathbf{z}:\mathbf{z}\in T_{V_{Z|UXY\tilde{X}}}(\mathbf{u},\mathbf{x}_i,\mathbf{y}_j,\mathbf{x}_k\}\big|
\end{aligned} \tag{73}$$

By properties of the codewords, mentioned in packing lemma, we can bound the right side of (73) by

$$\leq \exp\{-n\big[F_X(V)-H_V(Z|UXY\tilde{X})\big]\} \tag{74}$$

By simple calculation, the exponent in (74) can be rewritten as

$$\begin{aligned}
&I_V(X\wedge Y|U)+I_V(\tilde{X}\wedge Y|U)+I_V(\tilde{X}\wedge XZ|UY)\\
&\qquad -H_V(Z|UXY)-R_X
\end{aligned} \tag{75}$$

By using the fact that On the other hand, by the property of the codebook, the following bound for the second term, on the right side of (72), can be obtained

$$\frac{1}{M_X^* M_Y^*} \sum_{i,j} 1_{T_{V_{UXY}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j)$$
$$.\Big|\{\mathbf{z} : (\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k, \mathbf{z}) \in T_{V_{UXY\tilde{X}Z}} \text{for some } k \neq i\}\Big|$$
$$\leq \exp\big(-n[I_V(X \wedge Y|U) - H_V(Z|UXY) - 3\delta]\big) \tag{76}$$

By combining the exponents of (74) and (76), the right side of (72) can be bounded by

$$\leq 2^{-nE_X(R_X, R_Y, W, P_{XYU})} \tag{77}$$

where $E_X(R_X, R_Y, W, P_{XYU})$ is defined in (66). Similarly, by using a similar argument for the second term on the right side of (71), we can show that

$$\frac{1}{M_X^* M_Y^*} \sum_{i,j} W^n(\bigcup_{l \neq j} D_{il}|\mathbf{x}_i, \mathbf{y}_j) \leq 2^{-nE_Y(R_X, R_Y, W, P_{XYU})} \tag{78}$$

where $E_Y(R_X, R_Y, W, P_{XYU})$ is defined in (67). Now, consider the third term on the right side of (71). It can be written as

$$\frac{1}{M_X^* M_Y^*} \sum_{i,j} W^n\Big(\{\mathbf{z} : \alpha(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{z}) > \alpha(\mathbf{u}, \mathbf{x}_k, \mathbf{y}_l, \mathbf{z}),$$
$$\text{for some } (k,l) \neq (i,j)\}|\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j\Big)$$
$$= \sum_{V_{UXY\tilde{X}\tilde{Y}Z} \in \mathcal{V}_{XY}} 2^{-n[D(V_{Z|XYU}||W|V_{XYU}) + H_V(Z|XYU)]}$$
$$.\Big[\frac{1}{M_X^* M_Y^*} \sum_{i,j} 1_{T_{V_{UXY}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j)$$
$$.\Big|\{\mathbf{z} : (\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_l, \mathbf{z}) \in T_{V_{UXY\tilde{X}\tilde{Y}Z}}$$
$$\text{for some } (k,l) \neq (i,j)\}\Big|\Big] \tag{79}$$

The second term in (79) can be upper bounded by

$$\leq \frac{1}{M_X^* M_Y^*} \sum_{i,j} \sum_{\substack{k \neq i \\ l \neq j}} 1_{T_{V_{UXY\tilde{X}\tilde{Y}}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_l)$$
$$.\Big|\{\mathbf{z} : \mathbf{z} \in T_{V_{Z|UXY\tilde{X}\tilde{Y}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_l)\}\Big| \tag{80}$$

The second term is actually the cardinality of $T_{V_{Z|UXY\tilde{X}\tilde{Y}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_l)$, which is equal to $\exp\{nH_V(Z|UXY\tilde{X}\tilde{Y})\}$. By the properties of the codewords, the first term in (79) can be upper bounded by $\exp\{-n[F_{XY}(V)]\}$. Therefore (80) can be bounded by

$$\leq \exp\{-n\big[F_{XY}(V) - H_V(Z|UXY\tilde{X}\tilde{Y})\big]\} \tag{81}$$

By simple calculation, the exponent in (81) can be rewritten as

$$I_V(X \wedge Y|U) + I_V(\tilde{X} \wedge \tilde{Y}|U) + I_V(\tilde{X}\tilde{Y} \wedge XYZ|U)$$
$$-H_V(Z|UXY) - R_X - R_Y \tag{82}$$

By using the properties of codewords, the following bound for the second term on the right right side of (79) can be obtained as follows

$$\frac{1}{M_X^* M_Y^*} \sum_{i,j} 1_{T_{V_{UXY}}}(\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j)$$

$$\left|\{\mathbf{z} : (\mathbf{u}, \mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_l, \mathbf{z}) \in T_{V_{UXY\tilde{X}\tilde{Y}Z}} \text{for some } (k,l) \neq (i,j)\}\right|$$

$$\leq \exp\big(-n[I_V(X \wedge Y|U) - H_V(Z|UXY) - 3\delta]\big) \tag{83}$$

By combining the exponents of (81) and (83), the right side of (79) can be bounded by

$$\leq 2^{-nE_{XY}(R_X, R_Y, W, P_{XYU})} \tag{84}$$

where $E_{XY}(R_X, R_Y, W, P_{XYU})$ is defined in (68). Now, it follows from (77), (78), and (84), that the average probability of the given code is upper bounded by

$$e(C, W) \leq 2^{-n[E_{ex}(R_X, R_Y, W, P_{XYU}) - \delta]} \tag{85}$$

where $E_{ex}(R_X, R_Y, W, P_{XYU})$ is defined in (65). ■

In the following, we prove that the random coding bound in theorem 1 will result in a tighter bound in comparison to the best known random coding bound, found in [12]. For this purpose, let us use the minimum equivocation decoding rule.

Definition 3. *Given $\boldsymbol{u}$, for a multiuser code*

$$\mathcal{C} = \{(\boldsymbol{x}_i, \boldsymbol{y}_j, D_{ij}) : i = 1, ...M_X^*, j = 1, ...M_Y^*\}$$

we say that the D_{ij} are minimum equivocation decoding sets for $\boldsymbol{u}$ if $\boldsymbol{z} \in D_{ij}$ implies

$$H(\boldsymbol{x}_i \boldsymbol{y}_j | \boldsymbol{z}\boldsymbol{u}) = \min_{k,l} H(\boldsymbol{x}_k \boldsymbol{y}_l | \boldsymbol{z}\boldsymbol{u}).$$

It can be easily observed that these sets are equivalent to α-decoding sets, where $\alpha(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is defined as

$$\alpha(V_{UXYZ}) \triangleq H_V(XY|ZU). \tag{86}$$

Here, V_{UXYZ} is the joint empirical distribution of $(\boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$.

Theorem 2. *For every finite set $\mathcal{U}$, $\mathcal{P}_{XYU} \in \mathcal{P}(\mathcal{U})$, $R_X \geq 0$, $R_Y \geq 0$, and $W : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$,*

$$E_\beta(R_X, R_Y, W, P_{XYU}) \geq E^L_{r\beta}(R_X, R_Y, W, P_{XYU})$$
$$\beta = X, Y, XY \tag{87}$$

Hence

$$E_{ex}(R_X, R_Y, W, P_{XYU}) \geq E^L_r(R_X, R_Y, W, P_{XYU}) \tag{88}$$

for all $P_{XYU} \in \mathcal{P}(\mathcal{U})$ satisfying $X - U - Y$. Here, E^L_r is the random coding exponent of [12]. $E^L_{r\beta}$ are also defined in [12].

Proof: For any $V_{UXY\tilde{X}Z} \in \mathcal{V}_X$,

$$H_V(XY|ZU) \geq H_V(\tilde{X}Y|ZU), \tag{89}$$

therefore, by subtracting $H_V(Y|ZU)$ form both sides of (89), we can conclude that

$$H_V(X|U) - I_V(X \wedge YZ|U) \geq H_V(\tilde{X}|U) - I_V(\tilde{X} \wedge YZ|U),$$

Since $V_{XU} = V_{\tilde{X}U} = P_{XU}$, the last inequality is equivalent to

$$I_V(X \wedge YZ|U) \leq I_V(\tilde{X} \wedge YZ|U)$$

Since $I_V(\tilde{X} \wedge XZ|YU) + I(\tilde{X} \wedge Y|U) \geq I_V(\tilde{X} \wedge YZ|U)$, it can be seen that for any $V_{UXY\tilde{X}Z} \in \mathcal{V}_X$

$$I_V(\tilde{X} \wedge XZ|YU) + I(\tilde{X} \wedge Y|U) \geq I_V(X \wedge YZ|U)$$

Moreover, since

$$\begin{aligned} \mathcal{V}_X \subset \{V_{UXY\tilde{X}Z} : V_{UXYZ} \in \mathcal{V}(P_{UXY}) \\ I(X \wedge Y|U) \leq R_X + 3\delta\} \end{aligned} \tag{90}$$

it can be easily concluded that

$$E_X(R_X, R_Y, W, P_{XYU}) \geq E^L_{rX}(R_X, R_Y, W, P_{XYU}).$$

Similarly, for any $V_{UXY\tilde{Y}Z} \in \mathcal{V}_Y$,

$$H_V(XY|ZU) \geq H_V(X\tilde{Y}|ZU).$$

By using the fact that, $V_{YU} = V_{\tilde{Y}U} = P_{YU}$, it can be concluded that

$$I_V(\tilde{Y} \wedge YZ|XU) + I(X \wedge \tilde{Y}|U) \geq I_V(Y \wedge XZ|U).$$

Since

$$\begin{aligned} \mathcal{V}_Y \subset \{V_{UXY\tilde{Y}Z} : V_{UXYZ} \in \mathcal{V}(P_{UXY}) \\ I(X \wedge Y|U) \leq R_X + 3\delta\} \end{aligned} \tag{91}$$

we conclude that

$$E_Y(R_X, R_Y, W, P_{XYU}) \geq E^L_{rY}(R_X, R_Y, W, P_{XYU}).$$

Similarly, we can conclude that, for any $V_{UXY\tilde{X}\tilde{Y}Z} \in \mathcal{V}_{XY}$,

$$I_V(\tilde{X}\tilde{Y} \wedge XYZ|U) + I(\tilde{X} \wedge \tilde{Y}|U) \geq I_V(XY \wedge Z|U) + I(X \wedge Y|U).$$

Since

$$\begin{aligned} \mathcal{V}_{XY} \subset \{V_{UXY\tilde{X}\tilde{Y}Z} : V_{UXYZ} \in \mathcal{V}(P_{UXY}) \\ I(X \wedge Y|U) \leq R_X + 3\delta\}, \end{aligned} \tag{92}$$

it can be concluded that

$$E_{XY}(R_X, R_Y, W, P_{XYU}) \geq E^L_{rXY}(R_X, R_Y, W, P_{XYU}).$$

■

The last theorem shows that $E_{ex}(R_X, R_Y, W, P_{XYU})$ is at least as large as the Liu, Hughes [12] exponent. In the following, we show that at low rate pairs, we may have a strictly better result. To illustrate this, let us focus on the case where both codebooks have rate zero, $R_X = R_Y = 0$. For small δ, any $V_{UXY\tilde{X}Z} \in \mathcal{V}_X$ will satisfy the following relationships

$$X - U - Y, \quad \tilde{X} - U - Y, \quad \tilde{X} - UY - X \tag{93}$$

Therefore, any $V_{UXY\tilde{X}Z} \in \mathcal{V}_X$ can be written as

$$V_{Z|UXY\tilde{X}} P_{X|U} P_{Y|U} P_{X|U} P_U. \tag{94}$$

Similarly, any $V_{UXY\tilde{Y}Z} \in \mathcal{V}_Y$ can be written as

$$V_{Z|UXY\tilde{Y}} P_{X|U} P_{Y|U} P_{Y|U} P_U, \tag{95}$$

and any $V_{UXY\tilde{X}\tilde{Y}Z} \in \mathcal{V}_{XY}$ can be written as

$$V_{Z|UXY\tilde{X}\tilde{Y}} P_{X|U} P_{Y|U} P_{X|U} P_{Y|U} P_U. \tag{96}$$

For a moment, let us consider the point to point data communication. By using only a random coding argument, and without any expurgation, one can prove the following result.

Lemma 5. *For every $R > 0$, $\delta \geq 0$ and every type of $P \in P_n(\mathcal{X})$ satisfying $H(P) \geq R$, there exist $M \geq 2^{n(R-\delta)}$ sequences in T_P such that for every $P_{X\tilde{X}} \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$,*

$$\frac{1}{M} \sum_{i=1}^{M} \sum_{k \neq i} 1_{T_{P_{X\tilde{X}}}}(\boldsymbol{x}_i, \boldsymbol{x}_k) \leq 2^{n(R - I(X \wedge \tilde{X}))} \tag{97}$$

provided that $n \geq n_0(|\mathcal{X}|, |\mathcal{Y}|, \delta)$.

Now, let us multiply both sides of (97) by M. It can be shown that for every $1 \leq i \leq M$,

$$\sum_{k \neq i} 1_{T_{P_{X\tilde{X}}}}(\mathbf{x}_i, \mathbf{x}_k) \leq 2^{n(2R - I(X \wedge \tilde{X}))} \tag{98}$$

By using these sequences as our set of codewords, and using α-decoding, we will end up with a result very similar to [3]. The only difference is that our minimization would be taken over all distributions satisfying $I(X \wedge \tilde{X}) \leq 2R$, instead of $I(X \wedge \tilde{X}) \leq R$. Using the appropriate decoding rule, this bound would be exactly the same as the typical random coding bound that Barg and Forney found in [2]. As we can see, in point to point communications, even without doing any expurgation, we ended up with a strictly better bound in comparison to the usual random coding bound. Needless to say that if we eliminate half of the codewords in (97), the result would be equal to the expurgated bound [3].

REFERENCES

[1] R. Ahlswede. Multi-way communication channels. In *Proc. International Symposium on Information Theory*, 1971.

[2] A. Barg and D. Forney. Random codes: Minimum distances and error exponents. *IEEE Trans. Information Theory*, 48(9):2568–2573, Sept. 2002.

[3] I. Csiszar and J. Korner. Graph decomposition: A new key to coding theorems. *IEEE Trans. Information Theory*, 1:5–12, Jan. 1981.

[4] I. Csiszar and J. Korner. *Information theory: Coding theorems for Discrete memoryless Systems.* 1981.

[5] A. G. Dyachkov. Random constant composition codes for multiple-access channels. *Probl. of Control and Inform. Theory*, pages 357–369, 1984.

[6] R. Gallager. The random coding bound is tight for the average code. *IEEE Trans. Information Theory*, 23(2):244–246, Mar. 1973.

[7] R. Gallager. A perspective on multi-access channels. *IEEE Trans. Information Theory*, 31(2):124–142, Mar. 1985.

[8] R. G. Gallager. *Transmission of Information: A Statistical Theory of Communica- tion.* MIT Press, 1961.

[9] R. G. Gallager. *Information theory and Reliable Communications*. John Wiley & Sons, New York, 1968.

[10] E. A. Haroutunian. Lower bound for the error probability of multiple-access channels. *Problemy Peredachi Informatsii*, 11:23–36, June 1975.

[11] H. Liao. A coding theorem for multiple-access communications. In *Proc. International Symposium on Information Theory*.

[12] Y. Liu and B. L. Hughes. A new universal random coding bound for the multiple-access channels. *IEEE Trans. Information Theory*, 42(2):376–386, Mar. 1996.

[13] A. Nazari, S. S. Pradhan, and A. Anastasopoulous. A new sphere-packing bound for maximal error exponent for multiple-access channels. In *Proc. International Symposium on Information Theory*, 2008. Online: http://arxiv.org/abs/0803.3645.

[14] J. Pokorney and H. S. Wallmeier. Random coding bounds and codes produced by permutations for the multiple-access channels. *IEEE Trans. Information Theory*, 31(6):741–750, Nov. 1985.

[15] D. Slepian and J. K. Wolf. A coding theorem for multiple access channels with correlated sources. *bell Syst. tech. J.*, 52:1037–1076, 1973.