SIGNAL SPACES

We will view signals as points in certain mathematical spaces. The spaces have common structure, so it will be useful to study them in the abstract.

Metric Spaces

A <u>metric space</u> is a set $\mathcal{X}$ together w/ a <u>metric</u> (distance function) $d: \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ such that for all $x, y, z \in \mathcal{X}$

M1 $\quad d(x,y) = d(y,x)$

M2 $\quad d(x,y) \geq 0$

M3 $\quad d(x,y) = 0 \iff x = y$

M4 $\quad d(x,z) \leq d(x,y) + d(y,z)$ $\quad$ triangle inequality

<u>Example 1</u> ℓ_p space

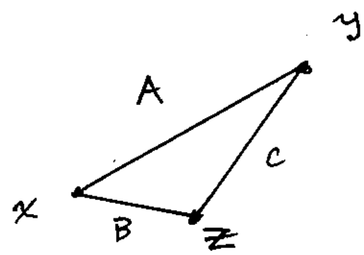
$\mathcal{X} = \mathbb{R}^n$

$$d_p(x,y) = \begin{cases} \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}, & 1 \leq p < \infty \\ \max_{i=1,\dots,n} |x_i - y_i|, & p = \infty \end{cases}$$

Each d_p is different and thus defines a different metric space. $p = 2 \iff$ Euclidean space.

The Δ ineq. follows from Minkowski's ineq. for $p < \infty$.

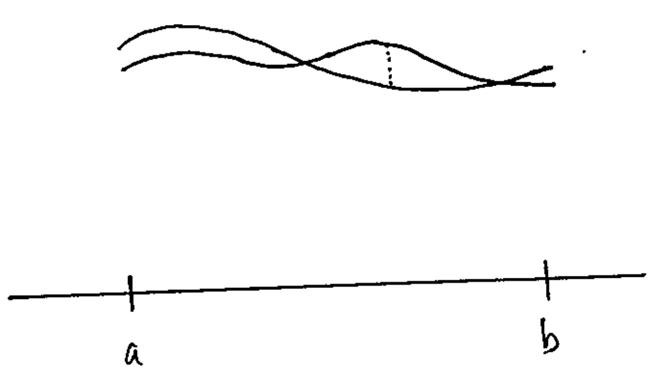
Picture for $n = 2$, $p = 2$:



$A \leq B + C$

<u>Example 1</u> $X = \mathcal{C}[a, b]$

$$d_\infty(x, y) := \max_{t \in [a,b]} |x(t) - y(t)|$$



Vector Spaces

Let K be a collection of scalars, $K = \mathbb{R}$ or $\mathbb{C}$

Let V be a set equipped with two operations

- vector addition: $x + y$
- scalar mult: $a \cdot x$

We say V is a vector space over K if,

VS1 (a) Closure of addition:

$\forall x, y \in V, \quad x + y \in V$

(b) Existence of additive identity:

$\exists\, 0 \in V$ s.t $\forall x \in V$,

$x + 0 = 0 + x = x.$

(c) Existence of additive inverse:

$\forall x \in V, \quad \exists\, y \in V$ s.t. $x + y = 0.$

Notation: $y = -x$

(d) Associativity of addition: $\forall x, y, z \in V$,

$(x + y) + z = x + (y + z)$

(e) Commutativity of addition: $\forall x, y \in V$,

$x + y = y + x.$

VS2 Properties of scalar multiplication: For all $x, y \in V$, $a, b \in K$,

- $a x \in V$
- $a(bx) = (ab)x$
- $(a+b)x = ax + bx$
- $a(x+y) = ax + ay$
- $\underline{1} \cdot x = x$

<u>Example</u> $\mathbb{R}^n$ over $\mathbb{R}$.

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$a \cdot x = a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} a x_1 \\ a x_2 \\ a x_3 \end{bmatrix}$$

Other examples:

$\mathbb{C}^n / \mathbb{C}$

$\mathbb{C}^n / \mathbb{R}$

not $\mathbb{R}^n / \mathbb{C}$ (scalar mult. not closed)

<u>Example</u> $V = C[a,b]$, $K = \mathbb{R}$

$$f + g \mapsto h, \quad h(t) := f(t) + g(t)$$

$$a \cdot f \mapsto h, \quad h(t) := a \cdot f(t)$$

The elements of a vector space are called <u>vectors</u>. Thus functions can be vectors. This is a key concept!

<u>Terminology</u> VS = linear space = linear vector space

Normed Vector Spaces

Let V be a vector space over K, $K = \mathbb{R}$ or $\mathbb{C}$.

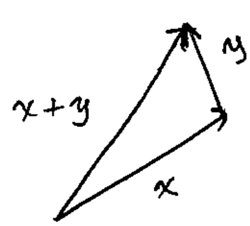
A <u>norm</u> is a function $\|\cdot\| : V \to \mathbb{R}$ such that

N1 $\quad \|x\| \geq 0 \quad \forall x \in V$

N2 $\quad \|x\| = 0 \iff x = 0$

N3 $\quad \|a \cdot x\| = |a| \cdot \|x\| \quad \forall x \in V, \ a \in K$

N4 $\quad \|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$ $\quad \boxed{\text{triangle inequality}}$



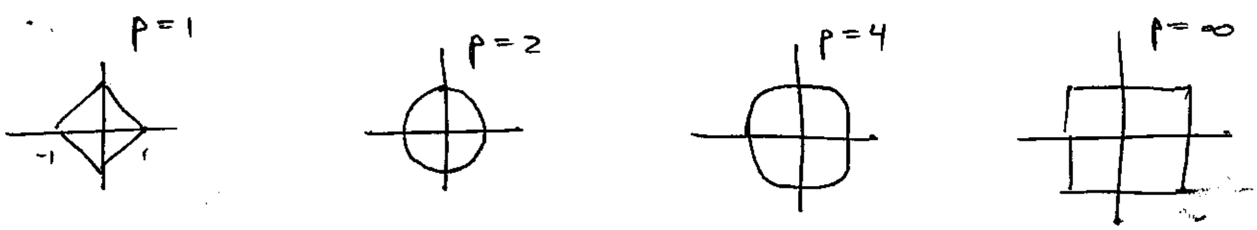
A vector space together with a norm is called a <u>normed vector space</u>, or <u>normed linear space</u>

<u>Example</u> ℓ_p norm on $V = \mathbb{R}^n$ over $\mathbb{R}$

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, & 1 \le p < \infty \\ \max_{i=1,\dots,n} |x_i|, & p = \infty \end{cases}$$

Different ℓ_p norms induce different geometries.

Consider the unit ball in $\mathbb{R}^2$, $\{x : \|x\|_p = 1\}$



<u>Note</u> A normed V.S. is a metric space,

with induced metric

$$d(x,y) = \|x-y\|.$$

Δ ineq: $\|x-z\| = \|x-y+y-z\| \le \|x-y\| + \|y-z\|$

$$\underset{d(x,y)}{\overset{\|}{\|x-z\|}} \qquad\qquad \underset{d(x,y)+d(y,z)}{\overset{\|}{\|x-y\|+\|y-z\|}}$$

Inner Product Spaces

Let V be a vector space over K, $K = \mathbb{R}$ or $\mathbb{C}$.

An <u>inner product</u> is a function $\langle \cdot , \cdot \rangle : V \times V \to K$ such that $\forall\, x, y, z \in V$, $a \in K$

IP1 $\quad \langle x, y \rangle = \overline{\langle y, x \rangle}$

IP2 $\quad \langle a \cdot x, y \rangle = a \cdot \langle x, y \rangle$

IP3 $\quad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

IP4 $\quad \langle x, x \rangle \geq 0$ with equality iff $x = 0$.

A vector space together with an inner product is called an <u>inner product space</u>.

<u>Example</u> Complex Euclidean space, $V = \mathbb{C}^n$, $K = \mathbb{C}$

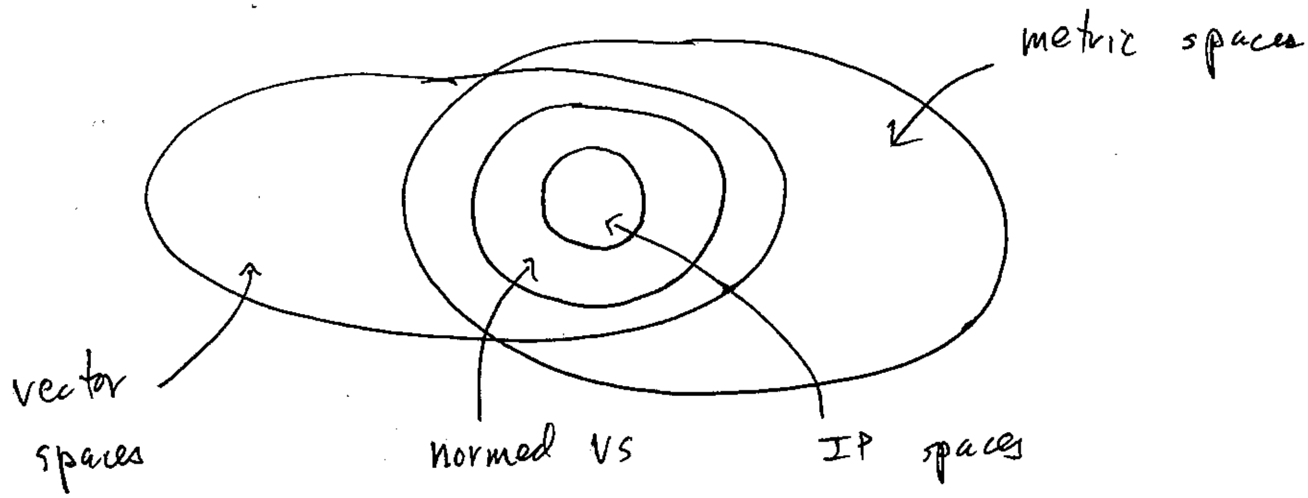
$$\langle x, y \rangle := \sum_{i=1}^{n} x_i \overline{y_i}$$

<u>Example</u> $V = \{ f : \mathbb{R} \to \mathbb{C} \}$, $K = \mathbb{C}$,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt$$

Note | An IPS is a NVS with induced norm $\|x\| = \sqrt{\langle x, x \rangle}$. Proof of the Δ ineq. relies on the Cauchy-Schwarz ineq, which states that for any inner prod, and any $x, y \in V$,

$$|\langle x, y \rangle| \le \sqrt{\langle x, x \rangle \cdot \langle y, y \rangle}$$



IP spaces have many more important properties that we will study in detail later.

Completeness

Let (X, d) be a metric space.

A sequence $x_1, x_2, \ldots$ in X <u>converges</u> if $\exists\ x \in X$ such that, $\forall \epsilon > 0$, $\exists N$ such that $n \geq N \implies d(x_n, x) < \epsilon$.

<u>Example</u> $X = \mathbb{R}$, $d(x, y) = |x - y|$,

$$x_n = \frac{1}{n} \longrightarrow 0 \quad \left(\text{choose } N = \frac{1}{\epsilon}\right)$$

A sequence $x_1, x_2, \ldots$ is called a <u>Cauchy sequence</u> if $\forall \epsilon > 0$, $\exists N$ such that

$$m, n \geq N \implies d(x_m, x_n) < \epsilon.$$

It can be shown that every convergent sequence is Cauchy. The converse, however, is false.

<u>Example</u> Consider $X = C[0,1]$, the real valued continuous functions on $[0,1]$, and $d(f, g) = \left(\int_0^1 f(t) g(t)\, dt\right)^{\frac{1}{2}}$.

Consider the sequence
$$f_n(t) = \begin{cases} nt, & 0 \leq t \leq \frac{1}{n} \\ 1, & \frac{1}{n} < t \leq 1 \end{cases}$$

Then the limit function $f(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \end{cases}$ is discontinuous, hence no limit in X exists. Yet $\{f_n\}$ is Cauchy.

<u>Example</u> $\mathbb{Q} = \{\text{rational numbers}\}$, $d(x,y) = |x-y|$.

$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415, \ldots \longrightarrow \pi \notin \mathbb{Q}$,

yet sequence is Cauchy.

A <u>complete</u> metric space is one for which every Cauchy sequence converges.

A complete normed VS is called a <u>Banach space</u>.

<u>Example</u> $p \geq 1$

$$L^p(\mathbb{R}) \doteq \{ f : \mathbb{R} \to \mathbb{C} \mid \|f\|_p < \infty \}$$

where
$$\|f\|_p = \begin{cases} \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p}, & 1 \leq p < \infty \\ \sup_{t \in \mathbb{R}} |f(t)| , & p = \infty \end{cases}$$

is complete.

A complete IPS is called a Hilbert space. Hilbert spaces will be extremely important in this course.

Example | ℓ_2 space

$$V = \{ x \in \mathbb{C}^n : \sum |x_i|^2 < \infty \}$$

$$\langle x, y \rangle = \sum_i x_i \overline{y_i}$$

is a Hilbert space, even when $n = \infty$.

Example | $L^2(\mathbb{R})$

$$V = \{ f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} f(t)^2 \, dt < \infty \}$$

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, dt$$

is a Hilbert space

CT FOURIER TRANSFORM

Fourier Transform in $L^1(\mathbb{R})$

Let $f : \mathbb{R} \to \mathbb{C}$. The <u>continuous time Fourier transform</u> of f is a function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} f(t) e^{-i\omega t}$$

<u>Example</u> If $f(t) = \mathbb{1}_{[-a,a]}(t) := \begin{cases} 1 & \text{if } -a \le t \le a \\ 0 & \text{else} \end{cases}$

then

$$\hat{f}(\omega) = \int_{-a}^{a} e^{i\omega t}\, dt = \frac{e^{i\omega a} - e^{-i\omega a}}{\omega i} = \frac{2 \sin \omega a}{\omega}$$

using the formula $\sin \theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ ▨

When is $\hat{f}$ well-defined, i.e. when does the integral converge?

If $f \in L^1(\mathbb{R})$, then

$$|\hat{f}(\omega)| = \left| \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt \right| \le \int_{-\infty}^{\infty} |f(t) e^{i\omega t}|\, dt$$

$$= \int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

Furthermore, if $f \in L^1(\mathbb{R})$, then $\hat{f}(\omega)$ is continuous (exercise).

~~Mention~~

$$\int |g| < \infty \Rightarrow \int g \text{ exists}$$

just like

$$\sum |a_n| < \infty$$

$$\Rightarrow \sum a_n \text{ exists}$$

Inverse Fourier Transform

When can we recover f from $\hat{f}$?

Theorem | If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega$$

Why do we need $\hat{f} \in L^1(\mathbb{R})$?

Proof | See Mallat Ch 2.

Interpretation: f = superposition of sinusoids, $\hat{f}(\omega)$ = frequency content of f at ω.

Conclusion: $f(t) \xrightarrow{CTFT} \hat{f}(\omega) \xrightarrow{CTFT} 2\pi f(-t)$

Therefore we often speak of FT pairs

Unfortunately, for many signals of interest, either $f \notin L^1(\mathbb{R})$ or $\hat{f} \notin L^1(\mathbb{R})$. In most signal processing books / courses, it is simply assumed that the forward and inverse formulas hold, w/o proof. While a comprehensive treatment of Fourier transforms is beyond the scope of the course, we will see how to extend the FT from $L^1(\mathbb{R})$ to $L^2(\mathbb{R})$, the space of finite energy signals.

Convolution Formula

The following result has broad implications in the context of LTI systems, which we will discuss later.

Theorem If $f, h \in L^1(\mathbb{R})$, and $g = h * f$, where

$$g(t) = (h * f)(t) = \int_{-\infty}^{\infty} h(u) f(t-u)\, du,$$

then $g \in L^1(\mathbb{R})$ and $\hat{g}(\omega) = \hat{h}(\omega) \hat{f}(\omega)$.

Proof

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(u) f(t-u)\, du \right) e^{-i\omega t}\, dt$$

$[v = t - u]$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(u) f(v)\, du \right) e^{-i\omega(u+v)}\, dv$$

$$= \left(\int_{-\infty}^{\infty} h(u) e^{-i\omega u}\, du \right) \cdot \left(\int_{-\infty}^{\infty} f(v) e^{-i\omega v}\, dv \right)$$

[Fubini's thm]

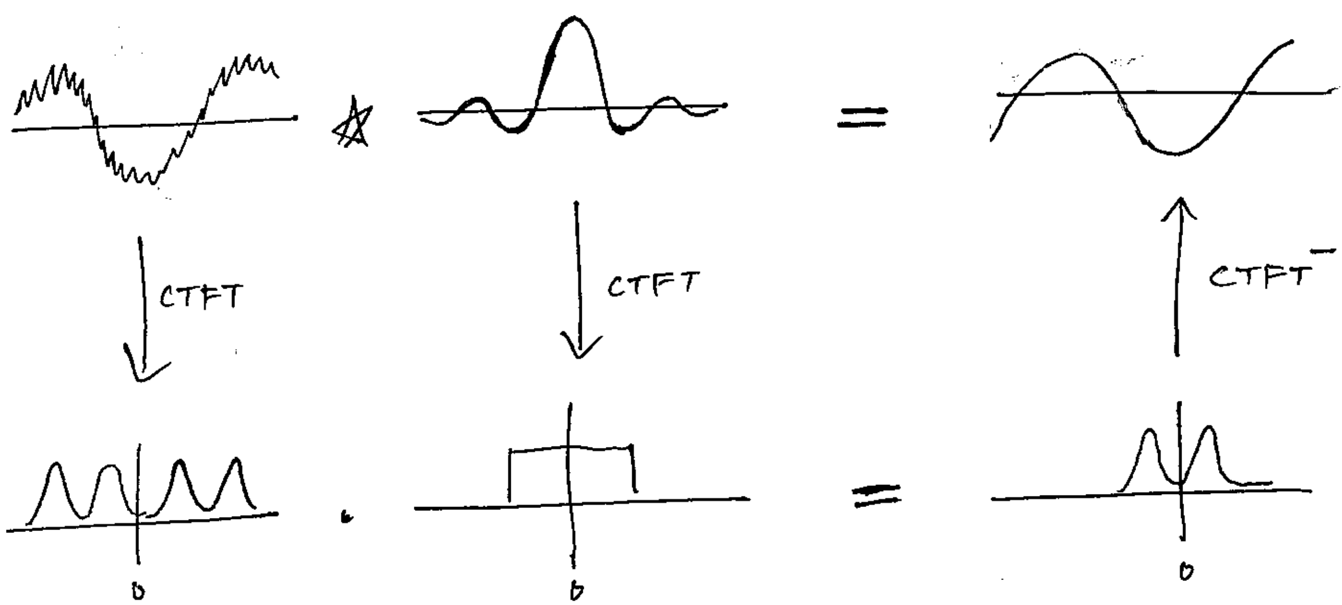
$$= \hat{h}(\omega) \cdot \hat{f}(\omega).$$

Implications

1. Frequency domain filtering

$$
\begin{tikzcd}
f(t) \arrow[r, "\star\, h(t)"] \arrow[d, "\mathrm{CTFT}"'] & L\{f(t)\} \\
\hat{f}(\omega) \arrow[r, "\cdot\, \hat{h}(\omega)"'] & \hat{h}(\omega)\cdot\hat{f}(\omega) \arrow[u, "\mathrm{CTFT}^{-1}"']
\end{tikzcd}
$$

$$L\{f(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(\omega)\,\hat{f}(\omega)\,e^{i\omega t}\,d\omega$$



$\hat{h}(\omega)$ is called the transfer function of L.

2. <u>Eigenfunctions</u>

$$L\{e^{i\omega t}\} = \int_{-\infty}^{\infty} h(u)\, e^{i\omega(t-u)}\, du$$

$$= e^{i\omega t} \int h(u) e^{-i\omega u}\, du$$

$$= \hat{h}(\omega)\, e^{i\omega t}$$

↑ eigenvalue = factor by which frequency ω is amplified / attenuated.

3. <u>Causality</u>: A system is causal if $L\{f\}$ at time t does not depend on $f(u)$, $u > t$.

Since

$$L\{f(t)\} = \int_{-\infty}^{\infty} h(u)\, f(t-u)\, du,$$

an LTI system is causal provided $h(u) = 0$ for $u < 0$.

4. <u>Stability</u>. A system is bound if $L\{f\}$ is bounded $f\} \in L^1(\mathbb{R})$ whenever f is bounded. A necessary and sufficient condition is

$$\int | h(u) |\, du < \infty$$

i.e., $h \in L^1(\mathbb{R})$. (see Mallat)

Fourier Transform in $L^2(\mathbb{R})$

Consider the function

$$f(t) = \frac{\sin t}{t} = \operatorname{sinc}(t)$$

Unfortunately, $f(t) \notin L^1(\mathbb{R})$. However,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt \le 2\left(1 + \int_1^{\infty} \frac{1}{t^2} dt\right) < \infty$$

thus $f \in L^2(\mathbb{R})$.

<u>Theorem</u> If f and h belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\int f(t)\,\overline{h(t)}\,dt = \frac{1}{2\pi}\int \hat{f}(\omega)\,\overline{\hat{h}(\omega)}\,d\omega \qquad \text{(Parseval)}$$

Therefore, if $f = h$,

$$\int |f(t)|^2 dt = \frac{1}{2\pi}\int |\hat{f}(\omega)|^2 d\omega \qquad \text{(Plancherel)}$$

<u>Proof</u> Set $g = f * h^-$ where $h^-(t) = \overline{h(-t)}$.

Then

$$\int f(t)\,\overline{h(t)}\,dt = \int f(t)\, h^-(-t)\,dt = g(0)$$

$$= \frac{1}{2\pi}\int \hat{g}(\omega)\,d\omega = \frac{1}{2\pi}\int \hat{f}(\omega)\cdot \hat{h^-}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int \hat{f}(\omega)\cdot \overline{\hat{h}(\omega)}\,d\omega$$

next time
swap the names,
since the second is
more often called Parseval
(is Mallat mistaken?)

Now suppose $f \in L^2(\mathbb{R})$ but $f \notin L^1(\mathbb{R})$.

Fact: $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, which means $\exists$ a sequence $f_1, f_2, \ldots \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

$$\lim_{n\to\infty} \| f_n - f \|_2 = 0.$$

Since $f_n \in L^1(\mathbb{R})$, $\hat{f}_n$ is well-defined.

By Plancherel, for any m, n,

$$\| \hat{f}_n - \hat{f}_m \|_2 = \sqrt{2\pi} \, \| f_n - f_m \|_2 .$$

Since $\{f_n\}$ converges, it is a Cauchy sequence. Therefore $\{\hat{f}_n\}$ is a Cauchy sequence. Since $L^2(\mathbb{R})$ is complete, $\exists$ a function, call it $\hat{f}$, such that

$$\lim_{n\to\infty} \| \hat{f}_n - \hat{f} \|_2 = 0.$$

We define $\hat{f}$ to be the CTFT of f.

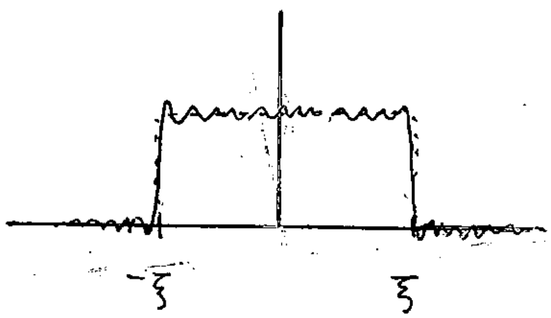
It can be shown that $\hat{f}$ satisfies the convolution, Plancherel, and Parseval formulas, as well as the other basic properties of the CTFT.

Uniqueness?
Properties? (parseval,
convolution, etc.)

Example

$$f(t) = \frac{\sin \xi t}{\pi t}, \quad \xi > 0$$

$$f_n(t) = \begin{cases} f(t), & |t| \le n \\ 0, & \text{else} \end{cases}$$



$$\Rightarrow \hat{f}(\omega) = 1_{[-\xi,\xi]}(\omega)$$

Dirac

The Dirac $\delta_\tau(t) = \delta(t-\tau)$ is defined by the property

$$\int f(t)\,\delta_\tau(t)\,dt = f(\tau).$$

Thus we define

$$\hat{\delta}_\tau(\omega) = e^{-i\omega\tau}$$

Aside

By analogy with the inversion formula, it is common to define the CTFT of $e^{i\xi t}$ to be $2\pi\delta_\xi(\omega)$

The interpretation of $\delta_\xi(\omega)$ is as a pure spike at $\omega = \xi$. Since $e^{i\xi t}$ is a pure (complex) sinusoid with frequency ξ, this is consistent with the interp. of $\hat{f}(\omega)$ as the frequency content at ω.

Note however that $e^{i\xi t}$ is in neither $L^1(\mathbb{R})$ nor $L^2(\mathbb{R})$, nor is it bounded! Fortunately, we won't formally require to compute its Fourier transform.

$\boxed{\text{Impulse Trains}}$

In sampling theory it is convenient to use <u>impulse trains</u>, or <u>Dirac combs</u>,

$$c_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

By linearity of the CTFT,

$$\hat{c}_T(\omega) = \sum_{n=-\infty}^{\infty} e^{-i\omega nT}$$

<u>Theorem</u> (Poisson formula)

$$\hat{c}_T(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

The equality in this theorem means that for any function $\hat{\phi}(w)$,

$$\int_{-\infty}^{\infty} \hat{\phi}(w) \cdot \left(\sum_{n=-\infty}^{\infty} e^{-iwnT} \right) dw = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \hat{\phi}\left(\frac{2\pi k}{T} \right).$$

<u>Proof</u> Mallat, Ch. 2.

CT LTI SYSTEMS

A <u>CT signal</u> is a function $f: \mathbb{R} \to \mathbb{C}$

A <u>CT system</u> is a function L mapping CT signals to CT signals, e.g.,

$$L\{f(t)\} = 2f(t) + f(t-1) + t^2$$

In many SP applications, systems are also

<u>linear</u>:

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

<u>time invariant</u>:

$$\text{if} \quad L\{f(t)\} = g(t)$$
$$\text{then} \quad L\{f(t-\tau)\} = g(t-\tau)$$

Impulse Response

Suppose L is LTI, and define the <u>impulse response</u> of L

$$h(t) = L\{\delta(t)\}.$$

We will show that $h(t)$ characterized L completely.

If $f(t)$ is any CT signal, then

$$f(t) = \int_{-\infty}^{\infty} f(u)\, \delta(t-u)\, du$$

Then

$$L\{f(t)\} = \int_{-\infty}^{\infty} f(u)\, L\{\delta(t-u)\}\, du \quad \text{(by linearity}^*\text{)}$$

$$= \int_{-\infty}^{\infty} f(u)\, h(t-u)\, du \qquad \text{(by time-invariance)}$$

$$= \int h(u)\, f(t-u)\, du$$

$$= (h * f)(t)$$

* By viewing the integral as the limiting value of a sequence of "Rieman sums," $\sum_i f(u_i)\, L\{\delta(t-u_i)\}\, \Delta u_i$. Technically, an additional continuity assumption of L is needed to justify passing the limit through L.

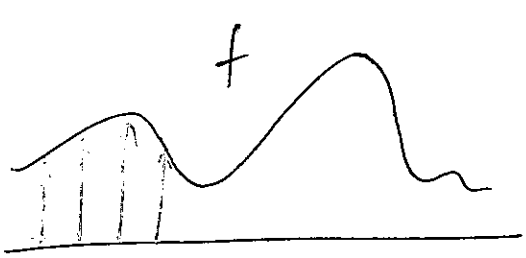
SAMPLING THEORY

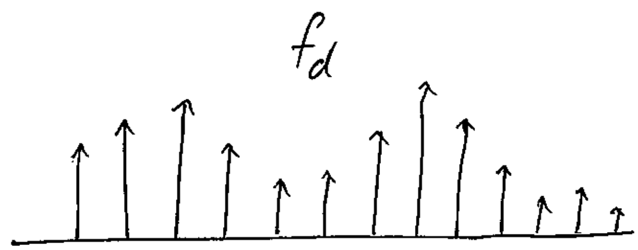
When can we recover $f(t)$ from discrete time samples $f(nT)$, $n \in \mathbb{Z}$?

$T > 0$ is called the sampling interval.

Define

$$f_d(t) = \sum_{n=-\infty}^{\infty} f(nT)\, \delta(t - nT).$$

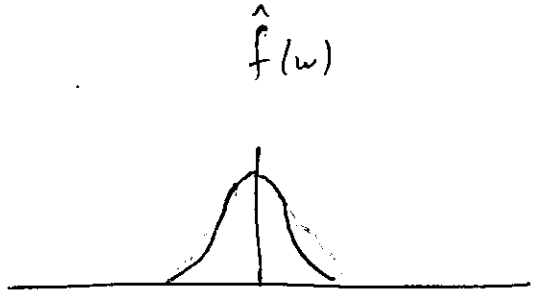


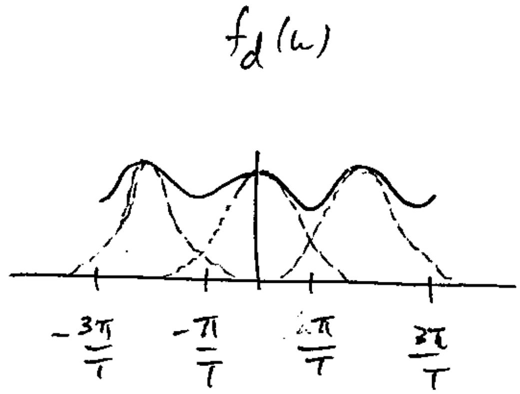


How is f_d related to f?

<u>Proposition</u>

$$\hat{f}_d(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\omega - \frac{2\pi k}{T}\right)$$





Proof

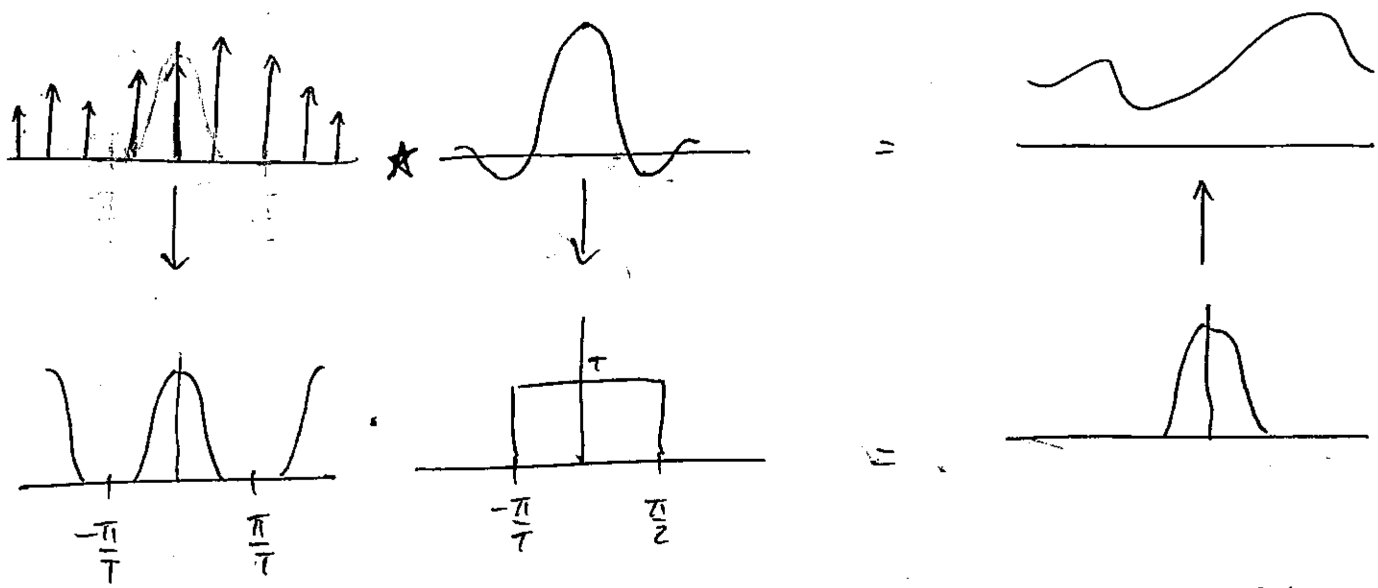
$$f_d(t) = f(t)\cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\text{impulse train } = \, c_T(t)} = f(t)$$

$$\Longrightarrow \hat{f}_d(\omega) = \frac{1}{2\pi}\hat{f}(\omega) \star \hat{c}_T(\omega)$$

$$= \frac{1}{2\pi}\hat{f}(\omega) \star \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\hat{f}\left(\omega - \frac{2\pi k}{T}\right) \qquad \blacksquare$$

Now suppose $\hat{f}(\omega) = 0$ for $|\omega| > \frac{\pi}{T}$.



Then we can recover $\hat{f}(\omega)$ with an <u>ideal low pass filter</u> $\hat{h}_T(\omega)$

$$= T\cdot \mathbb{1}_{[-\frac{\pi}{T},\frac{\pi}{T}]}(\omega) = \begin{cases} T, & |\omega| \le \frac{\pi}{T} \\ 0, & \text{else} \end{cases}.$$

In the time domain we have

$$f(t) = f_d(t) * h_T(t)$$

$$= \sum_{k=-\infty}^{\infty} f(nT)\, h_T(t - nT)$$

where

$$h_T(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

Thus we have proved:

<u>Theorem 1</u> (Whitaker sampling theorem)

If $|\hat{f}(\omega)| = 0$ for $|\omega| > \frac{\pi}{T}$, then

$$f(t) = \sum_{k=-\infty}^{\infty} f(nT) \cdot h_T(t - nT)$$

The Nyquist Rate

Suppose a signal has maximum frequency ω_0. At rate does it need to be sampled for perfect reconstruction?

$$\omega_0 \le \frac{\pi}{T} \iff \frac{1}{T} \geq \frac{\omega_0}{\pi}$$

sampling rate

It is also common to express frequency in terms of cycles per second,

$$\underbrace{2\pi f}_{\text{cycles/sec (Hz)}} = \underbrace{\omega}_{\text{radians/sec}}$$

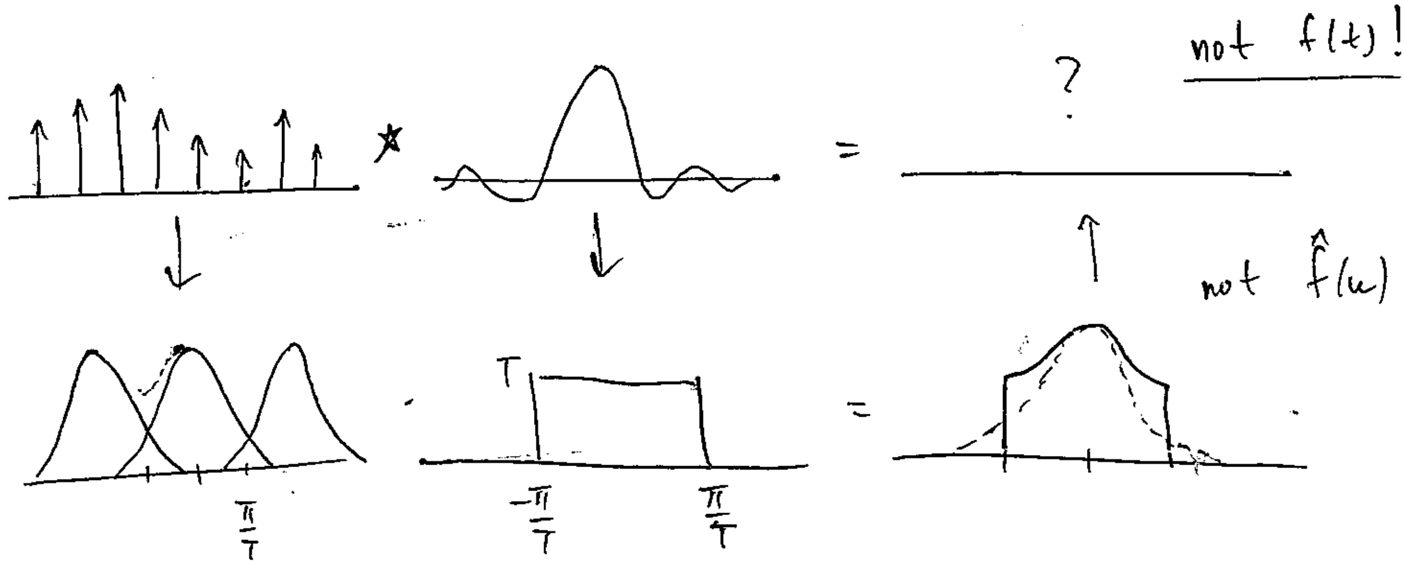
Then we require

$$2\pi f_0 \le \frac{\pi}{T} \iff \frac{1}{T} \ge 2 f_0$$

which says we need to sample at least twice the maximum frequency (in Hz). Then $2f_0$ is called the Nyquist rate.

Aliasing

What happens if we sample below the Nyquist rate?



When it is not possible to sample at the Nyquist rate, an <u>anti-aliasing</u> filter may be applied before sampling

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow \tilde{f}(t)$$

$$\hat{h}(\omega) = \mathbb{1}_{[-\omega_0, \omega_0]}(\omega),$$

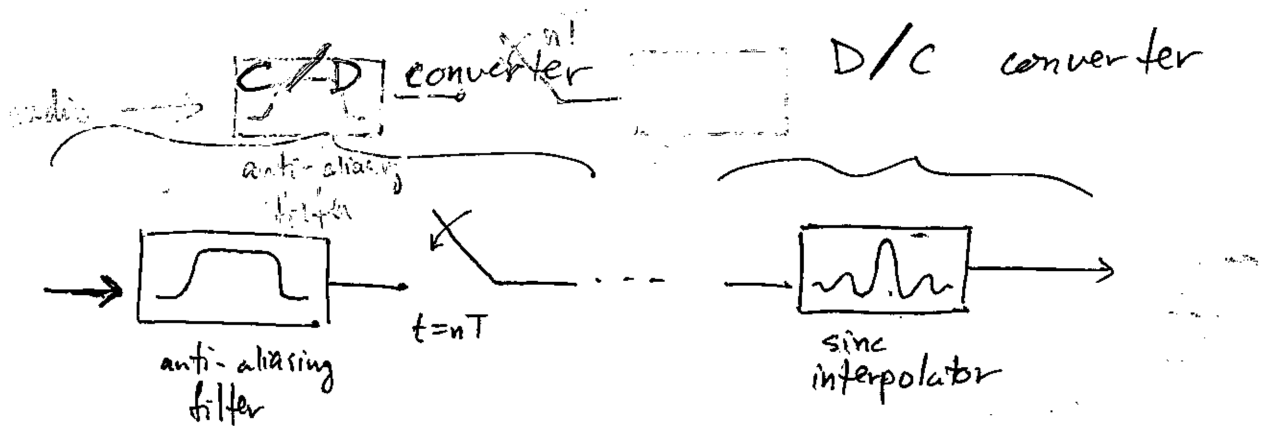
where $\omega_0 = \frac{\pi}{T}$.

<u>Example</u> | CDs

Human audio range: 20 kHz

CD sampling rate: 44.1k Hz

Why the surplus? In practice, ideal lowpass filter is impossible. Allows for a transition band.



DT FOURIER TRANSFORM

DT Signals

A DT (discrete-time) signal is a sequence

$$\ldots\ f[-2],\ f[-1],\ f[0],\ f[1],\ f[2],\ \ldots,$$

or $\{f[n]\}_{n\in\mathbb{Z}}$ for short.

Just like L^p spaces for CT signals, we have ℓ^p spaces for DT signals:

$$\ell^p(\mathbb{Z}) = \left\{ f : \mathbb{Z} \to \mathbb{C} \ \middle|\ \sum_{n\in\mathbb{Z}} |f[n]|^p < \infty \right\}$$

with norm

$$\|f\|_p := \begin{cases} \left(\sum_{n\in\mathbb{Z}} |f[n]|^p \right)^{1/p}, & 1 \le p < \infty \\ \sup_{n\in\mathbb{Z}} |f[n]|, & p = \infty \end{cases}$$

It can be shown that if $1 \le p < q \le \infty$, then $\ell^p \subseteq \ell^q$.

DTFT

The DTFT of a DT signal is

$$\hat{f}(\omega) := \sum_{n\in\mathbb{Z}} f[n] e^{-i\omega n}$$

Note that $\hat{f} : \mathbb{R} \to \mathbb{C}$. If $f \in \ell^1(\mathbb{Z})$, then

$$\sum_{n\in\mathbb{Z}} |f[n] e^{-i\omega n}| = \sum_n |f[n]| < \infty,$$ and therefore $\hat{f}(\omega)$ converges.

Since $e^{-i(\omega + 2\pi k)n} = e^{-i\omega n} \quad \forall k,$

$\hat{f}(\omega)$ is <u>periodic</u> with period 2π.

Therefore we view $\hat{f}$ as a function on $[-\pi, \pi]$.

This leads us to define the spaces

$$L^p([a,b]) := \left\{ f : [a,b] \to \mathbb{C} \;\middle|\; \int_a^b |f(t)|^p \, dt < \infty \right\}$$

When $p = 2$ we have the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, \overline{g(t)} \, dt, \qquad 1 \le p < \infty$$

Fact | If $1 \le p < q < \infty$, then $L^q([a,b]) \subset L^p([a,b])$

This is not true for $L^p(\mathbb{R})$.

$\boxed{\text{Inversion Formula}}$

Theorem | If $\hat{f}(\omega) \in L^2([-\pi, \pi])$, then

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\omega n} \, d\omega = \langle \hat{f}(\omega), e^{-i\omega n} \rangle$$

See Mallat, Ch 3.

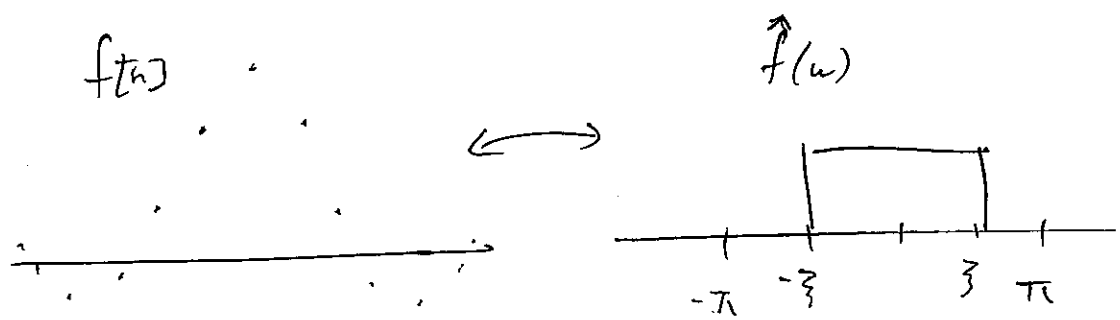
Interpretation: $e^{i\omega n}$ = discrete complex sinusoid with freq ω, $\hat{f}(\omega)$ = amount of that freq. component in $f[n]$

Example If $\hat{f}(\omega) = \mathbb{1}_{[-\xi,\xi]}(\omega)$, $0<\xi<\pi$,

then

$$f[n] = \frac{1}{2\pi}\int_{-\xi}^{\xi} e^{i\omega n}\,d\omega$$

$$= \frac{\sin \xi n}{\pi n}$$



Properties

The DTFT also has a Parseval formula:

$$\|f\|_2^2 = \sum |f[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{f}(\omega)|^2\,d\omega = \|\hat{f}\|_2^2$$

Properties

Note that if we define the CT signal

$$f_c(t) = \sum_{n=-\infty}^{\infty} f[n]\, \delta(t-n)$$

then

$$\hat{f}_c(\omega) = \int_{-\infty}^{\infty} \left(\sum f[n]\, \delta(t-n) \right) e^{-i\omega t}\, dt$$

$$= \sum f[n]\, e^{-i\omega n}$$

$$= \text{DTFT of } \{f[n]\}$$

Therefore, the DTFT inherits all the usual properties of the CTFT.

<u>Theorem</u> If $f, h \in \ell^1(\mathbb{Z})$, then $g = f * h \in \ell^1(\mathbb{Z})$,

where

$$g[n] := \sum_{k=-\infty}^{\infty} f[k]\, h[n-k],$$

and

$$\hat{g}(\omega) = \hat{f}(\omega) \cdot \hat{h}(\omega).$$

DT LTI SYSTEMS

Define the <u>delta</u> function (impulse)

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Suppose L is a DT, LTI system:

DT System: map $\{f : \mathbb{Z} \to \mathbb{C}\}$ to itself

Linear, Time-Invariant: same as CT, but with t restricted to $\mathbb{Z}$.

The <u>impulse response</u> of L is

$$h[n] = L\{\delta[n]\}.$$

If $f[n]$ is any DT signal, then

$$f[n] = \sum_{k=-\infty}^{\infty} f[k]\,\delta[n-k]$$

$$\Rightarrow L\{f[n]\} = \sum f[k]\, L\{\delta[n-k]\}$$
$$= \sum f[k]\, h[n-k]$$
$$= \sum h[k]\, f[n-k] \quad = h * f\,[n]$$

Therefore,

$$L\{f[n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}(\omega)\,\hat{f}(\omega)\,e^{i\omega n}\,d\omega$$

frequency domain filter

Causal system: $h[n] = 0$ for $n < 0$

Stable system: $h \in \ell^1(\mathbb{Z}) \iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

The design of discrete-time filters is taught in 451.

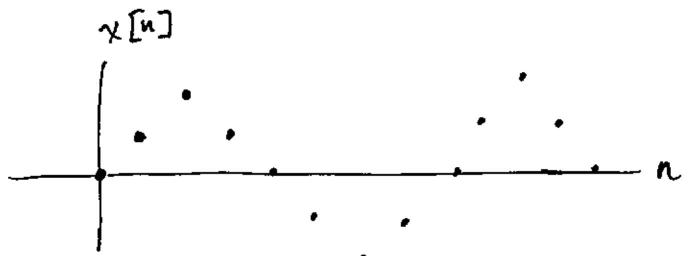
SAMPLING RATE CONVERSION

In many applications, especially audio, it is desirable to change the sampling rate. We could reconstruct an analog signal and resample, but in practice the appoximate interpolation introduces errors. We would like to operate entirely in the DT domain.

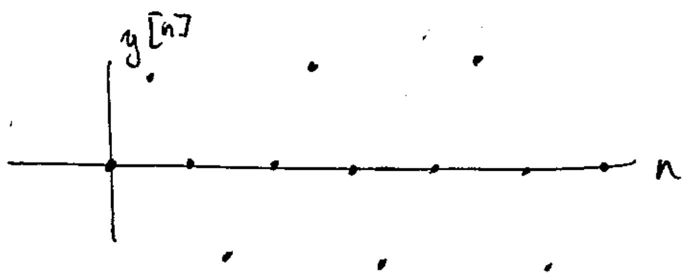
Downsampling

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y[n] \quad , \qquad M \geq 1 \text{ an integer}$$

$$y[n] = x[M \cdot n]$$



Is this system linear? time invariant?



How is $\hat{y}(\omega)$ related to $\hat{x}(\omega)$?

Consider the case $M=2$.

Define $r[m] = \frac{1}{2}(1+(-1)^m) = \begin{cases} 1 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$

Then

$$\hat{y}(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-i\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[2n] e^{-i\omega n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot r[m] e^{-i\left(\frac{\omega}{2}\right)\cdot m}$$

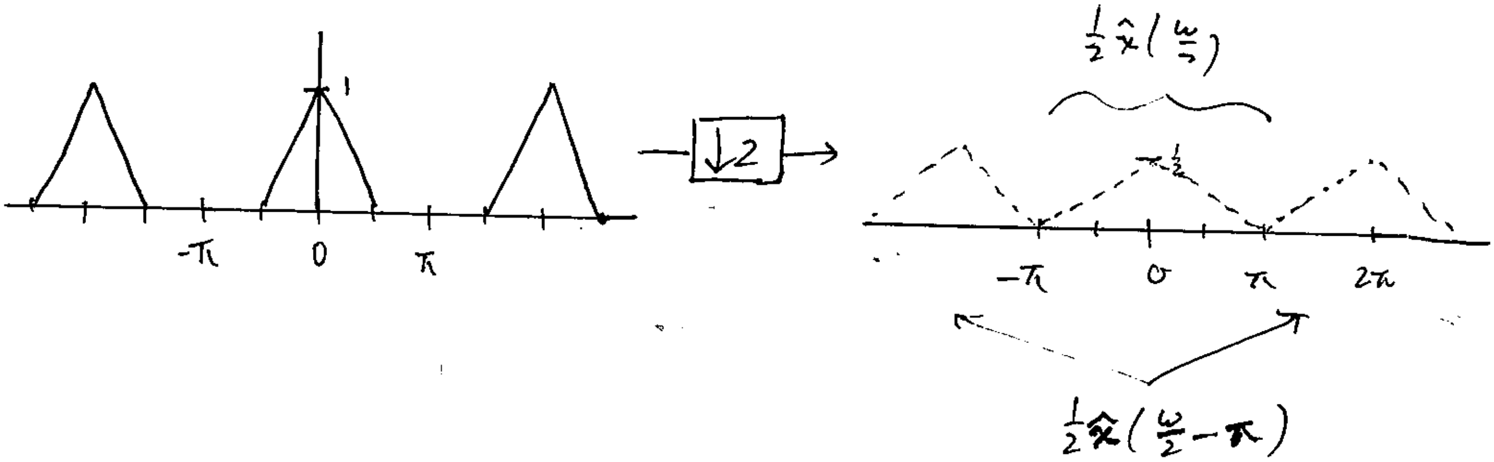
$$[m = 2n]$$

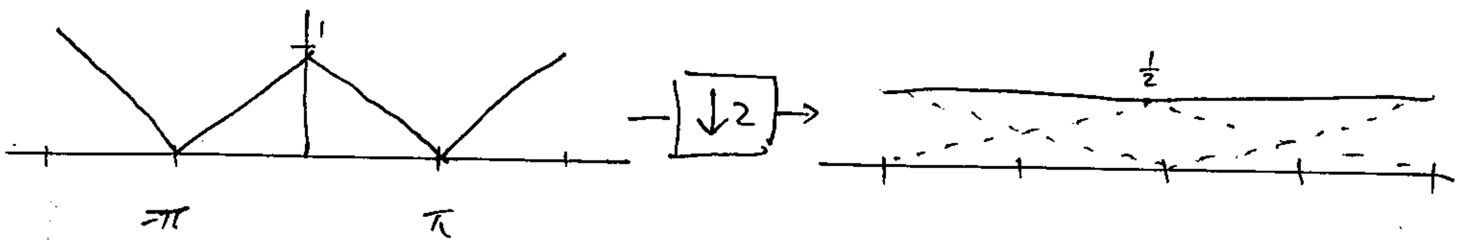
$$= \frac{1}{2} \sum_{m=-\infty}^{\infty} x[m] e^{-i\left(\frac{\omega}{2}\right)m} + \frac{1}{2} \sum x[m] (-1)^m e^{-i\left(\frac{\omega}{2}\right)m}$$

$$= \frac{1}{2}\hat{x}\left(\frac{\omega}{2}\right) + \frac{1}{2} \sum x[m] e^{-i\pi m} e^{-i\left(\frac{\omega}{2}\right)m}$$

$$= \frac{1}{2}\hat{x}\left(\frac{\omega}{2}\right) + \frac{1}{2} \sum x[m] e^{-i\left(\frac{\omega}{2}-\pi\right)m}$$

$$= \underbrace{\frac{1}{2}\hat{x}\left(\frac{\omega}{2}\right)}_{\text{dilated spectrum}} + \underbrace{\frac{1}{2}\hat{x}\left(\frac{\omega}{2}-\pi\right)}_{\text{dilated, shifted spectrum}}$$





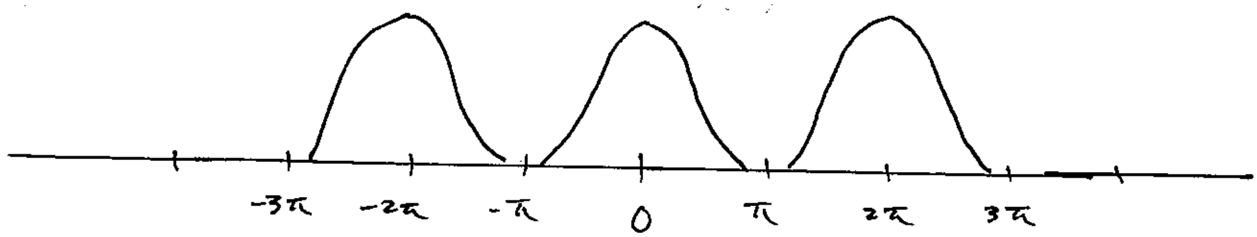
Consider now general M. How can the previous argument be generalized?

Introduce $r[m] = \begin{cases} 1 & \text{if } m = Mn \\ 0 & \text{else} \end{cases} = \frac{1}{M}\sum_{k=0}^{M-1} e^{i\cdot 2\pi\cdot\frac{mk}{M}}$

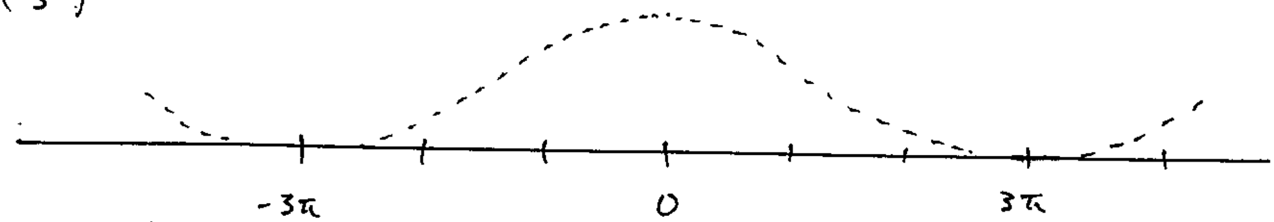
Then
$$\hat{y}(\omega) = \sum_n x[M\cdot n]\cdot e^{-i\omega n}$$
$$= \sum_m x[m]\, r[m]\, e^{-i(\frac{\omega}{M})m} \qquad [m = Mn]$$
$$= \frac{1}{M}\sum_{k=0}^{M-1}\sum_m x[m]\, e^{-i(\frac{\omega}{M}-\frac{2\pi k}{M})m}$$
$$= \frac{1}{M}\sum_{k=0}^{M-1}\hat{x}\left(\frac{\omega}{M}-\frac{2\pi k}{M}\right)$$
$$= \frac{1}{M}\sum \hat{x}\left(\frac{\omega-2\pi k}{M}\right)$$

$M=3$

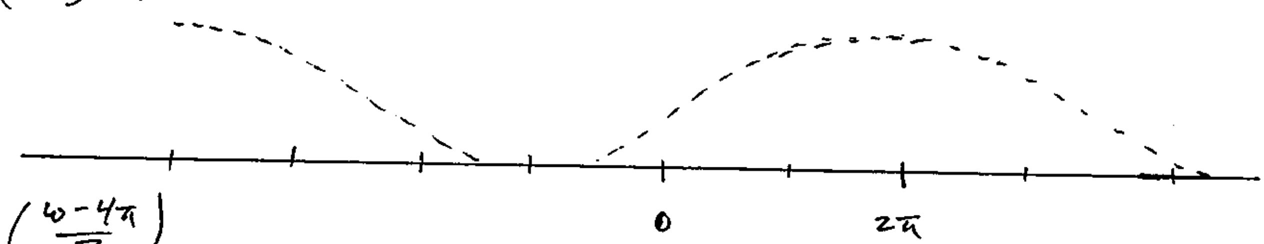
$\hat{x}(\omega)$



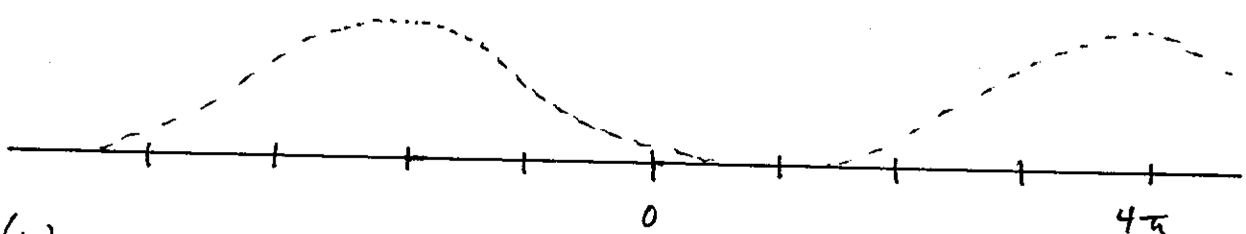
$\hat{x}\left(\frac{\omega}{3}\right)$



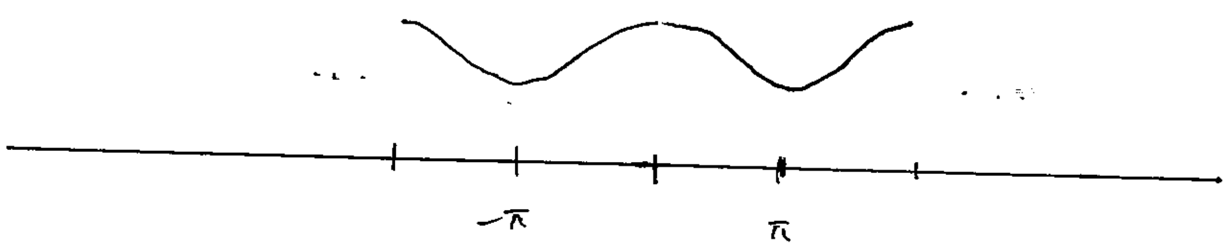
$\hat{x}\left(\frac{\omega-2\pi}{3}\right)$



$\hat{x}\left(\frac{\omega-4\pi}{3}\right)$

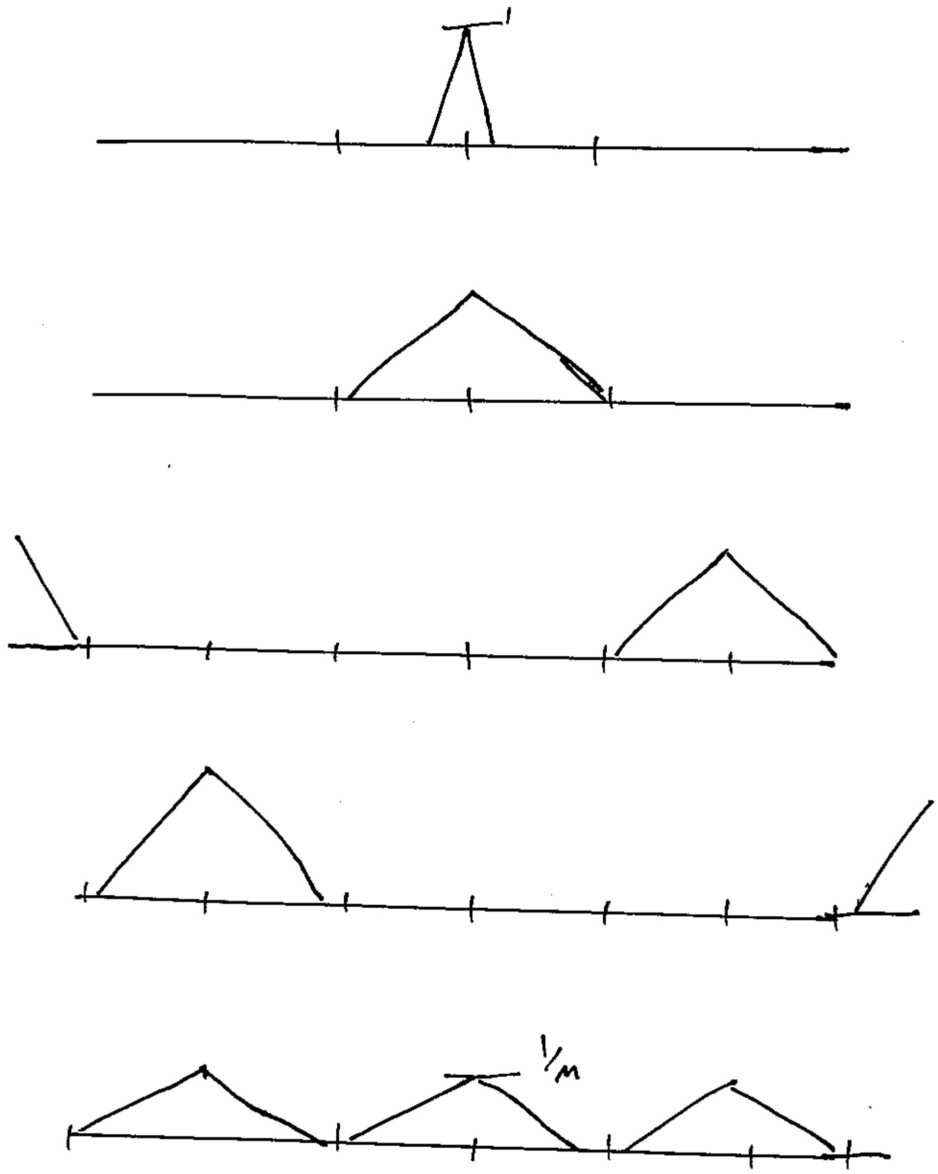


$\hat{y}(\omega)$



Suppose $\hat{x}(\omega) = 0$ for $|\omega| > \frac{\pi}{M}$

$M=3$



$\Rightarrow$ on $[-\pi, \pi]$,

$$\hat{y}(\omega) = \frac{1}{M} \hat{x}\left(\frac{\omega}{M}\right)$$

When x is bandlimited to $[-\frac{\pi}{M}, \frac{\pi}{M}]$, how can we recover x from y? We need a way compress the spectrum of y.

Upsampling

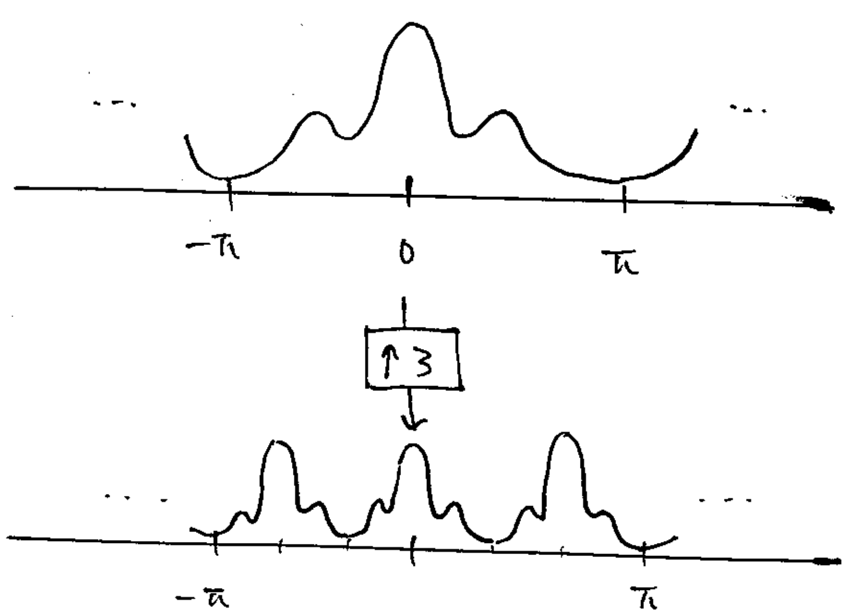
$$x[n] \longrightarrow \boxed{\uparrow M} \longrightarrow y[n]$$

$$y[n] := \begin{cases} x[n/M] & \text{if } n \text{ is a multiple of } M \\ 0 & \text{else} \end{cases}$$

$$\hat{y}(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-i\omega n}$$

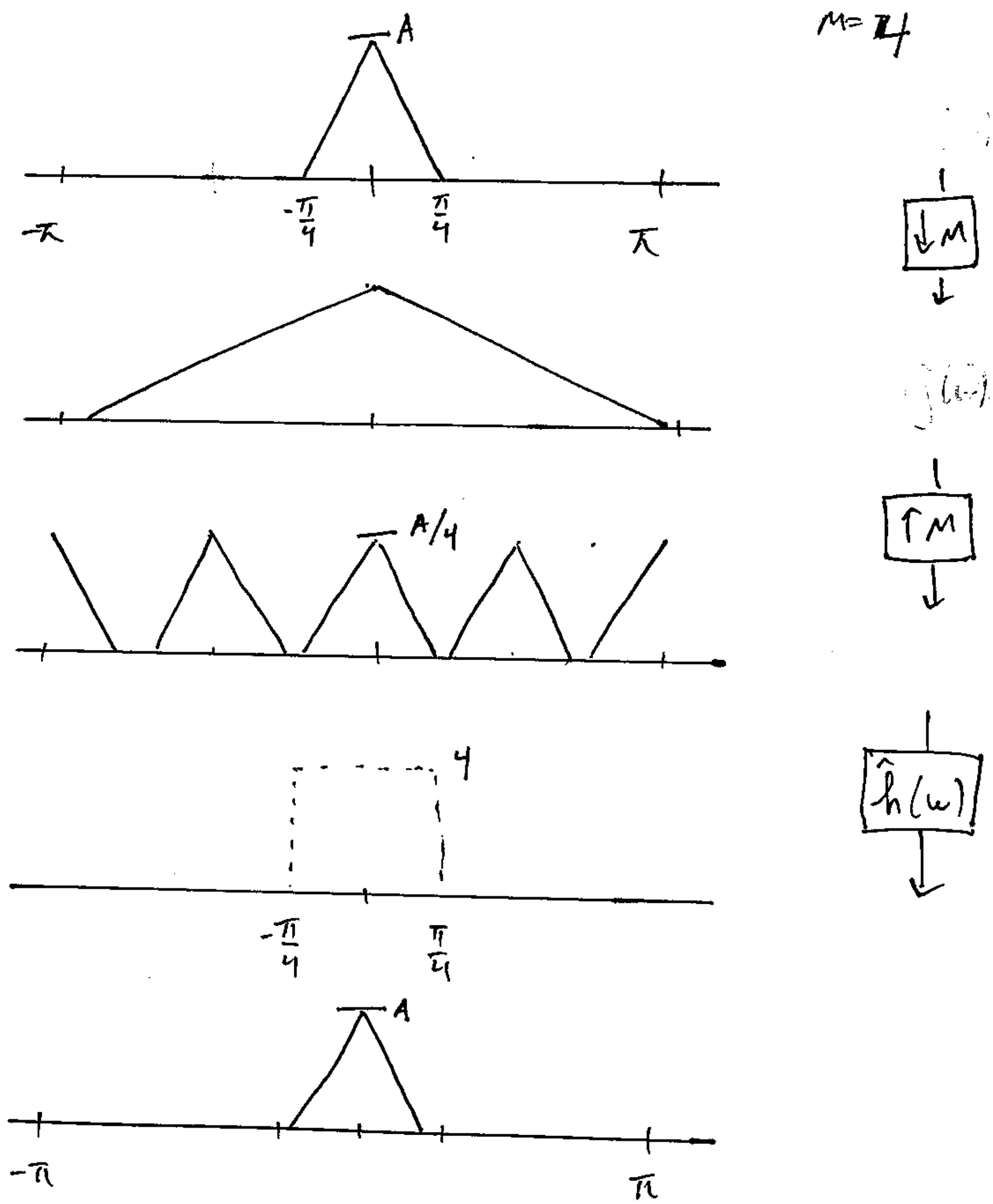
$$= \sum_{m=-\infty}^{\infty} x[m] e^{-i\omega \cdot m \cdot M}$$

$$= \hat{x}(\omega \cdot M)$$



So how can we recover x from a downsampled version?

Assume $\hat{x}(\omega) = 0$ for $|\omega| > \frac{\pi}{M}$.



What is this saying in the time domain?

$\boxed{\text{Interpolation}}$

Consider the following system

$$x[n] \longrightarrow \boxed{\uparrow M} \longrightarrow \boxed{\hat{h}_M(\omega)} \longrightarrow y[n]$$

where

$$\hat{h}_M(\omega) := M \cdot \mathbb{1}_{[-\frac{\pi}{M}, \frac{\pi}{M}]}(\omega).$$

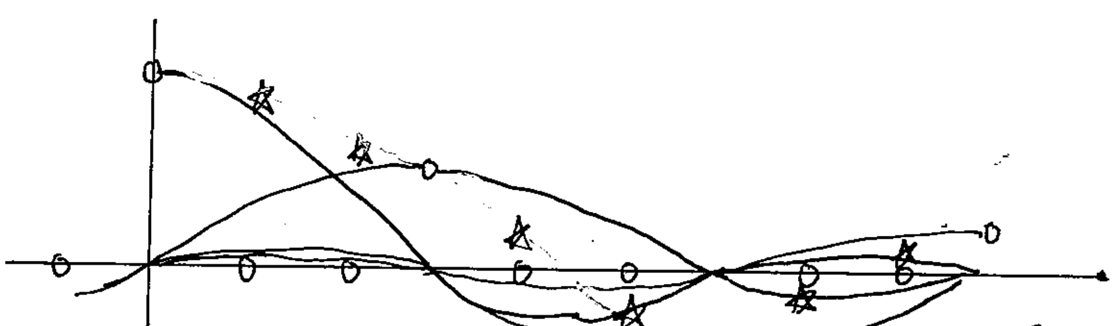
The impulse response of $\hat{h}_M(\omega)$ is

$$h_M[n] = \frac{\sin(\pi n / M)}{\pi n / M}.$$

Therefore

$$y[n] = \left(\sum_{k=-\infty}^{\infty} x[k]\, \delta[n - kM] \right) * h_M[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h_M[n - kM]$$



Therefore, this stage interpolates the upsampled signal to fill in the zero values. Note that at multiples of M, the interpolation is exact, i.e. $y[n] = x[n/M]$, because

$$h[0] = 1, \qquad h[n] = 0, \quad n = \pm M, \pm 2M, \pm 3M, \ldots$$

Now we can analyze the entire system:

$\boxed{\text{DT Sampling Thm}}$

Suppose $\hat{x}(\omega)$ is bandlimited to $[-\frac{\pi}{M}, \frac{\pi}{M}]$.

Then

$$x[n] = \sum_{k=-\infty}^{\infty} x[Mk] \frac{\sin[\pi(n-kM)/M]}{\pi(n-kM)/M}$$

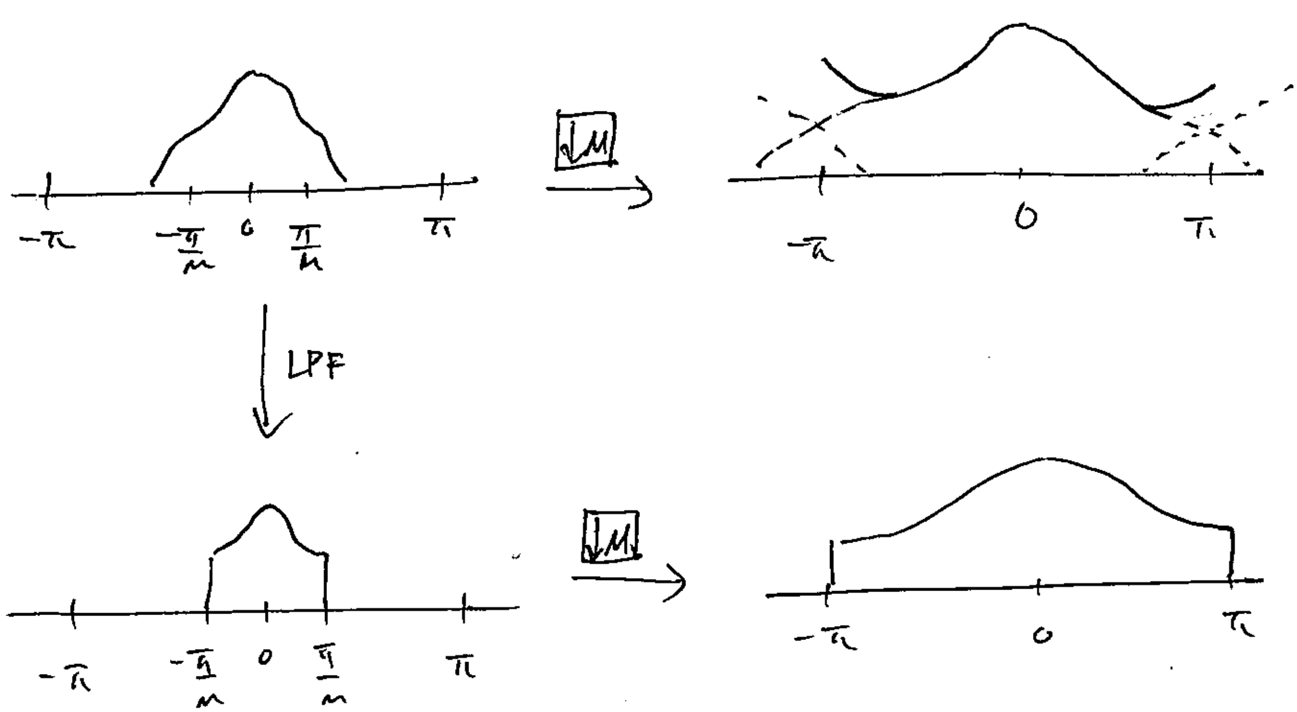
That is, x can be recovered after M-fold downsampling

$$x \longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow M} \longrightarrow \boxed{M \cdot \hat{h}_M(\omega)} \longrightarrow x$$

Aliasing

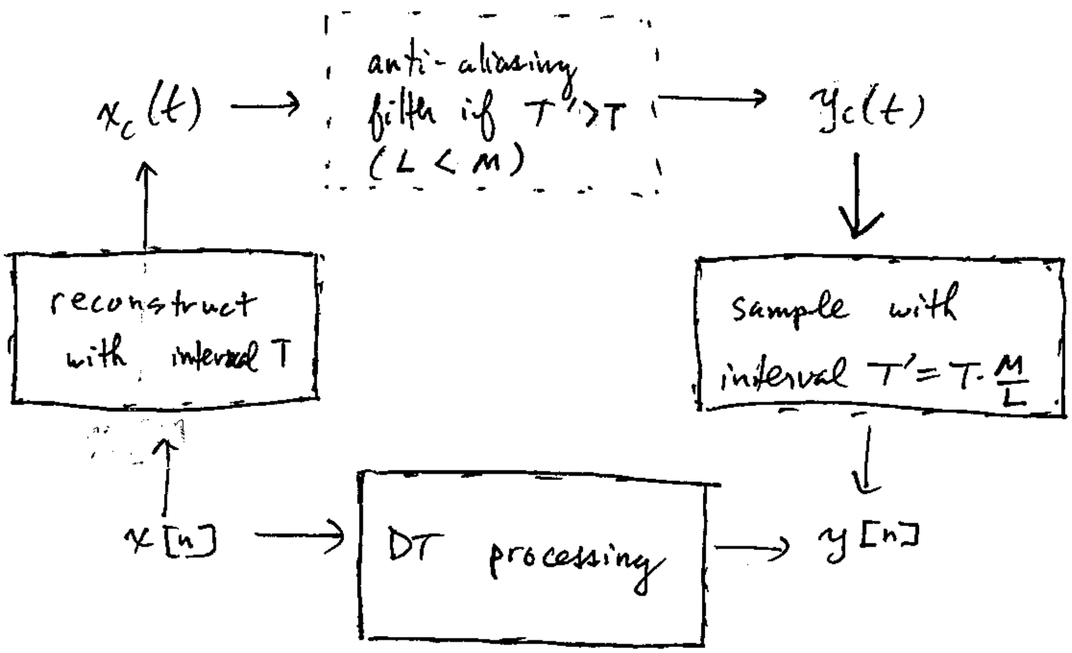
In practice, we may not know that a signal is bandlimited. If $\hat{x}(\omega) \neq 0$ for $\frac{\pi}{M} < |\omega| < \pi$, then downsampling will result in aliasing (high frequencies get folded down to low frequencies)

Therefore, as in CT sampling, we will pre-filter with an anti-aliasing filter with passband $\left[-\frac{\pi}{M}, \frac{\pi}{M}\right]$.



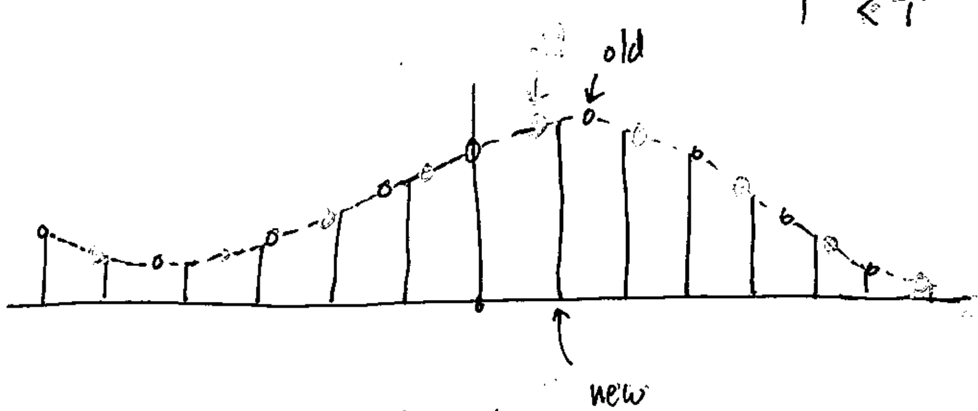
Changing the Sampling Rate by a Rational Factor

Our general goal is to change the sampling rate by a factor L/M.

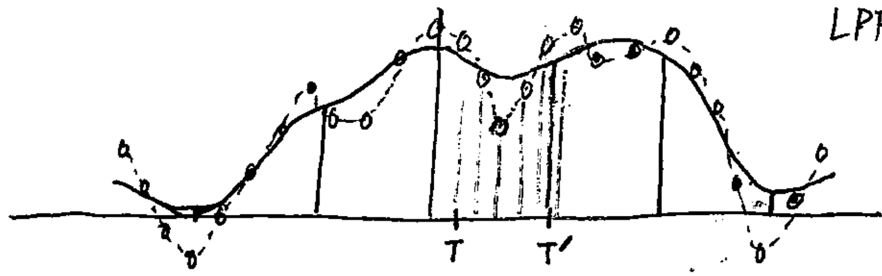


We would like to do our processing in the discrete domain to avoid the imprecision associated with the 2-3 analog stages.

$T' < T$



What does the picture look like for $T' > T$?



If $T' > T$, need to LPF to avoid aliasing

$M = 1$

$$x_c(t) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(t-kT)/T]}{\pi(t-kT)/T}$$

from Whitaker-Shannon interpolation formula

$T' < T$, so no need to apply anti-aliasing filter

$$\Rightarrow y_c(t) = x_c(t)$$

$$\Rightarrow y[n] = y_c\left(n\frac{T}{L}\right)$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left[\pi\left(\frac{nT}{L} - kT\right)/T\right]}{\pi\left(\frac{nT}{L} - kT\right)/T}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot \frac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L}$$

This is precisely the output of the system

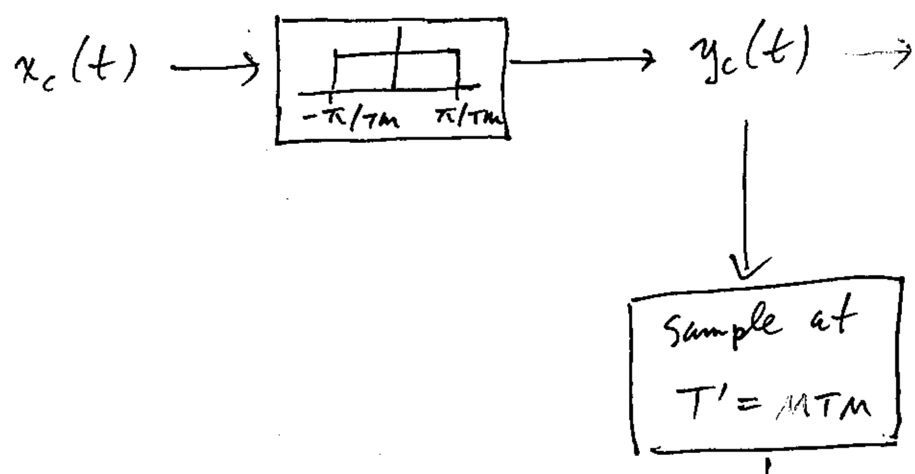
$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{\hat{h}_L(\omega)} \longrightarrow y[n]$$

where

$$\hat{h}_L(\omega) = L \cdot \mathbb{1}_{\left[-\frac{\pi}{L}, \frac{\pi}{L}\right]}(\omega)$$

$L = 1$

Here the anti-aliasing filter is necessary



$$y[n] = y_c(nTM)$$

On the homework you will show that this process is equivalent to

$$x[n] \longrightarrow \boxed{\mathbb{1}_{[-\frac{\pi}{M}, \frac{\pi}{M}]}(\omega)} \longrightarrow \boxed{\downarrow M} \longrightarrow y[n]$$

which is a downsampler preceded by a <u>discrete</u> anti-aliasing filter.

<u>General Case</u>

Decimator:

$$\longrightarrow \boxed{\mathbb{1}_{[-\frac{\pi}{M}, \frac{\pi}{M}]}(\omega)} \longrightarrow \boxed{\downarrow M} \longrightarrow \; =: \; -\boxed{D_M}\rightarrow$$

Interpolator:

$$\longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{L \cdot \mathbb{1}_{[-\frac{\pi}{L}, \frac{\pi}{L}]}(\omega)} \longrightarrow \; =: \; -\boxed{I_L}\rightarrow$$

Therefore

$$\xrightarrow{T} \boxed{D_M} \xrightarrow{T' = TM} \boxed{I_L} \longrightarrow T'' = T\frac{M}{L}$$

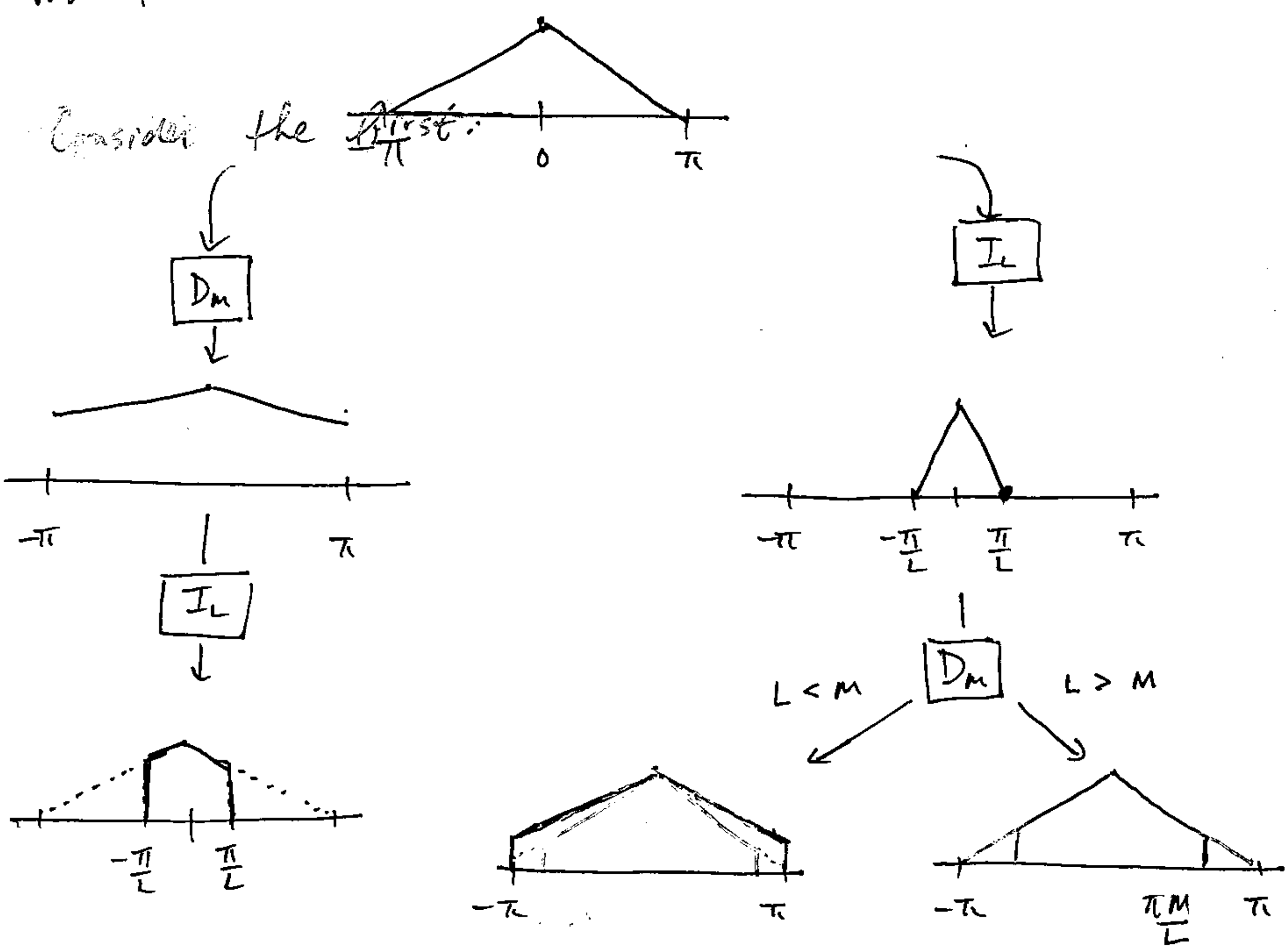
or

$$\xrightarrow{T} \boxed{I_L} \xrightarrow{T' = \frac{T}{L}} \boxed{D_M} \longrightarrow T'' = T\frac{M}{L}.$$

change the sampling rate by a factor of $\frac{M}{L}$.

Which order makes more sense? Does it depend on L, M?

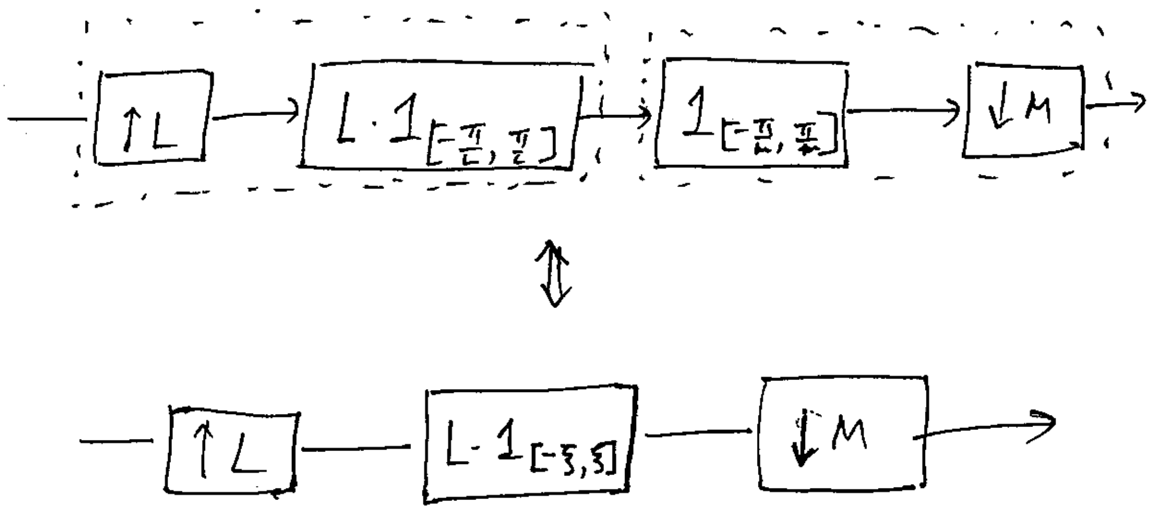
Consider the ~~first~~:



Thus, regardless of whether $L > M$ or $L < M$, we retain more of the signal by applying the interpolator before the decimator.

In short, interpolation shrinks the spectrum, thereby reducing the impact of the decimator's anti-aliasing filter.

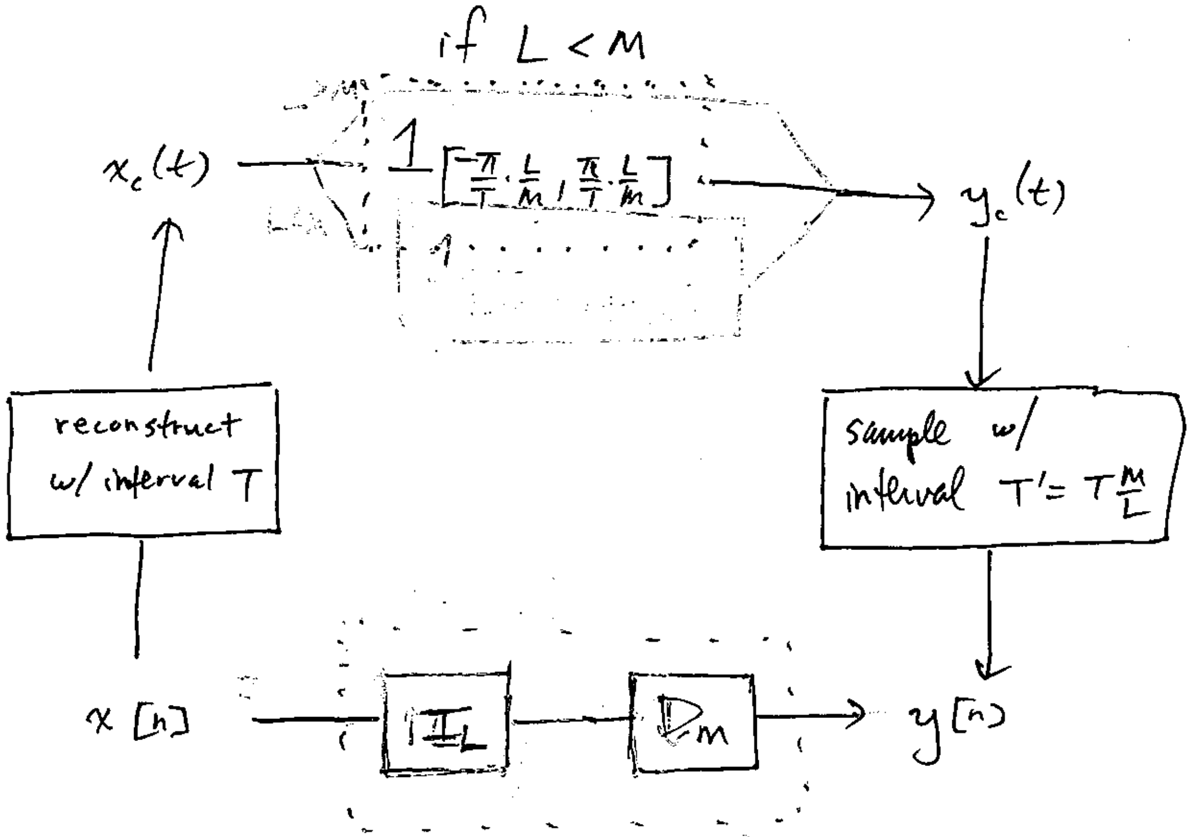
As a bonus, we can combine the interpolation and anti-aliasing filters.



where

$$\xi = \min\left(\frac{\pi}{M}, \frac{\pi}{L}\right)$$

The complete process now looks like this:



On the homework you will formally show that both paths from $x[n]$ to $y[n]$ are the same. The rates are obviously the same, but in fact the signals are identical. E.g, could

Practical Considerations

Suppose we wish to play a CD using a PC sound card. $\uparrow KL \to \downarrow KM$ We need to change rate from 44100 kHz to 8192 some.

The conversion factor is

$$\frac{44100}{8192} = \frac{11025}{2048} = \frac{M}{L}$$

$$\longrightarrow \boxed{\uparrow 2048} \longrightarrow \boxed{\begin{array}{l} \text{gain} = 2048 \\ \text{cutoff} = \frac{\pi}{11025} \end{array}} \longrightarrow \boxed{\downarrow 11025} \longrightarrow$$

Do you see any problems with this system?

- tiny passband $\Rightarrow$ huge filter length
- only 1 out of every 2048 interpolator outputs is nonzero
- only 1 out of every 11,025 decimator inputs is not discarded

Solution: change the rate in stages.

<u>Polyphase filterbanks</u> can be used to change rates very efficiently

Could show
example, e.g. from
Proakis + Manolakis.
See also Romberg notes.

LINEAR ALGEBRA

Linear algebra is the study of vector space (linear spaces) and linear transformations.

Linear Combinations

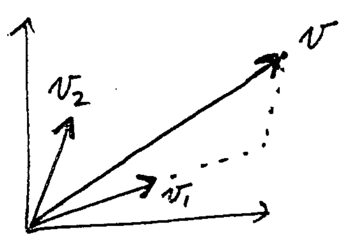
Let V be a vector space with scalars $K = \mathbb{R}$ or $\mathbb{C}$.

We say $v \in V$ is a <u>linear combination</u> of $v_1, \ldots, v_n \in V$ if $\exists\ a_1, \ldots, a_n \in K$ s.t. $v = \sum_{i=1}^{n} a_i v_i$.

Note that n is finite

Examples

(a)



(b) $f(t) = 2t^3 - 3t^2 + 9t - 4$

$f_i(t) = t^i, \quad i = 0, 1, 2, 3$

Consider writing a handout that gives more precise statements and proofs of lin. alg. in terms of basis, e.g. def of basis, max LI set, min span set, etc..

Linear Independence

Let $\mathcal{U} \subseteq V$. We say $\mathcal{U}$ is <u>linearly independent</u> if, for any n, and any $v_1, \dots, v_n \in \mathcal{U}$,

$$\sum a_i v_i = \underline{0} \implies a_i = 0 \quad i = 1, \dots, n.$$

If, on the other hand, $\exists\, v_1, \dots, v_n \in \mathcal{U}$ and $a_1, \dots, a_n \in K$, not all 0, s.t.

$$\sum a_i v_i = 0,$$

we say $\mathcal{U}$ is <u>linearly dependent</u>.

<u>Examples</u> (a) $V = \mathbb{R}^3$, $K = \mathbb{R}$, $\mathcal{U} = \left\{ \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix}, \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix}, \begin{bmatrix} u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} \right\}$

LI $\Leftrightarrow$ non-coplanar

(b) $f_1(t) = \cos(t)$, $f_2(t) = \sin(t)$, $f_3(t) = \cos(t + .2)$

$$f_3(t) = \cos(.2) \cdot \cos(t) - \sin(.2)\sin(t)$$
$$= a_1 f_1(t) + a_2 f_2(t)$$

$\implies$ LD

Span

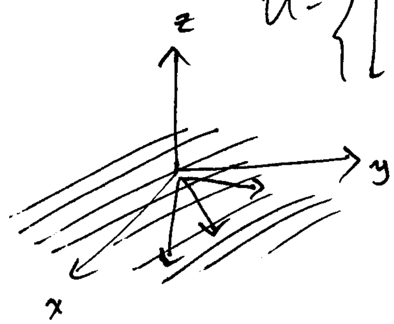
Let $\mathcal{U} \subseteq V$. The <u>span</u> of $\mathcal{U}$ is the set of all LC's of vectors in $\mathcal{U}$, i.e.

$$\text{span}(\mathcal{U}) = \langle \mathcal{U} \rangle := \left\{ \sum_{i=1}^{n} a_i v_i \;\middle|\; \begin{matrix} n \in \mathbb{N} \\ v_1, \dots, v_n \in \mathcal{U} \\ a_1, \dots, a_n \in K \end{matrix} \right\}$$

Example 1

(a)

$$\mathcal{U} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$



span $(\mathcal{U})$ = x-y plane

(b) $\mathcal{U} = \{ 1, t-2, 2t^2+3t-4, 3t^2-1 \}$

span $(\mathcal{U})$ = { polynomials in t of degree ≤ 2 }

Basis

A (Hamel) <u>basis</u> of V is a subset $\mathcal{U}$ s.t. $\mathcal{U}$ is LI and $V = \langle \mathcal{U} \rangle$.

<u>Example</u> (a) $V = \mathbb{R}^n$, $K = \mathbb{R}$

$$u_i = [0, \dots, 0, 1, 0, \dots, 0]^T$$

$\uparrow$ i^{th} position

$\boxed{\text{Standard basis}}$

(b) $V = \{$ all polynomials w/ complex coefficients of degree $\leq 5 \}$, $K = \mathbb{C}$

$$\mathcal{U} = \{1, t, t^2, t^3, t^4, t^5\}$$

$\hookrightarrow$ why are these LI?

fundamental thm of algebra: if $\sum_{i=0}^{5} a_i t^i = p(t)$ and some $a_i \neq 0$, then $p(t)$ has 5 roots.

Include

TFAE

(i) $\mathcal{U}$ is a basis

(ii) every $v \in V$ has unique ....

(iii) $\mathcal{U}$ is a maximal LI set

(iv) $\mathcal{U}$ is a minimal spanning set

don't prove, provide reference.

Theorem | $\mathcal{U}$ is a basis for V $\iff$ every $v \in V$ has a unique representation (up to order)

$$v = \sum_{i=1}^{n} a_i v_i$$

where $v_i \in \mathcal{U}$, $a_i \in K$, $a_i \neq 0$.

Proof |

- existence: since $V = \text{span}(\mathcal{U})$, and by definition of span
- uniqueness: assume $\exists\, v \in V$ s.t.

$$v = \sum_{i=1}^{m} a_i u_i = \sum_{j=1}^{n} b_j v_j$$

where $u_i, v_j \in V$, $a_i, b_j \in K$, $a_i, b_i \neq 0$.

Then

$$a_1 u_1 + \cdots + a_m u_m + (-b_1) v_1 + \cdots + (-b_n) v_n = \underline{0}$$

Since $\mathcal{U}$ is LI, this implies that (perhaps after reordering the v_i's),

$$u_i = v_i \quad \text{and} \quad a_i = b_i \quad \forall i$$

$\Rightarrow$ representation is unique.

Dimension

Theorem If $\mathcal{U}_1$ and $\mathcal{U}_2$ are bases for V, then $|\mathcal{U}_1| = |\mathcal{U}_2|$.

Therefore, we can define the *dimension* of a vector space to be the cardinality of any basis, denoted $\dim(V)$.

Facts Let V be a vector space

(a) If $\mathcal{U}_L \subseteq V$ is LI, and $\mathcal{U}_S \subseteq V$ spans V, then $|\mathcal{U}_L| \leq |\mathcal{U}_S|$

(b) If $\mathcal{U} \subseteq V$ is LI, then $\mathcal{U}$ can be extended to a basis

(c) If $\mathcal{U} \subseteq V$ spans V, then $\mathcal{U}$ contains a basis.

(d) If $\dim(V) = N < \infty$, $\mathcal{U}$ is LI, and $|\mathcal{U}| = N$, then $\mathcal{U}$ is a basis.

(e) If $\dim(V) = N < \infty$, $\operatorname{span}(\mathcal{U}) = V$, and $|\mathcal{U}| = N$, then $\mathcal{U}$ is a basis.

Example $V = \mathbb{C}^N$, $K = \mathbb{C}$

standard basis $\Rightarrow$ $\dim(V) = N$.

Consider the "DFT basis," $\mathcal{U} = \{u_0, \ldots, u_{N-1}\}$ where

$$u_k = \begin{bmatrix} 1 \\ e^{i2\pi \cdot \frac{k}{N}} \\ \vdots \\ e^{i2\pi \cdot \frac{k \cdot (N-1)}{N}} \end{bmatrix} = \begin{bmatrix} u_k[0] \\ \vdots \\ u_k[N-1] \end{bmatrix}$$

We have seen that for any $f = [f[0] \, \cdots \, f[N-1]]^T \in \mathbb{C}^N$,

$$f[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{f}[k] \, e^{i2\pi \frac{kn}{N}}, \qquad n = 0, \ldots, N-1.$$

Expressing all of these scalar equations in a single vector equation we have

$$f = \sum_{k=0}^{N} a_k u_k,$$

where $a_k = \frac{1}{2\pi} \hat{f}[k]$.

Thus, $\mathrm{span}(\mathcal{U}) = V$. Since $|\mathcal{U}| = N$, we deduce that $\mathcal{U}$ is a basis.

Subspaces

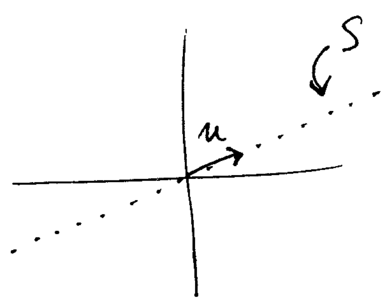
A <u>subspace</u> is a set $S \subseteq V$ that is closed under both vector addition and scalar multiplication.

Note that a subspace is a vector space in its own right. In particular, it has a dimension.

<u>Example 1</u> $V = \mathbb{R}^2$, $K = \mathbb{R}$

(a)

$$S = \langle u \rangle$$



(b) In general, if $\mathcal{U} \subseteq V$ is any set, and $S = \operatorname{span}(\mathcal{U})$, then S is a subspace.

(c) Bandlimited signals

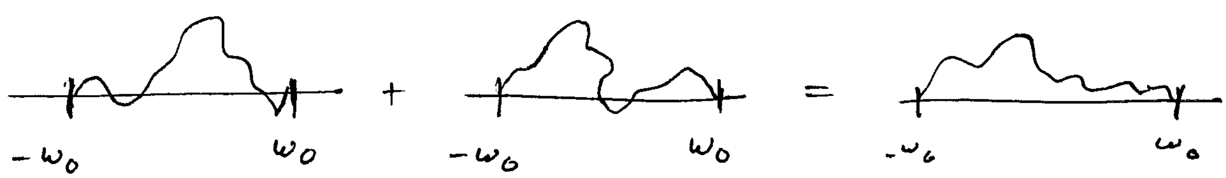
$V = L^2(\mathbb{R})$, $K = \mathbb{C}$

$$S = \{ f \in L^2(\mathbb{R}) \mid \hat{f}(\omega) = 0 \text{ for } |\omega| > \omega_0 \}$$

If $f_1, f_2 \in S$, $a_1, a_2 \in \mathbb{C}$, and $f = a_1 f_1 + a_2 f_2$,

then
$$\hat{f}(\omega) = a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega) = 0$$

for $|\omega| > \omega_0$.



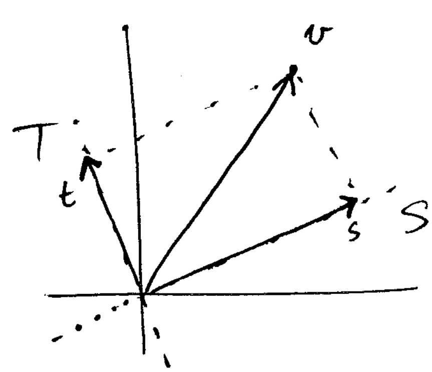
Direct Sum

Let $S, T \subseteq V$ be subspaces. We say V is the <u>(inner) direct sum</u> of S and T, written

$$V = S \oplus T$$

if, $\forall v \in V$, $\exists$ unique $s \in S$, $t \in T$, s.t.

$$v = s + t.$$



Fact | $V = S \oplus T$ iff

(i) $\forall v \in V$, $\exists\, s \in S, t \in T$ s.t. $v = s + T$

(ii) $S \cap T = \{0\}$

$\uparrow$ trivial subspace

If $V = S \oplus T$, we say S and T are <u>compliments</u>.

Example | $V = \mathbb{R}^{n \times n} := \{ n \times n \text{ matrices w/ real entries} \}$

$S = \{ A \in V \mid A^T = A \}$ (symmetric)

$T = \{ A \in V \mid A^T = -A \}$ (skew-symmetric)

Are S + T subspaces?

Let $A \in \mathbb{R}^{n \times n}$. We can write

$$A = \underbrace{\tfrac{1}{2}(A + A^T)}_{s} + \underbrace{\tfrac{1}{2}(A - A^T)}_{t}$$

This establishes (i).

Now suppose $A \in S \cap T$. Then $A^T = A$ and $A^T = -A$

$\Rightarrow A = -A \Rightarrow A = \underline{0}$. This establishes (ii).

Thus, $V = S \oplus T$.

If $V = S \oplus T$, and
$\mathcal{U}_S, \mathcal{U}_T$ are bases for
S, T, then
$\mathcal{U}_S \cup \mathcal{U}_T$ is a basis
for V. Conclude
$\dim(V) = \dim(S) + \dim(T)$

LINEAR TRANSFORMATIONS

Let V, W be vector spaces with the same set of scalars $K = \mathbb{R}$ or $\mathbb{C}$.

A function $L : V \to W$ is a <u>linear transformation</u> if, $\forall u, v \in V$, and $\forall a, b \in K$,

$$L(au + bv) = aL(u) + bL(v)$$

Why do we assume V, W have same scalars?

<u>Examples</u> (a) $V = W = L^2(\mathbb{R})$, $K = \mathbb{C}$

For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$:

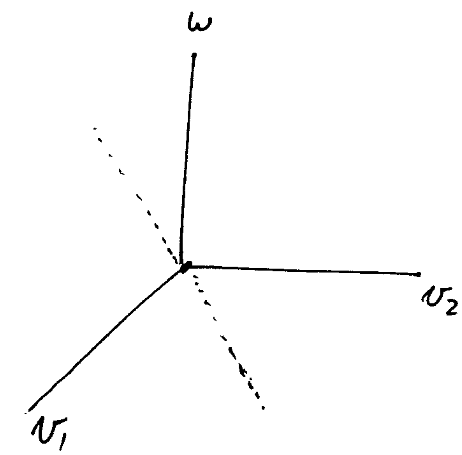
$$L\{af + bg\}(\omega) = \int_{-\infty}^{\infty} (af(t) + bg(t))e^{-i\omega t}\,dt$$

$$= a\int f(t)e^{-i\omega t}\,dt + b\int g(t)e^{-i\omega t}\,dt$$

$$= aL\{f\}(\omega) + bL\{g\}(\omega)$$

For $f, g \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$, use density argument.

(b) $V = \mathbb{R}^2, \quad W = \mathbb{R}, \quad K = \mathbb{R}$



$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longmapsto w = 2v_1 - 3v_2$$

why is this linear?

(c) More generally, suppose $V = K^m$, $W = K$

Consider the transformation

$$L : \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \longmapsto c_1 v_1 + \cdots + c_m v_m = \sum_{i=1}^{m} c_i v_i$$

where $c_1, \ldots, c_m \in K$ are fixed. Then

$$L(au + bv) = L\left(\begin{bmatrix} au_1 + bv_1 \\ \vdots \\ au_m + bv_m \end{bmatrix} \right)$$

$$= \sum c_i (au_i + bv_i)$$

$$= a \sum c_i u_i + b \sum c_i v_i$$

$$= a L(u) + b L(v)$$

What if we added a constant term? Quadratic?

(d) $V = K^m$, $W = K^m$

$$L : \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \longmapsto \begin{bmatrix} c_{11} v_1 + \cdots + c_{1m} v_m \\ \vdots \\ c_{n1} v_1 + \cdots + c_{nm} v_m \end{bmatrix}$$

Why is this linear? Apply previous case to each row.

Matrices

The last example can be represented as a matrix:

$$\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \\ c_{n1} & & c_{nm} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

So every matrix transformation is linear. When L maps one Euclidean space to another, the converse is true.

Theorem If $L : K^m \to K^n$ is linear, then L has a matrix representation.

Proof / Let $\{e_1, \dots, e_m\}$ be the standard basis of K^m.

Let $v \in K^m$ have the (unique). expansion

$$v = v_1 e_1 + \dots + v_m e_m$$

$$= v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + v_m \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

Since L is linear,

$$L(v) = \sum_{j=1}^{m} v_j L(e_j) = w$$

This is a vector equation in K^n.

Let's write it component-wise. Denote

$$L(e_j) = \begin{bmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{bmatrix}, \quad j = 1, \dots, m$$

Then

$$L(v) = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \\ c_{n1} & & c_{nm} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

▨

Inverses

Let $L: V \to W$ be linear. We say L is

- injective / 1-to-1 if

$$u \neq v \Rightarrow L(u) \neq L(v)$$

- surjective / onto if

$$\forall w \in W, \exists v \in V \text{ s.t. } L(v) = w.$$

- bijective if

L is one-to-one and onto,

If L is bijective it has an inverse

$$L^{-1}: W \to V$$

which is defined to be a function satisfying

$$L^{-1}(L(v)) = v \qquad \forall v \in V$$ "left inverse"

$$L(L^{-1}(w)) = w \qquad \forall w \in V$$ "right inverse"

In other words,

$$\left.\begin{aligned} L^{-1} \circ L &= I_V \\ L \circ L^{-1} &= I_W \end{aligned}\right\} \text{identity transformation}$$

<u>Examples</u> (a) CTFT on $L^2(\mathbb{R})$

$$L^{-1}\{\hat{f}\}(t) = \frac{1}{2\pi}\int \hat{f}(\omega)e^{i\omega t}\,d\omega$$

for $\hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

(b)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_W$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_V$$

The problem is that L is not <u>injective</u>

$$L\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

v_3 could be anything, output doesn't change.

(c) $L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $L^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad-bc \neq 0$.

Isomorphisms

If $L: V \to W$ is a bijection, we also call L an <u>isomorphism</u>, and say that V and W are <u>isomorphic</u>, denoted $V \approx W$.

<u>Theorem</u> If $L: V \to W$ is an isomorphism, and B is a basis for V, then $L(B) = \{L(b) \mid b \in B\}$ is a basis for V.

Proof: Must show (i) $L(B)$ is LI (ii) $\text{span}(L(B)) = w$.

(i) Suppose $\sum_{i=1}^{n} a_i L(b_i) = 0$. in W.

Apply L^{-1} to both sides. Since L^{-1} is linear (homework)

$0 = L^{-1}(0) = \sum a_i b_i$ in V

$\Rightarrow a_i = 0 \ \forall i$, since B is LI.

(ii) Let $w \in W$, and let $v \in V$ s.t. $L(v) = w$.

Since $V = \text{span}(B)$, we can write $v = \sum a_i b_i$.

Then $w = \sum a_i L(b_i)$.

<u>Corollary</u> If $V \approx W$, then $\dim(V) = \dim(W)$.

<u>Corollary</u> Only square matrices are invertible.

$\boxed{\text{Subspaces Associated with a LT}}$ $\quad L: V \to W$, linear

- <u>range / image</u>

$$\mathcal{R}(L) = \{ w \in W \mid w = L(v) \text{ for some } v \in V \}$$

- <u>nullspace / kernel</u>

$$\mathcal{N}(L) = \{ v \in V \mid L(v) = \underline{0} \}$$

Why are these subspace?

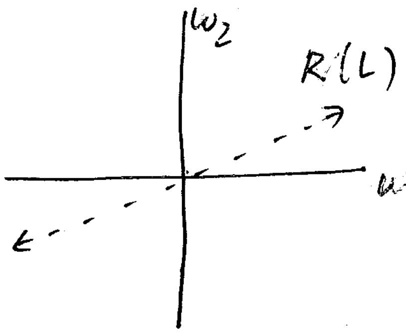
<u>Example</u> $\quad L = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$

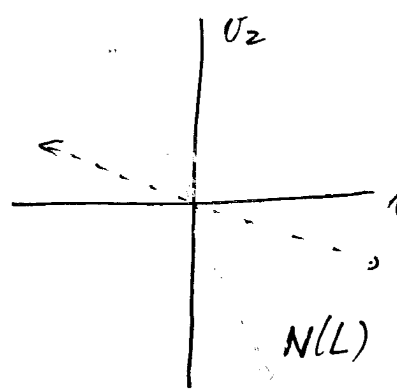
range: $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 6v_2 \\ v_1 + 3v_2 \end{bmatrix} = (v_1 + 3v_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow R(L) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$

nullspace: $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (v_1 + 3v_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$

$\Rightarrow v_1 = -3v_2 \Rightarrow N(L) = \text{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$





If L is a matrix, $L = [\underline{c}_1 \ \underline{c}_2 \ \dots \ \underline{c}_m]$, $\underline{c}_j = \begin{bmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{bmatrix}$

then $Lv = v_1 \cdot \underline{c}_1 + \dots + v_m \underline{c}_m$.

Therefore,

$$R(L) = \operatorname{span}(\underline{c}_1, \dots, \underline{c}_m) = \operatorname{colspan}(L).$$

which is also called the <u>column span</u> of L.

Therefore

- $N(L) = \{\underline{0}\} \iff \underline{c}_1, \dots, \underline{c}_m$ are LI
- Given $w \in W$, the equation

  $$Lv = w$$

  has a solution $\iff w \in \operatorname{colspan}(L)$.

More generally, for any LT $L: V \to W$,

- L is injective $\iff$ $N(L) = \{0\}$.
- L is surjective $\iff$ $R(L) = W$.

Let's prove the first statement.

($\Rightarrow$) Suppose L is injective. We know

$L(0) = 0 \cdot L(0) = 0$. So if $L(v) = 0$,

then $v = 0$. Thus $N(L) = \{0\}$.

($\Leftarrow$) Suppose $N(L) = \{0\}$. If $L(u) = L(v)$,

then $0 = L(u) - L(v) = L(u-v)$

$\Rightarrow u - v \in N(L) \Rightarrow u - v = 0 \Rightarrow u = v$.

$$\boxed{\text{Rank Plus Nullity Theorem}}$$

If $L : V \to W$ is linear, we define

$$\text{rank}(L) = \dim(R(L))$$
$$\text{null}(L) = \dim(N(L)).$$

any algebraic complement

Theorem/ For any LT $L : V \to W$, then $N(L)^c \approx R(L)$,
and therefore
$$\text{rank}(L) + \text{null}(L) = \dim(V).$$

Example/ $L = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 1 & 7 \end{bmatrix}$

$\dim(V) =$

$\text{rank}(L) =$

$\text{null}(L) =$

Proof: Let $N(L)^c$ be a complement of $N(L)$, so that
$$V = N(L) \oplus N(L)^c.$$

Then $\dim(V) = \text{rank}(L) + \dim(N(L)^c)$.

We will show $N(L)^c \approx R(L)$.

Let $L^c : N(L)^c \to R(L)$ be the restriction of L to $N(L)^c$. We will show it is an isomorphism.

(i) Injective: If $L^c(v) = 0$ for some $v \in N(L)^c$, then $v \in N(L) \Rightarrow v \in N(L) \cap N(L)^c = \{0\}$ $\Rightarrow v = 0$.

(ii) Surjective: Let $w \in R(L)$.. Then $\exists v \in V$ s.t. $L(v) = w$. Since $V = N(L) \oplus N(L)^c$, $\exists$ $s \in N(L)$, $t \in N(L)^c$ s.t. $v = s + t$.

Then $w = L(v) = L(s+t) = L(s) + L(t) = L(t)$.

<u>Corollary</u> Let $L : K^n \to K^n$, $K = \mathbb{R}$ or $\mathbb{C}$.

Then

$$\begin{aligned} L \text{ is invertible} &\Leftrightarrow \operatorname{rank}(L) = n \\ &\Leftrightarrow \operatorname{null}(L) = 0 \\ &\Leftrightarrow \text{the columns of } L \text{ are LI} \\ &\Leftrightarrow \text{'' '' '' '' span } K^n \end{aligned}$$

Do you know any other equivalent conditions?

- $\det(L) \neq 0$
- $\forall w \in \mathbb{K}^n$, $Lv = w$ has a unique solution
- Gaussian elimination $\Rightarrow$ identity

ORTHOGONALITY

Recall, an <u>inner product space</u> is a vector space V equipped with an <u>inner product</u>, which is function $\langle \cdot, \cdot \rangle : V \times V \to K$ such that $\forall\, x, y, z \in V$, $a \in K$

IP1	$\langle x, y \rangle = \overline{\langle y, x \rangle}$	conj- symm.
IP2	$\langle a x, y \rangle = a \langle x, y \rangle$	linearity in first variable
IP3	$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$	
IP4	$\langle x, x \rangle \geq 0$ with equality iff $x = 0$	pos. def.

<u>Theorem</u> (Cauchy-Schwarz inequality)

Let V be an IPS. For any $x, y \in V$,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

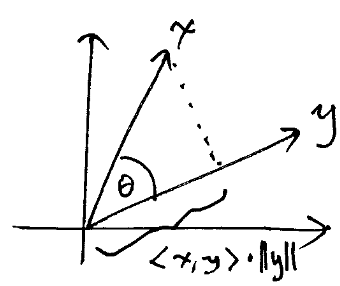
where $\|x\| := \sqrt{\langle x, x \rangle}$, with equality iff x, y are LD.

This implies that for all x, y

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|} \leq 1.$$

Recall that for $x, y \in \mathbb{R}^2$ with $\langle x, y \rangle = x_1 y_1 + x_2 y_2$,

$$\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos\theta.$$



~~Therefore we can define the angle between two~~

We generalize this formula to general $\overset{\text{real}}{\wedge}$ IPSs by defining the angle between two vectors.

$$\theta = \cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\cdot\|y\|}\right)$$

which makes sense by the CS ineq.

Thus we can define the notion of the angle between non-geometric quantities like polynomials.

We won't be too concerned about general angles, but we will focus on the case $\theta = \pm\frac{\pi}{2}$, i.e. $\langle x, y\rangle = 0$.

If $\langle x, y\rangle = 0$ we say x and y are orthogonal, and write $x \perp y$. ← { Pythag thm here

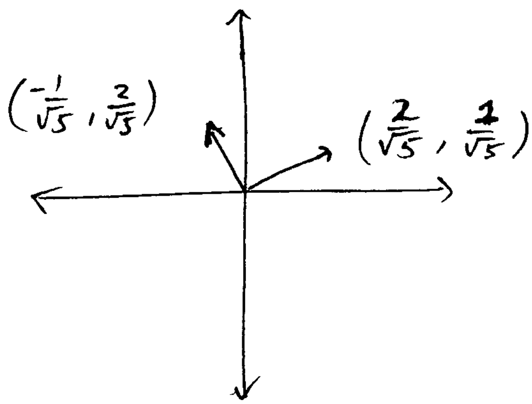
$\boxed{\text{Orthogonal / orthonormal sets/bases}}$

Let $\mathcal{U} \subseteq V$. We say $\mathcal{U}$ is an

- othogonal set if $\langle u, v\rangle = 0 \quad \forall\, u, v \in \mathcal{U}$.
- orthonormal set if $\mathcal{U}$ is an orthogonal set and $\|u\| = 1 \quad \forall\, u \in \mathcal{U}$.

- <u>orthogonal basis</u> if $\mathcal{U}$ is a basis and an orthog. set
- <u>orthonormal basis</u> " " " " basis " " orthonormal set.

<u>Example</u> $V = \mathbb{R}^2$, $\langle \cdot, \cdot \rangle$ = dot product



<u>Note</u> If $\mathcal{U}$ is orthogonal, then

$\left\{ \frac{u}{\|u\|} \mid u \in \mathcal{U} \right\}$ is orthonormal.

<u>Fact</u> If $\mathcal{U}$ is an orthogonal set, then $\mathcal{U}$ is LI.

(homework).

<u>Fact</u> (Pythagorean Thm)

If $u \perp v$, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Example | DFT vectors. $V = \mathbb{C}^N$, $K = \mathbb{C}$.

$$\mathcal{U} = \{ u_0, \dots, u_{N-1} \}, \qquad u_k = \begin{bmatrix} 1 \\ e^{2\pi i \frac{k}{N}} \\ \vdots \\ e^{2\pi i \frac{(N-1)k}{N}} \end{bmatrix}$$

$$\langle u_k, u_\ell \rangle = \sum_{j=0}^{N-1} e^{i 2\pi \frac{kj}{N}} e^{-i 2\pi \frac{\ell j}{N}}$$

$$= \sum_{j=0}^{N-1} e^{i \frac{2\pi j}{N}(k-\ell)} = \sum_{j=0}^{N-1} q^j, \quad q = e^{\frac{i2\pi}{N}(k-\ell)}$$

$$= \begin{cases} N & \text{if } k = \ell \\ \frac{1-q^N}{1-q} = 0 & \text{if } k \neq \ell \end{cases}$$

$\Rightarrow$ $\mathcal{U}$ is orthornmal

$\Rightarrow$ $\mathcal{U}$ is LI

$\Rightarrow$ $\mathcal{U}$ is a basis.

Also, $\tilde{\mathcal{U}} = \{ \tilde{u}_0, \dots, \tilde{u}_{N-1} \}$, $\tilde{u}_k = \frac{1}{\sqrt{N}} u_k$,

is an orthonormal basis.

$\boxed{\text{Fourier Expansions}}$

~~If $\mathcal{U}$ is an ONB for V~~

Theorem Suppose $\dim(V) = n < \infty$ and $\mathcal{U} = \{u_1, \ldots, u_n\}$ is an ONB for V. Then $\forall v \in V$,

$$\text{(i)} \quad v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$$

$$\text{(ii)} \quad \|v\|^2 = \sum_{i=1}^{n} |\langle v, u_i \rangle|^2$$

Proof (i) Since $\mathcal{U}$ is a basis, $\exists\, a_1, \ldots, a_n \in K$ s.t. $v = \sum a_i u_i$. Then

$$\langle v, u_j \rangle = \left\langle \sum a_i u_i, u_j \right\rangle = \sum a_i \langle u_i, u_j \rangle$$

$$= a_j.$$

(ii) $$\|v\|^2 = \langle v, v \rangle = \left\langle \sum_i \langle v, u_i \rangle u_i, \sum_j \langle v, u_j \rangle u_j \right\rangle$$

$$= \sum_{i,j} \langle v, u_i \rangle \cdot \overline{\langle v, u_j \rangle} \cdot \langle u_i, u_j \rangle$$

$$= \sum |\langle v, u_i \rangle|^2$$

or use Pythagorean Thm.

Example 1 $\tilde{\mathcal{U}}$ = normalized DFT basis $= \{\tilde{u}_0, \ldots, \tilde{u}_{N-1}\}$.

$V = \mathbb{C}^N$, $K = \mathbb{C}$

Then

$$\langle v, \tilde{u}_k \rangle = \sum_{n=0}^{N-1} v[n] \cdot \frac{1}{\sqrt{N}} e^{-i2\pi \frac{nk}{N}}$$

= Normalized DFT coeff.

Therefore, we conclude

reconstruction formula

$$v = \sum_k \langle v, u_k \rangle u_k$$

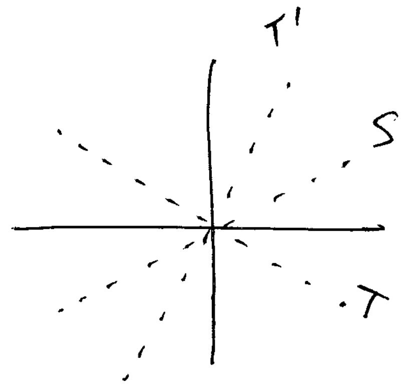
Parseval

$$\|v\|^2 = \sum_k |\langle v, v_k \rangle|^2$$

To recover the original formulas, we just rescale the above formulas.

PROJECTIONS

Let V be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, and let S be a subspace. If $V = S \oplus T$ for some T, we say that T is an algebraic complement of S. In general, every S has many complements.



Given S, T s.t. $V = S \oplus T$, we define the <u>projection onto S</u> (relative to T) to be the function

$$\pi_S : V \longrightarrow V$$

given by $\pi_S(v) = s$, where $v = s + t$, $s \in S, t \in T$.

Properties

- $\pi_S^2 = \pi_S$ "idempotent"
- $\pi_S + \pi_T = I_V$
- $\pi_S \circ \pi_T = 0$
- $R(L) = S$
- $N(L) = T$

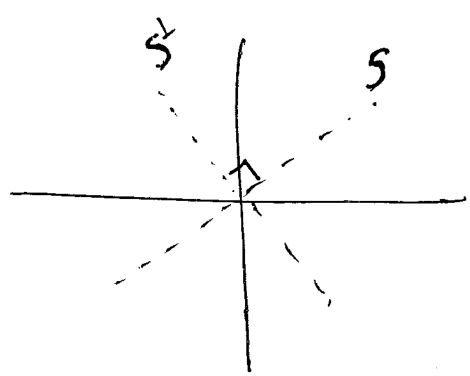
It would be nice if there was a unique complement we could associate to S.

Orthogonal Complements Now assume V is an IPS.

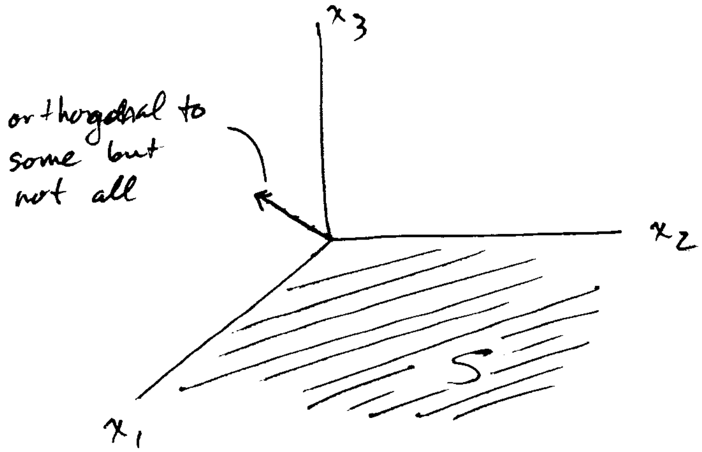
For any subspace S, we define its <u>orthogonal complement</u>

$$S^\perp := \{ v \in V \mid v \perp s \quad \forall s \in S \}$$

Examples (i) $V = \mathbb{R}^2$



(ii)



$S^{\perp} = x_3\text{-axis}$

(iii) $\quad V = L^2(\mathbb{R}), \quad \langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt$

$S = \{\text{odd functions}\}, \quad T = \{\text{even functions}\}$

Suppose $f \in S$, $g \in T$. Then

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt$$

$$= \int_{-\infty}^{0} f(t)\,\overline{g(t)}\,dt + \int_{0}^{\infty} f(t)\,\overline{g(t)}\,dt$$

$$= 0$$

So $T \subseteq S^{\perp}$. Is $T = S^{\perp}$?

We know $V = S \oplus T$. Let $v \in S^{\perp}$, and write $v = s + t$. Then $0 = \langle v, s\rangle = \langle s + t, s\rangle$

$= \langle s, s\rangle \implies s = 0 \implies v \in T$.

Thus $S^{\perp} = T$.

Properties of $S^\perp$:

- $S^\perp$ is a subspace: if $a, b \in K$, and

if $\langle u, s \rangle = 0 \quad \forall s \in S$

$\langle v, s \rangle = 0 \quad \forall s \in S$

then $\langle au + bv, s \rangle = a\langle u, s \rangle + b\langle v, s \rangle$

$= 0 \quad \forall s \in S.$

- $S \cap S^\perp = \{\underline{0}\}$

If $s \in S \cap S^\perp$, then $\langle s, s \rangle = 0$

$\Longrightarrow s = \underline{0}.$

- If $\mathcal{U}$ is a basis for S, and $v \perp u \ \forall u \in \mathcal{U}$, then $v \in S^\perp$.

Orthogonal Projections

If $V = S \oplus S^\perp$, we can define the <u>orthogonal projection</u> onto S to be the projection onto S relative to $S^\perp$.

Is it always the case that $V = S \oplus S^\perp$? That is, is the orthog. comp. always an alg. comp?

Example | $V = \mathbb{R}^\infty$, $K = \mathbb{R}$

$S = \{ \underline{v} \in V \mid v_i = 0$ for all but finitely many $i \}$

Note $\underline{e}_i \in S$ where e_i has a 1 in the i^{th} position 0 elsewhere.

If $\underline{v} \in S^\perp$, then $\underline{v} \perp \underline{e}_i \ \forall i$

$\Rightarrow \langle \underline{v}, \underline{e}_i \rangle = v_i = 0 \Rightarrow \underline{v} = 0$

So $S^\perp = \{\underline{0}\} \Rightarrow V \neq S \oplus S^\perp$.

Theorem (Projection Theorem, version 1)

If V is an IPS, $S \subseteq V$ a subspace, and $\dim(S) < \infty$, then $V = S \oplus S^\perp$.

Proof: Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be an ONB for S.

Let $v \in V$, and set $s = \sum_{i=1}^{n} \langle v, u_i \rangle u_i \in S$

and $t = v - s$. Now for any i,

$\langle t, u_i \rangle = \langle v - s, u_i \rangle = \langle v, u_i \rangle - \langle v, u_i \rangle = 0$

$\Rightarrow t \in S^\perp$. This shows $\{S, S^\perp\}$ spans V.

Since $S \cap S^\perp = \{0\}$, we conclude $V = S \oplus S^\perp$.

Where did we use the assumption $\dim(S) = n < \infty$?

When we claimed $s := \sum_{i=1}^{n} \langle v, u_i \rangle u_i \in S$, which it may not be if $n = \infty$. Consider the previous example.

Nearest Point Property

Theorem If $V = S \oplus S^\perp$, then $\forall\ v \in V$, $\Pi_S(v)$ is the unique solution of

$$\min_{s' \in S} \| v - s' \|$$

Proof: Let $v \in V$. Write $v = s + t$, where $s \in S$, $t \in S^\perp$, so that $\Pi_S(v) = s$. Then for any $s' \in S$,

$$\| v - s' \|^2 = \| s - s' + t \|^2$$

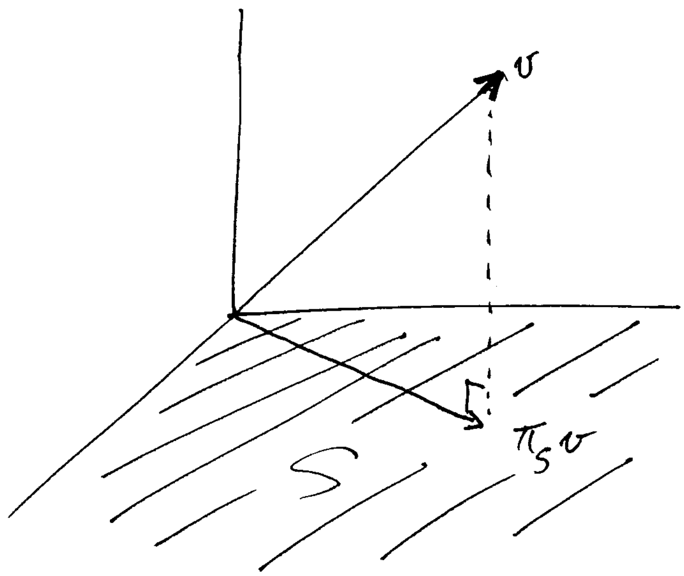
$$= \| s - s' \|^2 + \| t \|^2$$

which is minimized only when $s' = s$.

Orthogonality Principle

$v - \Pi_S v \perp s \quad \forall s \in S$. If we take $s \in \{s_1, s_2, \ldots\}$ = basis for S, we can often set up a system of equations to solve for $\Pi_S v$.

From now on, when we
mention projection, we
~~mean~~ mean orthog. proj.
unless otherwise indicated



It turns out that many important problems in signal processing can be understood using projections.

- Filtering: project onto S to estimate a noisy signal known to belong to S
- Approximation: find the best fit to a signal using a signal from S.

PROJECTION MATRICES AND LEAST SQUARES

Let's now focus on the case $V = K^N$, and

$<x, y> = y^H x$ (standard dot product)

Let S be an arbitrary subspace, spanned by linearly indep. vectors $\underline{a}_1, \dots, \underline{a}_p \in K^N$. Denote

$$A = [\underline{a}_1 \ \cdots \ \underline{a}_p] \qquad (N \times p)$$

<u>Theorem</u>

$$\Pi_S = A \cdot (A^H A)^{-1} A^H$$

Proof: $A^H A$ is invertible since $\operatorname{rank}(A) = p$ (homework)

Let $\underline{b}_1, \dots, \underline{b}_{N-p}$ span $S^\perp$ and set $B = [\underline{b}_1 \cdots \underline{b}_{N-p}]$.

For any $v \in V$, $v = s + t$, $s \in S$, $t \in S^\perp$,

$$= A\theta + B\phi, \quad \theta \in K^p, \ \phi \in K^{N-p}.$$

Then $$A(A^H A)^{-1} A^H v = A(A^H A)^{-1} A^H (A\theta + B\phi)$$

$$= A\underbrace{(A^H A)^{-1} A^H A}_{I}\theta + A(A^H A)^{-1} \underbrace{A^H B}_{0}\phi$$

$$= A\theta = s$$

∎

If $\underline{a}_1, \dots, \underline{a}_p$ are orthogonal, then

$$\Pi_S = A \cdot A^H$$

and we obtain

$$\Pi_S(\underline{v}) = A \cdot A^H \underline{v}$$

$$= A \cdot \begin{bmatrix} \langle \underline{v}, \underline{a}_1 \rangle \\ \vdots \\ \langle \underline{v}, \underline{a}_p \rangle \end{bmatrix}$$

$$= \sum_{i=1}^{p} \langle \underline{v}, \underline{a}_i \rangle \, \underline{a}_i$$

which is a Fourier expansion.

We saw this previously in the proof of the projection Thm.

Note] $A^H A = \begin{bmatrix} \langle \underline{a}_1, \underline{a}_1 \rangle & \cdots & \langle \underline{a}_1, \underline{a}_p \rangle \\ \vdots & & \\ \langle \underline{a}_p, \underline{a}_1 \rangle & \cdots & \langle \underline{a}_p, \underline{a}_p \rangle \end{bmatrix}$ is called the <u>Grammian</u>. We saw on the homework that $A^H A$ is invertible when $\text{rank}(A) = p$.

The problem

$$\min_{\underline{\theta}} \| \underline{v} - A\underline{\theta} \|^2$$

is called a <u>least squares</u> problem.

From the nearest point property of orthog. projections, and the previous theorem, we deduce that the unique least squares solution is

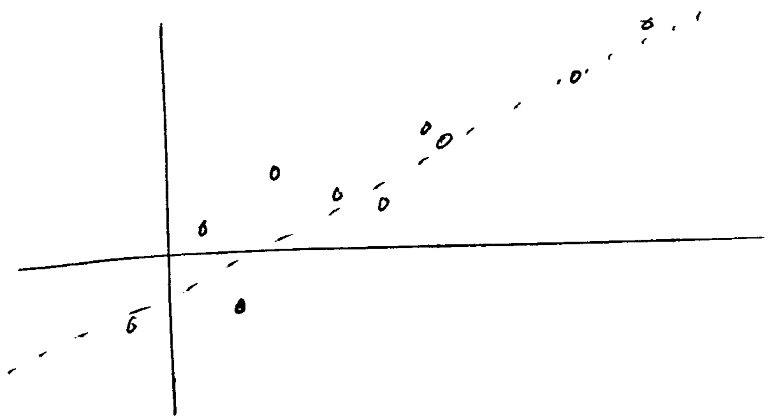
$$(A^H A)^{-1} A^H \underline{v}.$$

The matrix

$$A^+ := (A^H A)^{-1} A^H$$

is called the <u>pseudoinverse</u> of A.

$\boxed{\text{Linear Regression}}$ Observe (x_i, y_i), $i = 1, \dots, n$



Suppose $y_i = a x_i + b + e_i$. How can we estimate a, b? Let's minimize the sum of squared errors:

$$(\hat{a}, \hat{b}) = \underset{(a,b)}{\arg\min} \sum_{i=1}^{n} \left[y_i - (a x_i + b) \right]^2$$

$$= \|\underline{e}\|^2$$

Now

$$\underline{e} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \underline{y} - A \underline{\theta}$$

So $\underline{\hat{\theta}} = \underset{\theta}{\arg\min} \|\underline{y} - A\theta\|^2 = (A^T A) A^T \underline{y}$

Denoising Sinusoidal Signals

Suppose we observe a signal in noise

$$x[n] = s[n] + e[n], \qquad n = 0, \dots, N-1.$$

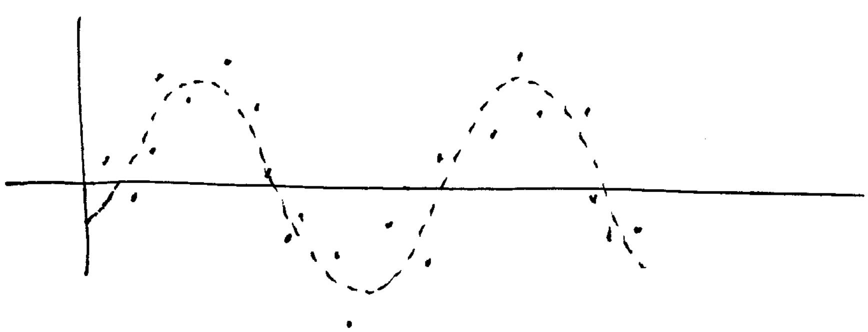
where

$$s[n] = C \cdot \cos[2\pi f n + \phi]$$

$C \in \mathbb{C}$, unknown

$\phi \in [-\pi, \pi]$, unknown

$f \in [-\frac{1}{2}, \frac{1}{2}]$, known



How can we estimate $s[n]$?

Denote $\underline{x} = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$, $\underline{s} = \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix}$, $\underline{e} = \begin{bmatrix} e[0] \\ \vdots \\ e[N-1] \end{bmatrix}$

so that $\underline{x} = \underline{s} + \underline{e}$.

Notice

$$\cos(2\pi f n + \phi) = \frac{e^{i(2\pi f n + \phi)} + e^{-i(2\pi f n + \phi)}}{2}$$

$$= \frac{1}{2} e^{i\phi} e^{i2\pi f n} + \frac{1}{2} e^{-i\phi} e^{-i2\pi f n}$$

Therefore

$$\underline{s} = \frac{c}{2} e^{i\phi} \underbrace{\begin{bmatrix} 0 \\ e^{i2\pi f} \\ \vdots \\ e^{i2\pi f(N-1)} \end{bmatrix}}_{\underline{a}_1} + \frac{c}{2} e^{-i\phi} \underbrace{\begin{bmatrix} 0 \\ e^{-i2\pi f} \\ \vdots \\ e^{-i2\pi f(N-1)} \end{bmatrix}}_{\underline{a}_2}$$

$$= c_1 \underline{a}_1 + c_2 \underline{a}_2$$

$$\in \operatorname{span}\{\underline{a}_1, \underline{a}_2\} = S$$

Furthermore, we can recover C, ϕ from c_1, c_2:

$$C = \sqrt{4 c_1 c_2}$$

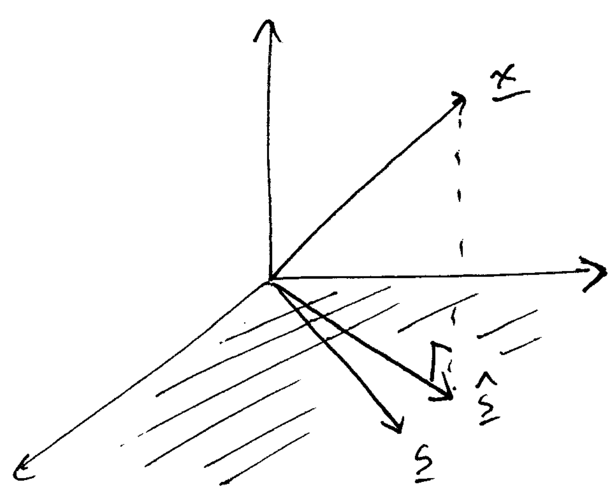
$$\phi = \frac{1}{2} \arg\left(\frac{c_1}{c_2}\right)$$

So signals like $s[n]$ coincide exactly span$\{\underline{a}_1, \underline{a}_2\}$.

So let's estimate $\underline{s}$ by projecting onto S

$$\underline{\hat{s}} = \Pi_S \underline{x} = A\,(A^H A)^{-1} A^H \underline{x}$$

Will this recover $\underline{s}$ exactly? No, because in general, $\Pi_S \underline{e} \neq 0$, so $\underline{\hat{s}} = \underline{s} + \Pi_S \underline{e} \neq \underline{s}$



Least Squares FIR Filter Design

Consider an LTI system with impulse response $h[n]$ having length N:

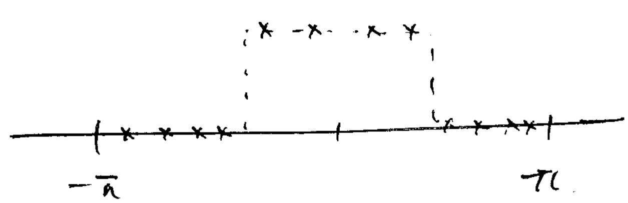
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = \sum_{k=0}^{N-1} h[k]\, x[n-k]$$

Suppose we with to design $\{h[n]\}$ to have a certain frequency response. In particular, suppose we wish $\hat{h}(\omega)$ to match some desired response $\hat{h}_d(\omega)$ at a finite set of chosen frequencies $\omega_0, \dots, \omega_{L-1} \in [-\pi, \pi]$, $L > N$.

How can we minimize

$$\sum_{k=0}^{L-1} \left| \hat{h}(\omega_k) - \hat{h}_d(\omega_k) \right|^2$$



Let's write

$$\hat{h}(\omega) = \sum_{n=0}^{N-1} h[n] e^{-i\omega n}$$

$$= \langle \underline{h}, \underline{v}(\omega) \rangle$$

where

$$\underline{h} = \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[N-1] \end{bmatrix} \qquad \underline{v}(\omega) = \begin{bmatrix} 1 \\ e^{i\omega} \\ \vdots \\ e^{i\omega(N-1)} \end{bmatrix}$$

Then

$$\sum_{k=0}^{L-1} |\hat{h}(\omega_k) - \hat{h}_d(\omega_k)|^2$$

$$= \left\| \begin{bmatrix} \hat{h}_d(\omega_0) \\ \vdots \\ \hat{h}_d(\omega_{L-1}) \end{bmatrix} - \begin{bmatrix} \underline{v}(\omega_0)^H \\ \vdots \\ v(\omega_{L-1})^H \end{bmatrix} \begin{bmatrix} h[0] \\ \vdots \\ h[N-1] \end{bmatrix} \right\|^2$$

$$= \| \underline{\hat{h}}_d - A \cdot \underline{h} \|^2$$

Therefore the LS optimal filter coefficients are given by

$$\underline{h}^* = A^+ \hat{\underline{h}}_d = (A^H A)^{-1} A^H \hat{\underline{h}}_d$$

What happens if $\omega_k = \frac{2\pi k}{L}$? Then

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-i2\pi \frac{1}{L}} & & e^{-i2\pi \frac{N-1}{L}} \\ \vdots & & & \\ 1 & e^{-i2\pi \frac{L-1}{L}} & \cdots & e^{-i2\pi \frac{(L-1)(N-1)}{L}} \end{bmatrix}$$

$(L \times N)$

$$= \begin{bmatrix} \underline{u}_0 & \underline{u}_1 & \cdots & \underline{u}_{N-1} \end{bmatrix}$$

DFT basis vectors for length-L DFT

$$\Rightarrow \quad A^H A = L \cdot I_{N \times N}$$

$$\Rightarrow \quad \underline{h}^* = \frac{1}{L} A^H \hat{\underline{h}}_d = \frac{1}{L} \sum_{k=0}^{L-1} \hat{h}_d\left(\frac{2\pi k}{L}\right) \cdot \underline{u}\left(\frac{2\pi k}{L}\right) = \frac{1}{L} \begin{bmatrix} \langle \hat{\underline{h}}_d, \underline{u}_0 \rangle \\ \vdots \\ \langle \hat{\underline{h}}_d, \underline{u}_{N-1} \rangle \end{bmatrix}$$

$\Longrightarrow$ Fast implementation w/ FFT.

Rethink this.
Not worth much
class time, if
any.

In general, $h[n] \in \mathbb{C}$ even if $\hat{h}_d \in \mathbb{R}$.

What if we wanted $h[n] \in \mathbb{R}$?

If we enforce the constraint $h[n] = h[N-n]$, $n \geq 1$

then A is real $\Rightarrow$ $\underline{h}^*$ is real.

LINEAR PREDICTION

Random Variables as Vectors

Consider the set V of all complex-valued, zero mean random variables.

<u>Aside</u>: A complex-valued random variable has the form $X = X_R + iX_I$ where X_R, X_I are jointly distributed real RVs.

Then
$$E[X] := E[X_R] + i\,E[X_I]$$
$$\mathrm{Var}(X) := E[|X - E[X]|^2]$$

Then V is a vector space over $\mathbb{C}$.

Now consider the inner product

$$\langle X, Y \rangle = E[X\overline{Y}]$$

Is this a valid inner product?

The induced norm is

$$\|X\| = \sqrt{\langle X, X\rangle} = \sqrt{E[|X|^2]} = \sqrt{\mathrm{Var}(X)}$$

By Cauchy-Schwarz, $-1 \le \rho_{XY} \le 1$, where

$$\rho_{XY} := \frac{E[XY]}{\sqrt{V(X)V(Y)}}$$ is the <u>correlation coefficient</u>.

$$= \mathrm{Var}(Y_R) + \mathrm{Var}(X_I)$$

Note that in this norm, $X = Y \iff E[|X-Y|^2] = 0$

$\boxed{\text{Linear Prediction}}$

Consider a random process $\{x[n]\}_{n=-\infty}^{\infty}$,

which is zero-mean and wide-sense stationary, i.e.

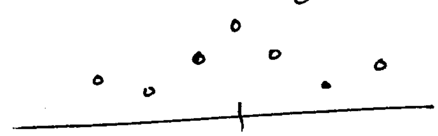
$$E[x[n]] = 0 \quad \forall n$$

$$r_{xx}[k] := \langle x[n], x[n-k] \rangle$$

auto-correlation function (ACF)

$$= E\left[x[n]\, \overline{x[n-k]}\right]$$

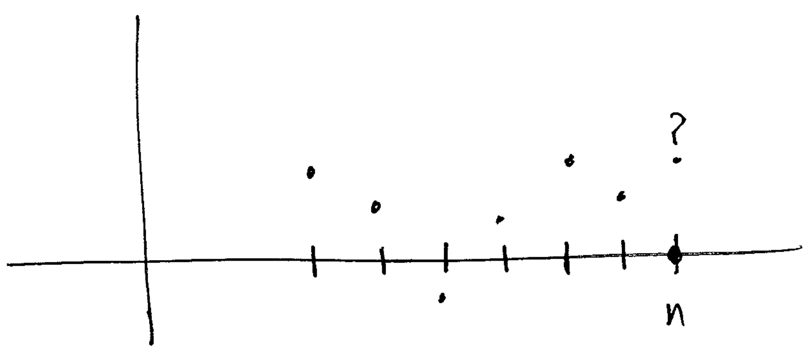
for real case



is independent of n.

Suppose we observe $x[n-1], \dots, x[n-p]$

and we wish to predict $x[n]$.



Unless the $x[n]$ are uncorrelated, we expect to be able do better than random guessing.

Let's look for a <u>linear estimator</u>

$$\hat{x}[n] = \sum_{k=1}^{p} h_p[k]\, x[n-k]$$

In other words, we want to determine $\underline{h}_p = \begin{bmatrix} h_p[1] \\ \vdots \\ h_p[p] \end{bmatrix}$

to have the smallest <u>prediction error</u>,

$$e[n] = x[n] - \hat{x}[n]$$

When we say "small," we mean in the sense of small

$$\|e[n]\|^2 = E\left[\,|x[n] - \hat{x}[n]|^2\right]$$

Define the subspace

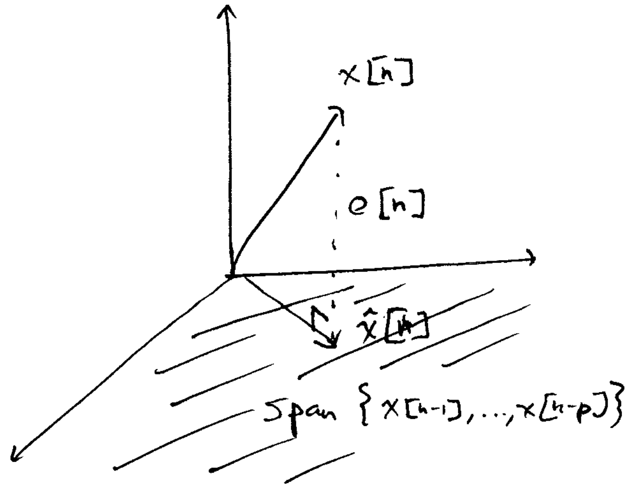
$$S = \operatorname{span}\{x[n-1], \dots, x[n-p]\}$$

Then we seek the point $\hat{x}[n]$ in S that is closest to $x[n]$. Since S is finite dim, we know $V = S \oplus S^{\perp}$. Therefore $\hat{x}[n]$ is the projection of $x[n]$ onto S, and

$$e[n] = x[n] - \hat{x}[n] \perp s \quad \forall s \in S.$$

$\leftarrow$ orthogonality principle

$\boxed{\text{Solution via Orthogonality Principle}}$



By the orthogonality principle we know

$$x[n] - \hat{x}[n] \perp x[n-k],$$

$k = 1, \ldots, p$. That is

$$\begin{aligned}
0 &= \langle x[n] - \hat{x}[n], x[n-k] \rangle \\
&= \left\langle x[n] - \sum_{\ell=1}^{p} h_p[\ell]\, x[n-\ell],\ x[n-k] \right\rangle \\
&= \langle x[n], x[n-k] \rangle - \sum_{\ell=1}^{p} h_p[\ell] \langle x[n-\ell], x[n-k] \rangle \\
&= r_{xx}[k] - \sum_{\ell=1}^{p} h_p[\ell] \cdot r_{xx}[k-\ell]
\end{aligned}$$

This holds for each k. As a matrix equation

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[-1] & \cdots & r_{xx}[-(p-1)] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[-(p-2)] \\ \vdots & & & \\ r_{xx}[p-1] & r_{xx}[p-2] & \cdots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} h_p[1] \\ \vdots \\ h_p[p] \end{bmatrix} = \begin{bmatrix} r_{xx}[1] \\ \vdots \\ r_{xx}[p] \end{bmatrix}$$

or

$$R_p \cdot \underline{h}_p = \underline{r}_p$$

Therefore the optimal prediction filter is the solution of a linear system of equations.

This solution can be arrived at by other means, e.g. gradients, but none are as simple and elegant as the orthogonality principle.

$\boxed{\text{Algorithms}}$

Since R_p is Toeplitz, the system can be solved efficiently.

In addition, it is possible to update h_{p+1} from h_p, once $x[n]$ is observed, using the Levinson-Durbin algorithm.

FOURIER SERIES IN HILBERT SPACE

Goal of this lecture and the next: Generalize Fourier series and projections to infinite dimensional subspaces.

We have seen that if V is a vector space with an ONB $\mathcal{O} = \{\underline{e}_1, \dots, \underline{e}_n\}$, then

$$\underline{v} = \sum_{k=1}^{n} \langle \underline{v}, \underline{e}_k \rangle \underline{e}_k$$

and

$$\|\underline{v}\|^2 = \sum_{k=1}^{n} |\langle \underline{v}, \underline{e}_k \rangle|^2$$

for all $v \in V$.

What if a finite ONB does not exist?

$\boxed{\text{Orthonormal Sequences}}$

An <u>orthonormal sequence</u> is an orthonormal set, $\{\underline{e}_1, \underline{e}_2, \dots\}$, indexed by $\mathbb{N}$.

Examples

- standard basis in

$$\ell^2 = \{(x_1, x_2, \ldots) \in \mathbb{C}^\infty \mid \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$$

$$\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$$

- $\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \right\}_{n \in \mathbb{Z}}$ in

$$L^2(-\pi, \pi)$$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)}\, dt$$

Theorem (Bessel's inequality)

If $\{e_i\}$ is an orthonormal sequence, then

$$\sum_{i=1}^{\infty} |\langle v, e_i \rangle|^2 \le \|v\|^2$$

for all $v \in V$.

Proof: Denote $v_N = \sum_{i=1}^{N} \langle v, e_i \rangle e_i$. Then

$$\|v - v_N\|^2 = \left\langle v - \sum \langle v, e_i \rangle e_i,\ v - \sum \langle v, e_i \rangle e_i \right\rangle$$

$$= \langle v, v \rangle - \sum \overline{\langle v, e_i \rangle} \langle v, e_i \rangle - \sum \langle v, e_i \rangle \langle e_i, v \rangle$$

$$+ \sum |\langle v, e_i \rangle|^2$$

$$= \|v\|^2 - \sum |\langle v, e_i \rangle|^2$$

So $\displaystyle\sum_{i=1}^{N} |\langle v, e_i\rangle|^2 = \|v\|^2 - \|v - v_N\|^2$

$$\leq \|v\|^2.$$

$\boxtimes$

Now let $N \to \infty$.

What does $\displaystyle\sum_{i=1}^{\infty} \lambda_i e_i$ mean? When does it converge?

If $x \in V$, $\in \mathbb{N}$, we say that the series $\displaystyle\sum_{=1}^{\infty} x$, converges to x if

$$\sum_{i=1}^{n} x_i \to x \quad \text{as } n \to \infty.$$

(i.e., $\forall \epsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow \left\| \sum_{i=1}^{n} x_i - x \right\| < \epsilon$.)

In a Hilbert space, there is a precise characterization of when $\sum \lambda_i e_i$ converges.

<u>Thm</u>) Let $\mathcal{H}$ be a Hilbert space and let $\{e_1, e_2, \ldots\}$ be an orthonormal sequence. Then

$$\sum_{i=1}^{\infty} \lambda_i e_i \text{ converges} \iff \sum |\lambda_i|^2 < \infty.$$

Proof: Denote

$$x_n = \sum_{i=1}^{n} \lambda_i e_i , \qquad r_n = \sum_{i=1}^{n} |\lambda_i|^2.$$

Then (assuming $m < n$)

$$\| x_n - x_m \|^2 = \Big\| \sum_{i=m+1}^{n} \lambda_i e_i \Big\|^2$$

$$= \sum_{i=m+1}^{n} |\lambda_i|^2$$

$$= | r_n - r_m |$$

Therefore $\{x_n\}$ is Cauchy $\Leftrightarrow$ r_n is Cauchy.

$(\Rightarrow)$ $\{x_n\}$ converges $\Rightarrow$ $\{x_n\}$ is Cauchy

$\Rightarrow$ $\{r_n\}$ is Cauchy

$\Rightarrow$ $\{r_n\}$ converges (since $\mathbb{R}$ is complete)

$(\Leftarrow)$ $\{r_n\}$ converges $\Rightarrow$ $\{r_n\}$ is Cauchy

$\Rightarrow$ $\{x_n\}$ is Cauchy

$\Rightarrow$ $\{x_n\}$ converges (since H is complete)

$\blacksquare$

Complete Orthonormal Sequences

Let $\{e_1, e_2, \dots\}$ be an ONS in a Hilbert Space $\mathcal{H}$.

We would like to write

$$v = \sum \langle v, e_i \rangle e_i$$

for arbitrary $v \in \mathcal{H}$. Is this possible?

By Bessel's ineq.,

$$\sum_{i=1}^{\infty} |\langle v, e_i \rangle|^2 \le \|v\|^2 < \infty$$

$\Longrightarrow$ $\sum \langle v, e_i \rangle e_i$ converges. But does it converge to v?

Examples

- $e_1 = (0, 1, 0, \dots)$

  $e_2 = (0, 0, 1, 0, \dots)$

  $e_3 = (0, 0, 0, 1, 0, \dots)$

  $\vdots$

  in ℓ^2

  If $v = (1, 0, 0, \dots)$ then $\sum \langle v, e_i \rangle e_i = \underline{0} \neq v$

- $e_i = e^{i(2n)t}$, $n \in \mathbb{Z}$, in $L^2(-\pi, \pi)$

An ON set is said to be <u>complete</u> if

$$v \perp e_i \quad \forall i \implies v = 0.$$

<u>Theorem</u> If $\{e_1, e_2, \dots\}$ is an ONS in a Hilbert space H, then TFAE:

(1) $\{e_i\}$ is complete

(2) $v = \sum \langle v, e_i \rangle e_i \quad \forall\, v \in H$

(3) $\|v\|^2 = \sum |\langle v, e_i \rangle|^2 \quad \forall\, v \in H.$

<u>Proof</u>

$(1 \Rightarrow 2)$ Let $u = v - \sum_{i=1}^{\infty} \langle v, e_i \rangle e_i.$

Then

$$\begin{aligned}
\langle u, e_j \rangle &= \langle v, e_j \rangle - \Big\langle \sum_{i=1}^{\infty} \langle v, e_i \rangle e_i \Big\rangle \\
&= \langle v, e_j \rangle - \lim_{n \to \infty} \Big\langle \sum_{i=1}^{n} \langle v, e_i \rangle, e_j \Big\rangle \\
&= \langle v, e_j \rangle - \lim_{n} \sum_{i=1}^{n} \langle v, e_i \rangle \langle e_i, e_j \rangle \\
&= \langle v, e_j \rangle - \sum_{i=1}^{\infty} \langle v, e_i \rangle \langle e_i, e_j \rangle = 0
\end{aligned}$$

$\implies u \perp e_j \quad \forall j \implies u = 0.$

<u>Lemma</u>: If $x_n \to x$, then $\langle x_n, y \rangle \to \langle x, y \rangle$.

Proof: $|\langle x_n, y \rangle - \langle x, y \rangle| = |\langle x_n - x, y \rangle| \le \|x_n - x\| \cdot \|y\|$

$\to 0$ as $n \to 0$. ↖ cauchy-schwarz

Now apply with $x_n = \sum_{i=1}^{n} \langle v, e_i \rangle e_i$

$(2 \Rightarrow 3)$

<u>Lemma:</u> If $x_n \to x$, then $\|x_n\| \to \|x\|$.

By Δ ineq:

$$\|x\| \le \|x - x_n\| + \|x_n\| \Rightarrow \|x\| - \|x_n\| \le \|x - x_n\|$$

$$\|x_n\| \le \|x_n - x\| + \|x\| \Rightarrow \|x_n\| - \|x\| \le \|x - x_n\|$$

So $\big| \|x\| - \|x_n\| \big| \le \|x - x_n\| \to 0$ as $n \to \infty$.

Then

$$\|x\|^2 = \Big\| \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \Big\|^2$$

$$= \Big\| \lim_n \sum_{i=1}^{n} \langle x, e_i \rangle e_i \Big\|^2$$

$$= \lim_{n \to \infty} \Big\| \sum_{i=1}^{n} \langle x, e_i \rangle e_i \Big\|^2$$

$$= \lim_n \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$$

$$= \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2$$

$(3 \Rightarrow 1)$ Suppose (3) holds but $\{e_i\}$ is not complete.

Then $\exists\, v \neq \underline{0}$ s.t. $v \perp e_i$ $\forall i$. Then

$0 \neq \|v\|^2 = \sum |\langle v, e_i \rangle|^2 = 0$, a contradiction.

A Hilbert space is said to be <u>separable</u> if it contains a complete ONS (indexed by $\mathbb{N}$ or finite)

<u>Examples</u>

- $\mathbb{C}^N$
- ℓ^2
- $L^2(\mathbb{R})$ – Hermite polynomials
  – Wavelets
- $L^2(-\pi, \pi)$ – $\{e^{int}\}_{n \in \mathbb{Z}}$ (proof of completeness in Mallat)
  – Legendre polynomials

Complete orthonormal sequences are very similar to orthonormal Hamel bases, except that we allow infinite linear combos. In fact, complete orthonormal sequences are examples of <u>Hilbert bases</u>. A Hilbert bases is a maximal orthonormal set. Some Hilbert bases are uncountable, so not every Hilbert space is separable. Every Hilbert basis has the same dimension, so we can define the <u>Hilbert dimension</u>. For finite dimensional spaces this agrees w/ the Hamel dimension, but otherwise it generally does not.

Parseval Theorem If $\mathcal{H}$ is a Hilbert space with a complete ONS $\{e_1, e_2, \ldots\}$ then

$$\langle x, y\rangle = \sum_{i=1}^{\infty} \langle x, e_i\rangle \overline{\langle y, e_i\rangle}$$

Proof: $\langle x, y\rangle = \left\langle \sum_i \langle x, e_i\rangle e_i \, , \, \sum_j \langle y, e_j\rangle e_j \right\rangle$

$$= \sum_{i=1}^{\infty} \langle x, e_i\rangle \left\langle e_i \, , \sum_j \langle y, e_j\rangle e_j \right\rangle$$

by continuity of inner product

$$= \sum_i \sum_j \langle x, e_i\rangle \cdot \overline{\langle y, e_j\rangle} \, \langle e_i, e_j\rangle$$

$$= \sum_i \langle x, e_i\rangle \cdot \overline{\langle y, e_i\rangle}$$

And of course we already know

$$\|x\|^2 = \sum |\langle x, e_i\rangle|^2$$

In $\mathbb{C}^N$ or ℓ^2 with standard basis, this doesn't tell us anything we didn't already know. But with different bases, or in different spaces, we do get interesting results.

Example $\mathcal{H} = L^2(\mathbb{R})$, $\langle f, g \rangle = \int f(t)\overline{g(t)}\,dt$

$$S_T := \{ f \in L^2(\mathbb{R}) \mid \hat{f}(\omega) = 0 \text{ for } |\omega| > \pi/T \}$$

$$h_T(t) = \frac{\sin \pi t/T}{\pi t/T}$$

Then $\left\{ \frac{1}{\sqrt{T}} h_T(t - nT) \right\}_{n \in \mathbb{Z}}$ is a complete orthonormal sequence for S_T.

$$\left\langle \tfrac{1}{\sqrt{T}} h_T(t-nT), \tfrac{1}{\sqrt{T}} h_T(t-mT) \right\rangle$$

$$= \frac{1}{T} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} T \cdot \mathbb{1}_{[-\frac{\pi}{T}, \frac{\pi}{T}]}(\omega) \cdot e^{-nT\omega}\; T \mathbb{1}_{[-\frac{\pi}{T}, \frac{\pi}{T}]}(\omega)\, e^{+mT\omega}\, d\omega$$

$$= \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{-(n-m)T\omega}\, d\omega$$

$$= \delta[n-m].$$

From the Whitaker sampling Thm

$$f(t) = \sum f(nT)\, h_T(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\left(\sqrt{T}\, f(nT)\right)}_{\text{Fourier coefficient}} \frac{1}{\sqrt{T}} h_T(t-nT) \implies \text{completeness}$$

$\boxed{v = \sum \lambda_i e_i,\ \{e_i\} \text{ ONS},\ \Rightarrow \lambda_j = \langle v, e_j \rangle}$

$$f(nT) = \frac{1}{T}\langle f(t), h_T(t-nT) \rangle = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \hat{f}(\omega)\, e^{inT\omega}\, d\omega$$

and $\sum |f[n]|^2 = \frac{1}{T} \int |f(t)|^2 dt$

$\boxed{\text{Example}}$ $\mathcal{H} = L^2(-\pi, \pi)$. $\quad \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) \overline{g(\omega)}\, d\omega$

We can use the previous example (with $T=1$) to show that $\{e^{-in\omega}\}_{n \in \mathbb{Z}}$ is a complete ONB for $\mathcal{H}$.

$$\langle e^{-in\omega}, e^{-im\omega} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n-m)\omega}\, d\omega$$

$$= \begin{cases} 1 & \text{if } m = n \\ \frac{1}{2\pi(n-m)} \left[e^{-i(n-m)\pi} - e^{+i(n-m)\pi} \right] = 0, & m \neq n \end{cases}$$

What about completeness? Let $\hat{f} \in L^2(-\pi, \pi)$.

Extend $\hat{f}$ to $\mathbb{R}$: $\hat{f}(\omega) = 0$ for $|\omega| > \pi$.

Let $f(t)$ be the inverse CTFT of $\hat{f}(\omega)$. Since $f \in S_1$ we know

$(T=1)$

$$f(t) = \sum_{n=-\infty}^{\infty} f[n]\, h_1(t-n)$$

where

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{in\omega}\, d\omega$$

(from previous example)

$$= \langle \hat{f}, e^{-in\omega} \rangle$$

Then

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} f[n] \, \widehat{h_1(t-n)}(\omega)$$

$$= \sum f[n] \, e^{-in\omega} \qquad \text{(DTFT)}$$

$$= \sum_{n=-\infty}^{\infty} \langle \hat{f}, e^{-in\omega} \rangle \, e^{-in\omega}$$

$\Longrightarrow$ $\{e^{-in\omega}\}_{n\in\mathbb{Z}}$ is complete.

We can apply Parsevals thm to conclude

$$\sum |f[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 \, d\omega$$

where $\hat{f}(\omega)$ = DTFT of $f(t)$

Also note that

$$f[n] = \langle \hat{f}(\omega), e^{-in\omega} \rangle$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \, e^{in\omega} \, d\omega$$

establishes the DTFT inversion formula.

PROJECTIONS IN HILBERT SPACE

Previously we have seen that if V is an IPS and $S \subseteq V$ is a finite dim. subspace, then

$$V = S \oplus S^{\perp}$$

and we can define the orthogonal projection onto S.

We will now generalize this result. Our generalization requires

V is a Hilbert space

S is a <u>closed</u> subspace

A subspace S is said to be <u>closed</u> if

$$x_n \to x, \quad x_n \in S \ \forall n \implies x \in S.$$

When $\dim(S) < \infty$, S is always closed. When $\dim(S) = \infty$, this is not the case.

<u>Example</u> | $V = \ell^2(\mathbb{N})$, $\quad x_n = (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, 0, 0, \ldots)$

$S = \{\text{finitely many nonzero coordinates}\}$ $\quad x = (1, \frac{1}{2}, \cdots \quad)$

$$\| x_n - x \|^2 = \sum_{i=n+1}^{\infty} \frac{1}{i^2} \to 0 \quad \text{as } n \to \infty$$

$\implies$ not closed.

Theorem (Projection Theorem – general version)

Let $\mathcal{H}$ be a Hilbert space, and $S \subseteq \mathcal{H}$ a closed subspace. Then $V = S \oplus S^{\perp}$.

Proof: book

Implications

- can define orthogonal projections
- nearest point property
- orthogonality principle

} as we have already seen, these follow as long as you can define orthogonal projections

Example $\mathcal{H} = L^2(\mathbb{R})$, $S_T = \{ f \in \mathcal{H} \mid \hat{f}(\omega) = 0 \text{ for } |\omega| > \frac{\pi}{T} \}$.

Is S_T closed? If $f_n \to f$, $f_n \in S_T$,

then $\hat{f}_n \to \hat{f}$, and $\hat{f}_n(\omega) = 0$ for $|\omega| > \frac{\pi}{T}$, so

$\hat{f}(\omega) = 0$ for $|\omega| > \frac{\pi}{T} \Rightarrow f \in S_T$.

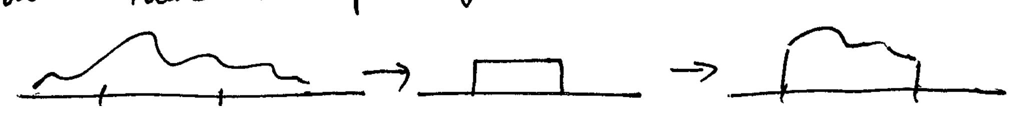
For arbitrary $f \in \mathcal{H}$, what is the proj. of f onto S_T?

If $g \in S_T$, then

$$\|f-g\|^2 = \int_{-\infty}^{-\frac{\pi}{T}} |\hat{f}(\omega)|^2 \, d\omega + \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} |\hat{f}(\omega) - \hat{g}(\omega)|^2 \, d\omega + \int_{\frac{\pi}{T}}^{\infty} |\hat{f}(\omega)|^2 \, d\omega$$

minimized when $\hat{g}(\omega) = \hat{f}(\omega) \cdot \mathbb{1}_{[-\frac{\pi}{T}, \frac{\pi}{T}]}(\omega)$

So projection = ideal low-pass filter



Can we represent the projection in terms of $\{\frac{1}{\sqrt{T}} h_T(t-nT)\}$?

Projections and Fourier Series

Suppose $\{\underline{e}_1, \underline{e}_2, \dots\}$ is an orthonormal sequence in $\mathcal{H}$ which is complete w.r.t. the closed subspace S, i.e. if $s \in S$, $s \perp e_i \ \forall i$, then $s = \underline{0}$.

Let $\underline{v} \in \mathcal{H}$.
Set $\underline{s} = \sum_{i=1}^{n} \langle \underline{v}, \underline{e}_i \rangle \underline{e}_i$. Since S is closed, $\underline{s} \in S$. Also, $\forall j$,

$$\langle \underline{v} - \underline{s}, \underline{e}_j \rangle = \langle \underline{v} - \sum \langle \underline{v}, \underline{e}_i \rangle \underline{e}_i, \underline{e}_j \rangle$$

$$= \langle \underline{v}, \underline{e}_j \rangle - \sum_{i=1}^{\infty} \langle \underline{v}, \underline{e}_i \rangle \langle \underline{e}_i, \underline{e}_j \rangle$$

$$= \langle \underline{v}, \underline{e}_j \rangle - \langle \underline{v}, \underline{e}_j \rangle = 0$$

$\Rightarrow \underline{v} - \underline{s} \in S^\perp$

So $\underline{v} = \underline{s} + (\underline{v} - \underline{s})$

$\Rightarrow \sum \langle \underline{v}, \underline{e}_i \rangle \underline{e}_i$ is the projection onto S.

Example, cont'd. In the case of bandlimited signals,

$$(\Pi f)(t) = \sum_{n\in\mathbb{Z}} \langle f, \tfrac{1}{\sqrt{T}} h_T(t-nT)\rangle \cdot \tfrac{1}{\sqrt{T}} h_T(t-nT)$$

$$= \sum \underbrace{\tfrac{1}{T}\langle f(t), h_T(t-nT)\rangle}_{} h_T(t-nT)$$

$$\frac{1}{T}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, T\cdot 1_{[-\frac{\pi}{T},\frac{\pi}{T}]}(\omega)\cdot e^{inT\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \hat{f}(\omega) e^{inT\omega}\, d\omega$$

$$= \tilde{f}(nT), \quad \tilde{f} = \text{low-pass filtered version of } f$$

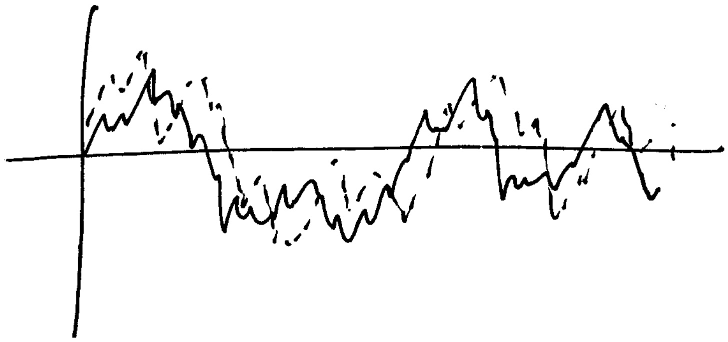
The Infinite Weiner Smoother

Let $x[n] = s[n] + w[n]$, $n\in\mathbb{Z}$, be an observed signal. The goal is to estimate $s[n]$, $n\in\mathbb{Z}$.

Assume:

- $x[n]$ is zero mean, WSS with ACF $r_x[k] = E\left[x[n]\overline{x[n-k]}\right]$
- $s[n]$ is zero mean, WSS with ACF $r_s[k]$
- $x[n]$ and $s[n]$ are jointly WSS with cross-correlation fun. (CCF) $r_{sx}[k] = E\left[s[n]\overline{x[n-k]}\right]$
- $r_x, r_s, r_{sx} \in \ell^2(\mathbb{Z})$

Students didn't get this. Leave it for 564. See Yonina Eldar's work for other potential apps. of this theory.
(Still, could briefly preview 564, set up problem)



We seek a linear estimator of $s[n]$ (n fixed):

$$\hat{s}[n] = \sum_{k \in \mathbb{Z}} h[k]\, x[n-k]$$

which minimizes $\| s[n] - \hat{s}[n] \|^2 = E\left[|s[n] - \hat{s}[n]|^2 \right]$.

just say, don't try to formalize

$\mathcal{H} = \{$ zero mean variables w/ finite variance $\}$

$\mathcal{S} = \left\{ \sum_k h[k]\, v[n-k] \in \mathcal{H} \;\middle|\; v[n] \in \mathcal{H}\ \forall n \quad h[n] \in \ell^2(\mathbb{Z}) \right\}$

Facts: $\mathcal{H}$ is a Hilbert space and $\mathcal{S}$ is a closed subspace.

Therefore, by the orthogonality principle, for each $\ell \in \mathbb{Z}$,

$$\begin{aligned} 0 &= \langle s[n] - \hat{s}[n],\ x[n-\ell] \rangle \\ &= \langle s[n] - \sum_k h[k]\, x[n-k],\ x[n-\ell] \rangle \\ &= r_{sx}(\ell) - \sum h[k]\, r_x[\ell-k] \qquad \forall \ell \in \mathbb{Z} \end{aligned}$$

$\Rightarrow$ h is independent of n, (LTI filter)

Furthermore we can solve for h in the Fourier domain:

$$\hat{r}_{sx}(\omega) = \hat{h}(\omega) \cdot \hat{r}_{xx}(\omega) \qquad \forall \omega \in [-\pi, \pi]$$

$$\Rightarrow \quad \hat{h}(\omega) = \frac{\hat{r}_{sx}(\omega)}{\hat{r}_{xx}(\omega)}$$

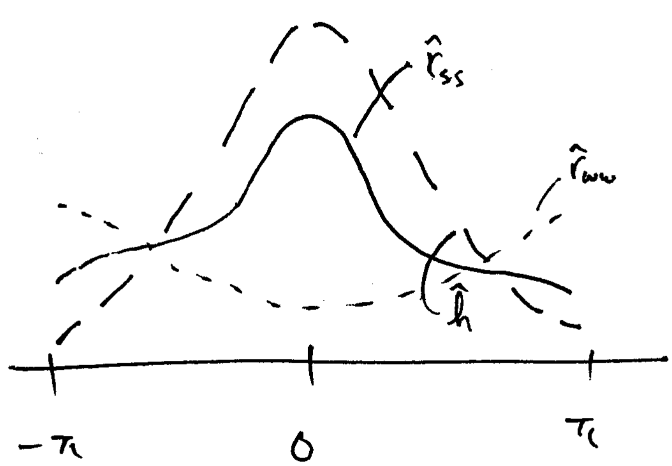
$\leftarrow$ cross spectral density.

$\leftarrow$ spectral density

If $s[n]$ and $w[n]$ are independent, then

$$r_{sx} = r_{ss}, \quad r_{xx} = r_{ss} + r_{ww}$$

$$\Rightarrow \quad \hat{h}(\omega) = \frac{\hat{r}_{ss}(\omega)}{\hat{r}_{ss}(\omega) + \hat{r}_{ww}(\omega)}$$



low frequency signal
high frequency noise
$\Rightarrow$ low-pass filter to estimate s

EIGENVALUES AND EIGENVECTORS

Let V be a vector spaces over a field K, and let $L : V \to V$ be linear. If $\lambda \in K$ and $\underline{0} \neq \underline{v} \in V$ are such that

$$L\underline{v} = \lambda \underline{v},$$

we say λ is an *eigenvalue* of L, and $\underline{v}$ is an *eigenvector* corresponding to λ.

We will focus on $V = \mathbb{C}^N$ (or $\mathbb{R}^N$) so that L is an $N \times N$ matrix, which we will usually denote A.

How to determine eigenvalues/vectors? Note

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } A &\iff \exists\, \underline{v} \neq 0 \text{ s.t. } A\underline{v} = \lambda \underline{v} \\
&\iff \exists\, \underline{v} \neq 0 \text{ s.t. } (A - \lambda I)\underline{v} = 0 \\
&\iff A - \lambda I \text{ is singular (noninvertible)} \\
&\iff \det(A - \lambda I) = 0 \\
&\iff \lambda \text{ is root of the poly eqn. } \det(A - \lambda I) = 0.
\end{aligned}$$

$\boxed{\text{characteristic poly}}$

Since $\det(A - \lambda I)$ is a degree N poly, there are N roots $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$.

Example 1)

$$A = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & .5 \\ .5 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - .25 = 0$$

$$\Rightarrow \quad \lambda = 1 \pm 0.5$$

$$\Rightarrow \quad \lambda_1 = 1.5\,, \quad \lambda_2 = 0.5$$

$\lambda_1 = 1.5$

$$\begin{bmatrix} -.5 & .5 \\ .5 & -.5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad v_{11} = v_{12}$$

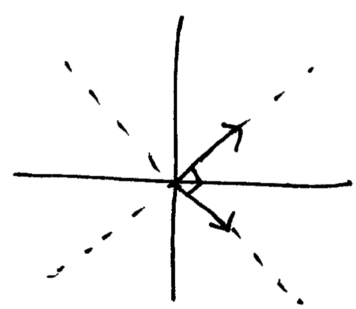
Choose $\underline{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ as a normalized representative

$\lambda_2 = 0.5$

$$\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad v_{21} = -v_{22}$$

Choose $\underline{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.



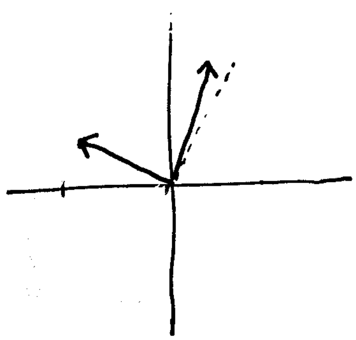
Note: Eigenvectors corresponding to distinct eigenvalues are orthogonal.

Example 2 $\qquad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = (1-\lambda)\cdot(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2 = 0$$

$$\Rightarrow \lambda = \frac{5 \pm \sqrt{33}}{2}$$

$$\underline{v}_1 = \begin{bmatrix} .416 \\ .909 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} -.825 \\ .566 \end{bmatrix}$$



$$\langle \underline{v}_1, \underline{v}_2 \rangle = 0.172 \neq 0$$

Example 3 $\qquad A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = (1-\lambda)(4-\lambda) + 6 = \lambda^2 - 5\lambda + 10 = 0$$

$$\Rightarrow \lambda = \frac{5 \pm \sqrt{-15}}{2} \notin \mathbb{R}$$

Example 4 $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$

Then $\quad A\underline{v} = \underline{v} \quad \forall \underline{v} \in \mathbb{C}^2$

$\Rightarrow \lambda = 1$ is the only eigenvalue, and all vectors are eigenvectors

$$\det(A - \lambda I) = (1-\lambda)^2 = 0$$

$\Rightarrow \lambda = 1$ (multiplicity 2).

Example 5 $\longrightarrow A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$. Eigenvectors depend on whether λ_i are distinct.

Characteristic Polynomial

The polynomial $p(\lambda) = \det(A - \lambda I)$ is a degree N polynomial. The coefficient of λ^N is $(-1)^N$, and therefore

$$p(\lambda) = \prod_{i=1}^{N} (\lambda_i - \lambda)$$

If we set $\lambda = 0$ we get

$$\det(A) = \prod_{i=1}^{N} \lambda_i \Rightarrow \boxed{A \text{ is invertible iff } \lambda_i \neq 0 \ \forall i}$$

The coefficient of λ^N is

$$\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} a_{ii} =: tr(A)$$

Terminology: The set of eigenvalues of a lin. trans are called its spectrum.

Note that if we group common factors, $p(\lambda)$ can be written

$$p(\lambda) = (\lambda_1 - \lambda)^{e_1} \cdots (\lambda_n - \lambda)^{e_n}$$

where $n \leq N$, $e_i \geq 1$, and $\lambda_1, \dots, \lambda_n$ are distinct. We refer to e_i as the <u>algebraic multiplicity</u> of λ_i.

<u>Eigenspaces</u>

Let $A \in \mathbb{C}^{N \times N}$ and let λ be an eigenvalue of A.

The set

$$\mathcal{E}_\lambda = \{ \underline{v} \in \mathbb{C}^N \mid A\underline{v} = \lambda \underline{v} \}$$

is called the <u>eigenspace</u> corresponding to λ.

Note $\mathcal{E}_\lambda = N(A - \lambda I)$ is a subspace.

$\dim(\mathcal{E}_\lambda) = \text{null}(A - \lambda I)$ is called the <u>geometric</u> multiplicity of λ.

<u>Fact</u> geometric multiplicity of $\lambda \leq$ algebraic multiplicity of λ

Proof: See Mardia, Kent, + Bibby

$\boxed{\text{Diagonalizability}}$

Let $A, B \in \mathbb{C}^{N\times N}$. We say A and B are <u>similar</u> if $\exists$ an invertible matrix T such that

$$A = TBT^{-1}.$$

<u>Exercise 1</u> Show that if A, B are similar, then they have the same eigenvalues.

Soln: Suppose $B\underline{v} = \lambda \underline{v}$. Set $\underline{w} = T\underline{v}$.

Then $A\underline{w} = TB\underline{v} = \lambda T\underline{v} = \lambda \underline{w}$.

A is said to be <u>diagonalizable</u> if it is similar to a diagonal matrix, i.e. $\exists\, T$ s.t.

$$A = T\Lambda T^{-1}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$, $\{\lambda_i\}$ = eigenvalues of A.

Not all matrices are diagonalizable.

Diagonalizable matrices are nice to work with in many settings. We can often translate to the diagonalized coordinate system and computations and concepts become easier.

Application: Matrix Powers

If A is diagonalizable, then

$$A^k = T\Lambda T^{-1} \cdot T\Lambda T^{-1} \cdots T\Lambda T^{-1}$$

$$= T\Lambda^k T^{-1}$$

$$= T \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_N^k \end{bmatrix} T^{-1}$$

Similarly, we can define

$$\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum \frac{T\Lambda^k T^{-1}}{k!}$$

$$= T \left(\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} \right) T^{-1}$$

$$= T \begin{bmatrix} \sum \frac{\lambda_1^k}{k!} & & \\ & \ddots & \\ & & \sum \frac{\lambda_N^k}{k!} \end{bmatrix} T^{-1} = T \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_N} \end{bmatrix} T^{-1}$$

Example | Suppose a system evolves according to

$$\underline{x}[t+1] = A \cdot \underline{x}[t]. \quad \text{Then} \quad \underline{x}[t] = A^t \cdot \underline{x}[0].$$

Now $$A^t = T\Lambda^t T^{-1} = T \begin{bmatrix} \lambda_1^t & & \\ & \ddots & \\ & & \lambda_N^t \end{bmatrix} T^{-1}$$

so the system is stable provided $|\lambda_i| \leq 1 \ \forall i$.

When is a matrix diagonalizable, and how can we determine T?

Observation | A is diagonalizable $\Leftrightarrow$ $\exists$ a basis of eigenvectors

Proof: ($\Rightarrow$) If A is diagonalizable, then $A = T\Lambda T^{-1}$. Write $T = [\underline{v}_1 \cdots \underline{v}_N]$. From $AT = T\Lambda$, reading off each column, we see that $A\underline{v}_i = \lambda_i \underline{v}_i$ for each i. Since T is invertible, $\{\underline{v}_1, \ldots, \underline{v}_N\}$ are LI and hence constitute a basis. ($\Leftarrow$). Suppose $\{\underline{v}_1, \ldots, \underline{v}_N\}$ is a basis of $\mathbb{C}^N$. Then $A\underline{v}_i = \lambda_i \underline{v}_i \ \forall i$, and hence $A \cdot T = \Lambda \cdot T$ where $T = [\underline{v}_1 \cdots \underline{v}_N]$. Since $\{\underline{v}_1, \ldots, \underline{v}_N\}$ is LI, T^{-1} exists $\Rightarrow A = T\Lambda T^{-1}$.

So when do the eigenvalues form a basis?

Theorem | If $\lambda_1, \dots, \lambda_n$ are distinct eigenvalues of a linear transformation L, and $\underline{v}_1, \dots, \underline{v}_n$ are corresp. eigenvectors, then $\{\underline{v}_1, \dots, \underline{v}_n\}$ is LI.

Corollary | If $A \in \mathbb{C}^{N \times N}$ has distinct eigenvalues $\lambda_1, \dots, \lambda_N$, then A is diagonalizable.

Proof: Suppose not, and that

$r_1 \underline{v}_1 + \dots + r_m \underline{v}_m = \underline{0}$ is the shortest nontrivial

LC (all $r_i \neq 0$), renumbering the $\underline{v}_i$ if necessary.

Then

$$r_1 \lambda_1 \underline{v}_1 + r_2 \lambda_1 \underline{v}_2 + \dots + r_m \lambda_1 \underline{v}_m = \underline{0} \qquad (1)$$

and

$$r_1 \lambda_1 \underline{v}_1 + r_2 \lambda_2 \underline{v}_2 + \dots + r_m \lambda_m \underline{v}_m = 0 \qquad (2)$$

by taking L of both sides. Then (1) − (2) is

$$r_2 (\lambda_1 - \lambda_2) \underline{v}_2 + \dots + r_m (\lambda_1 - \lambda_m) \underline{v}_m.$$

Since the λ_i are distinct, we have a shorter nontrivial LC $= \underline{0}$, a contradiction. Note: If eigvals not distincts, say $\lambda_1 = \lambda_2$, then $m = 2$ because $\lambda_1 v_1 - \lambda_2 v_1 = \underline{0}$ and argument fails.

THE SPECTRAL THEOREM

The spectral theorem is an extremely important and powerful result on the diagonizability of Hermetian matrices. Note: $A = A^H \Rightarrow \langle Ax, y \rangle = \langle x, Ay \rangle$.

<u>Proposition</u> If A is an $N \times N$ Hermetian matrix, then

(i) the eigenvalues of A are real

(ii) eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof: Suppose $A\underline{v} = \lambda \underline{v}$, $v \neq \underline{0}$. Then

(i)
$$
\begin{aligned}
\lambda \|\underline{v}\|^2 &= \underline{v}^H A \underline{v} \\
&= \underline{v}^H A^H \underline{v} \\
&= (A\underline{v})^H \underline{v} \\
&= (\lambda \underline{v})^H v \\
&= \bar{\lambda} \|\underline{v}\|^2 \quad \Rightarrow \quad \lambda = \bar{\lambda}.
\end{aligned}
$$

(ii) Suppose $A\underline{v}_1 = \lambda_1 \underline{v}_1$, $A\underline{v}_2 = \lambda_2 \underline{v}_2$, $\lambda_1 \neq \lambda_2$. Then

$$
\begin{aligned}
\langle A\underline{v}_1, \underline{v}_2 \rangle &= \langle \underline{v}_1, A\underline{v}_2 \rangle = \langle \underline{v}_1, \lambda_2 \underline{v}_2 \rangle = \lambda_2 \langle \underline{v}_1, \underline{v}_2 \rangle \\
&\parallel \\
\langle \lambda_1 \underline{v}_1, \underline{v}_2 \rangle &= \lambda_1 \langle \underline{v}_1, \underline{v}_2 \rangle
\end{aligned}
$$

$\Rightarrow (\lambda_1 - \lambda_2) \langle \underline{v}_1, \underline{v}_2 \rangle = 0 \Rightarrow \underline{v}_1 \perp \underline{v}_2$ since $\lambda_1 \neq \lambda_2$.

Therefore, if the eigenvalues of A are distinct, then A is diagonalizable and there exists an orthonormal basis of eigenvectors. → with real eigenvalues.

This is also true even when the eigenvalues are not distinct.

<u>Spectral Theorem</u> If A is Hermitian, then there exists an orthonormal basis of $\mathbb{C}^N$ of eigenvectors of A.

Proof: Let λ_1 be an eigenvalue, and $\underline{v}_1$ a corresponding normalized eigenvector. Define

$$S_1 := \operatorname{span}(\underline{v}_1)^\perp$$

If $\underline{v} \in S_1$, then

$$\begin{aligned} \langle A\underline{v}, \underline{v}_1 \rangle &= \langle \underline{v}, A\underline{v}_1 \rangle \\ &= \langle \underline{v}, \lambda \underline{v}_1 \rangle \\ &= \lambda \langle \underline{v}, \underline{v}_1 \rangle \\ &= 0 \end{aligned}$$

$$\Longrightarrow A\underline{v} \in S_1$$

There we may consider $A_1 : S_1 \to S_1$, the restriction of A to S_1. This is also a Hermitian matrix, so we can recursively construct a basis of $S_1 \approx \mathbb{C}^{N-1}$ to obtain a basis for $\mathbb{C}^N$.

A linear trans $L: V \to V$, V an IPS, is <u>self-adjoint</u> $\Leftrightarrow$ $\langle Lx, y\rangle = \langle x, Ly\rangle \ \forall x,y \in V$.

<u>Spectral Thm</u> Suppose V is a fin. dim. complex IPS, and $L: V \to V$ is self-adjoint. Then ~~∃ an~~ the eigenvalues of L are real, and V has an ONB consisting of eigen~~values~~ vectors of L.

Proof: Let λ_1 be an eigenvalue of L, and $\underline{u}_1$ a corresp. normalized eigenvector. Define

Proof by induction: If $N = 1$, done. Suppose true for $N-1$, let's establish for N.

$$S_1 = \text{span}(\underline{u}_1)^\perp.$$

If $\underline{v} \in S_1$, then

$$\langle L\underline{v}, \underline{u}_1\rangle = \langle \underline{v}, L\underline{u}_1\rangle$$

$$= \langle \underline{v}, \lambda_1 \underline{u}_1\rangle$$

$$= \lambda_1 \langle \underline{v}, \underline{u}_1\rangle$$

$$= 0$$

$\Longrightarrow L\underline{v} \in S_1$. Thus consider

$L_1: S_1 \to S_1$, the restriction of L to S_1.

This is also self-adjoint. By induction, $\exists$ ONB of S_1, say $\{\underline{u}_2, \dots, \underline{u}_N\}$, consisting of eigenvectors of L_1. But EV of L_1 are EVs of L, and $S_1 \perp \underline{u}_1$, so $\{\underline{u}_1, \dots, \underline{u}_N\}$ is an ONB of EVs of L.

We also need to establish the existence of at least one eigenvalue / vector of L. Let Φ be an isomorphism $\Phi: V \to \mathbb{C}^N$. Then $\Phi \circ L \circ \Phi^{-1} : \mathbb{C}^N \to \mathbb{C}^N$ is linear and thus has a matrix representation, say A. We know A has N eigenvalues from char poly and fund. thm. of alg. Say $A\underline{v} = \lambda \underline{v}$. Then

$$\Phi \circ L \circ \Phi^{-1} \underline{v} = \lambda \underline{v}$$

$$\Leftrightarrow \quad L(\Phi^{-1}\underline{v}) = \lambda(\Phi^{-1}\underline{v})$$

so λ is an eigenvalue, $\Phi^{-1}\underline{v}$ an eigenvector.

Matrix Interp

$$A = A^H \Rightarrow A = U \Lambda U^H$$

where $U^H U = U U^H = I_{n \times n}$,
$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{R}$.

Spectral Thm, matrix form

From the spectral theorem, we know that

$$A \cdot \mathcal{U} = \mathcal{U} \cdot \Lambda \cdot \mathcal{U}$$

where $\mathcal{U}^H \mathcal{U} = I_{N \times N}$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$, $\lambda_i \in \mathbb{R}$

Claim: $\mathcal{U} \cdot \mathcal{U}^H = I_{N \times N}$. Need to show

$\mathcal{U}\mathcal{U}^H \underline{v} = \underline{v} \quad \forall \underline{v} \in \mathbb{C}^N$. Write $\underline{v} = \mathcal{U} \cdot \underline{w}$.

Then
$$\mathcal{U} \cdot \mathcal{U}^H \underline{v} = \mathcal{U}\mathcal{U}^H \mathcal{U} \underline{w}$$
$$= \mathcal{U} \underline{w}$$
$$= \underline{v} \qquad \square.$$

If $\mathcal{U}^H \mathcal{U} = \mathcal{U}\mathcal{U}^H = I_{N \times N}$, we say $\mathcal{U}$ is <u>unitary</u>. Unitary matrices are norm preserving:

$$\| \mathcal{U}x - \mathcal{U}y \|^2 = (\mathcal{U}x - \mathcal{U}y)^H (\mathcal{U}x - \mathcal{U}y)$$
$$= (x-y)^H \mathcal{U}^H \mathcal{U} (x-y) = \|x-y\|.$$

So if $A^H = A$, then

$$A = \mathcal{U} \Lambda \mathcal{U}^H$$

for some unitary $\mathcal{U}$, real diagonal Λ.

Projection Interpretation

$$A\underline{x} = \mathcal{U}\Lambda\mathcal{U}^H\underline{x}$$

$$= \mathcal{U}\Lambda\begin{bmatrix}\underline{u}_1^H\underline{x}\\ \vdots\\ \underline{u}_N^H\underline{x}\end{bmatrix} = \mathcal{U}\begin{bmatrix}\lambda_1\underline{u}_1^H\underline{x}\\ \vdots\\ \lambda_N\underline{u}_N^H\underline{x}\end{bmatrix}$$

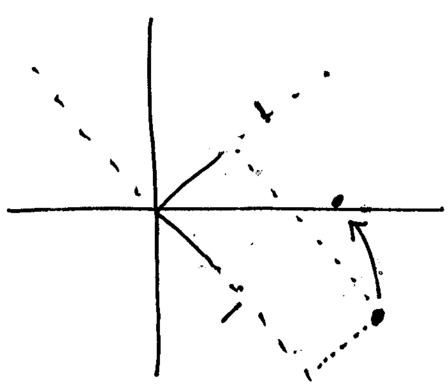
$$= \sum_{i=1}^{N}\underline{u}_i\cdot\lambda_i\underline{u}_i^H\underline{x}$$

$$= \Big(\sum_{i=1}^{N}\lambda_i\underbrace{\underline{u}_i\underline{u}_i^H}_{\text{proj. onto } \underline{u}_i}\Big)\underline{x}$$

Example

$$A = \begin{bmatrix}1 & .5\\ .5 & 1\end{bmatrix}$$

$$\mathcal{U} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix},\quad \Lambda = \begin{bmatrix}1.5 & 0\\ 0 & .5\end{bmatrix}$$



Real Matrices

If $A^T = A$, then $\exists$ a basis of $\mathbb{R}^N$ consisting of orthogonal real eigenvectors.

Thus, we can write

$$A = U \cdot \Lambda \cdot U^T,$$

where $U^T U = U U^T = I$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$, $\lambda_i \in \mathbb{R}$.

The proof is the same as for complex matrices, except now it requires some effort to show that $\exists$ an eigenvalue with a real eigenvector.

If $U^T U = U U^T = I$, $U \in \mathbb{R}^{N \times N}$, we say U is an <u>orthogonal</u> matrix. If $N = 2$,

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ for some } \theta \in [0, 2\pi).$$

So $\quad A = U \Lambda U^T$

$\quad\quad = $ rotate, scale, rotate back

Proof that ~~every real, symmetric matrix has~~ if $A \in \mathbb{R}^{N\times N}$, ~~+~~ $A^T = A$, then A has N real eigenvalues, and ~~the~~ corresponding real eigenvectors.

Since $A^T = A$, $A^H = A$, and thus the complex eigenvalues $\lambda_1, \dots, \lambda_N$ of A are real. Suppose λ is an eigenvalue of A. Let's find a real eigenvector. Let $\underline{v}$ be a complex eigenvector. Two cases:

(i) $\underline{v}$ is purely imaginary: Then set $\tilde{v} = i \cdot \underline{v}$. Then $\tilde{v} \in \mathbb{R}^N$ and

$$A\tilde{v} = i \cdot Av = i\lambda v = \lambda \tilde{v}.$$

(ii) $\underline{v}$ is not purely imaginary. Then set

$$\tilde{v} = v + \bar{v} \in \mathbb{R}^N, \quad \tilde{v} \neq 0. \text{ Then}$$

$$\begin{aligned} A\tilde{v} = Av + A\bar{v} &= \lambda v + \overline{Av} \\ &= \lambda v + \overline{\lambda v} \\ &= \lambda v + \lambda \bar{v} \\ &= \lambda(v + \bar{v}) \\ &= \lambda \tilde{v}. \end{aligned}$$

POSITIVE (SEMI –) DEFINITE MATRICES

Let $A \in \mathbb{C}^{N\times N}$ and $A^H = A$. Notice that for any $x \in \mathbb{C}^N$, $x^H A x \in \mathbb{R}$, because

$$(x^H A x)^H = x^H A^H x = x^H A x. \quad (1\times 1)$$

We say A is <u>positive definite</u> if

$$x^H A x > 0 \quad \forall x \neq \underline{0}.$$

We say A is <u>positive semi-definite</u> (or nonnegative definite) if

$$x^H A x \geq 0 \quad \forall x$$

<u>Notation</u> | $A > 0$ (PD)

$A \geq 0$ (PD)

Eigenvalues of PD/PSD Matrices

Theorem

A is PD (PSD) $\Longleftrightarrow$ the eigenvalues of A are all positive (nonnegative)

Proof: By the spectral thm, $\exists\ \lambda_i \in \mathbb{R}$, $\underline{u}_i \in \mathbb{C}^N$ s.t. $\underline{u}_i^H \underline{u}_j = \delta_{ij}$, s.t.

$$A \underline{u}_i = \lambda_i \underline{u}_i, \qquad i = 1, \ldots, N.$$

$(\Rightarrow)$
$$\begin{aligned}\lambda_i &= \lambda_i \underline{u}_i^H \underline{u}_i \\ &= \underline{u}_i^H \lambda_i \underline{u}_i \\ &= \underline{u}_i^H A \underline{u}_i \begin{cases} > 0 & \text{if } A \text{ PD} \\ \geq 0 & \text{if } A \text{ PSD}\end{cases}\end{aligned}$$

$(\Leftarrow)$ Suppose $\underline{x} \neq \underline{0}$. Then

$$\begin{aligned}\underline{x}^H A \underline{x} &= \underline{x}^H \left(\sum_{i=1}^{N} \lambda_i \underline{u}_i \underline{u}_i^H \right) \underline{x} \\ &= \sum_{i=1}^{N} \lambda_i\, \underline{x}^H \underline{u}_i \cdot \underline{u}_i^H \underline{x} \\ &= \sum_i \lambda_i \| \underline{u}_i^H \underline{x} \|^2 \\ &\begin{cases} > 0 & \text{if } \lambda_i > 0 \ \forall i \\ \geq 0 & \text{if } \lambda_i \geq 0 \ \forall i\end{cases}\end{aligned}$$

Implications

① If A is PSD, then $\det A = \prod \lambda_i \geq 0$

② A is PD $\Rightarrow$ $\det A > 0$

③ A is PD $\Rightarrow$ A is invertible (since $\det A \neq 0$, and since the rank of a diagonalizable matrix is the number of nonzero eigenvalues)

④ If A is PD, $A = U \Lambda U^H$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$, $\lambda_i > 0 \ \forall i$. Then

$$A^{-1} = U \Lambda^{-1} U^H, \quad \Lambda^{-1} = \text{diag}(\lambda_1^{-1}, \dots, \lambda_N^{-1}),$$

because

$$\begin{aligned} U \Lambda U^H \cdot U \Lambda^{-1} U^H &= U \Lambda \Lambda^{-1} U^H \\ &= U U^H \\ &= I_{N \times N} \end{aligned}$$

(true for any nonsingular Hermitian matrix)

and therefore A^{-1} is also PD.

Application: Covariance matrix of multivariate Gaussian.

⑤ If A is real and PSD, then $\exists$ a full rank real matrix B s.t.

$$A = B^H B \qquad \text{(matrix square root).}$$

$$A = \left[\begin{array}{c:c} A_1 & A_2 \end{array}\right] \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_p & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \left[\begin{array}{c} A_1^H \\ \hdashline A_2^H \end{array}\right]$$

where $\lambda_1, \dots, \lambda_p > 0$.

$$\Rightarrow \quad B = A_1 \cdot \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_p} \end{bmatrix}$$

⑥ If $A = (a_{ij})$ is PD (PSD), then the diagonal entries of A, a_{ii}, are positive (nonnegative).

The converse is not true.

Grammians

Let $v_1, \dots, v_N \in V$, an IPS. Set

$$G = \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_N \rangle \\ \langle v_2, v_1 \rangle & \cdots & \langle v_2, v_N \rangle \\ \vdots & \ddots & \vdots \\ \langle v_N, v_1 \rangle & \cdots & \langle v_N, v_N \rangle \end{bmatrix} \in \mathbb{C}^{N \times N}$$

Arises in projections, least squares : $G = A^H A$

<u>Theorem</u> G is PSD, and G is PD $\Leftrightarrow$ $v_1, \dots, v_N$ are LI.

Proof : Let $x = [x_1 \ \dots \ x_N]^T \in \mathbb{C}^N$. Then

$$x^H G x = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \bar{x}_j \langle v_j, v_i \rangle$$

$$= \sum_i x_i \left\langle \sum \bar{x}_j v_j , v_i \right\rangle$$

$$= \left\langle \sum \bar{x}_j v_j , \sum_i \bar{x}_i v_i \right\rangle$$

$$\geq 0$$

with equality iff $\sum \bar{x}_i v_i = \underline{0}$.

Corollary | $|\langle v_1, v_2 \rangle| \leq \sqrt{\langle v_1, v_1 \rangle \cdot \langle v_2, v_2 \rangle}$, w/ equality iff v_1, v_2 are LI.

Pf: Consider $G = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{bmatrix}$

G is PSD $\Rightarrow$ $\det G \geq 0$

$$\det G = \langle v_1, v_1 \rangle \langle v_2, v_2 \rangle - \underbrace{\langle v_2, v_1 \rangle \langle v_1, v_2 \rangle}_{|\langle v_1, v_2 \rangle|^2} \geq 0$$

w/ equality iff v_1, v_2 LI.

THE MULTIVARIATE GAUSSIAN DISTRIBUTION

The random vector $X \in \mathbb{R}^N$ has a MVG dist. if it has the joint density

$$f_{\mu,\Sigma}(x) = (2\pi)^{-\frac{N}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\}$$

where $\mu \in \mathbb{R}^N$, and Σ is P.D. Note that Σ^{-1} is also PD.

Using the spectral theorem and properties of PD matrices, we can prove several important properties

- $\int_{\mathbb{R}^N} f_{\mu,\Sigma}(x)\,dx = 1$
- $\{x : f_{\mu,\Sigma}(x) = c\}$ is an ellipse
- $E[(X-\mu)(X-\mu)^T] = \Sigma$
- generate realizations of X from independent univariate Gaussian RVs.

$\boxed{\text{Contours}}$ Contours of $f_{\mu,\Sigma} \Leftrightarrow$ Contours of $(x-\mu)^T \Sigma^{-1} (x-\mu)$

To ease visualization, let's focus on the case $N=2$.

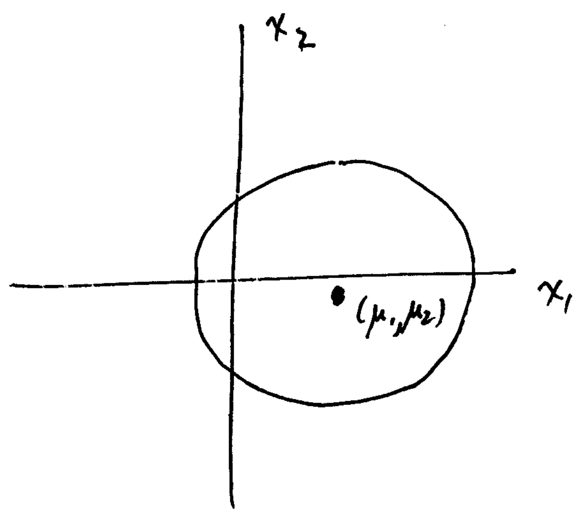
We'll consider 3 cases that increase in generality.

Σ = multiple of identity

$$\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{pmatrix}$$

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = \frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} = c$$

$\Rightarrow$ circle



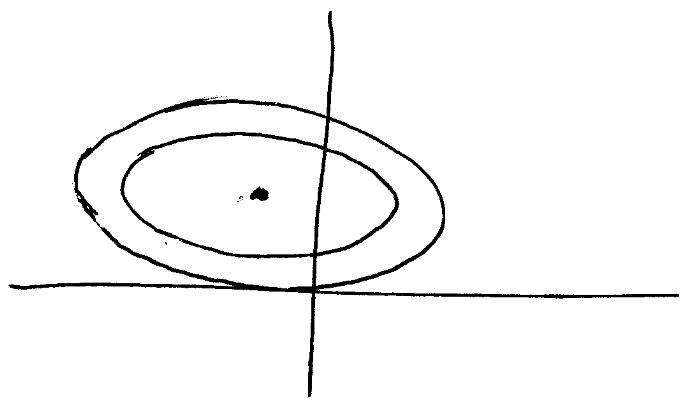
$f_{\mu,\Sigma}$ = 2d bell surface, radially symmetric, circular contours

Σ = diagonal

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \Rightarrow \Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{pmatrix}$$

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = \frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} = c$$

$\Rightarrow$ ellipse



- axes are coordinate axes
- slice along any axis = bell curve

Σ = arbitrary PD matrix

$$\Sigma = U \Lambda U^T \Rightarrow \Sigma^{-1} = U \Lambda^{-1} U^T$$

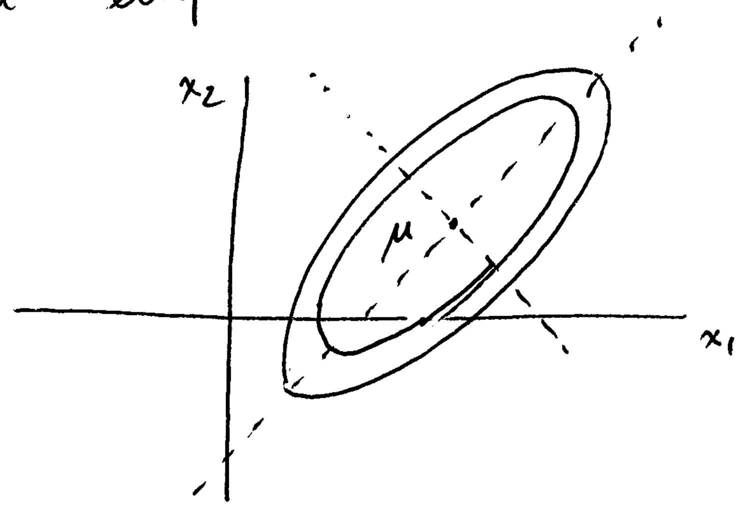
$$\begin{aligned}(x-\mu)^T \Sigma^{-1} (x-\mu) &= (x-\mu)^T U \Lambda U^T (x-\mu) \\ &= [U^T(x-\mu)]^T \Lambda [U^T(x-\mu)] \\ &= (\tilde{x}-\tilde{\mu})^T \Lambda (\tilde{x}-\tilde{\mu})\end{aligned}$$

where $\tilde{x} = U^T x$, $\tilde{\mu} = U^T x$

and U^T is a rotation

$$= \frac{(\tilde{x}_1 - \tilde{\mu}_1)^2}{\lambda_1} + \frac{(\tilde{x}_2 - \tilde{\mu}_2)^2}{\lambda_2} = c$$

$\Rightarrow$ rotated ellipse



Note: $U^T x = \begin{bmatrix} u_1^T x \\ \vdots \\ u_N^T x \end{bmatrix}$ = expansion coefficients of x

in the basis $\{u_1, \dots, u_N\}$.

$\Rightarrow$ u_i are the axes of the ellipses,

λ_i determine relative length

$\underline{\underline{N > 2}}$

contours are arbitrary ellipsoids

$\boxed{\text{Covariance}}$

Thm) If $X \sim N(\mu, \Sigma)$, then

$$E\left[(X-\mu)(X-\mu)^T\right] = \Sigma.$$

Proof: First, consider the case $\mu = 0$, $\Sigma = I_{N\times N}$.

Then
$$E\left[X X^T\right] = \begin{bmatrix} E[X_1^2] & E[X_1 X_2] & & \\ E[X_2 X_1] & E[X_2^2] & & \\ & & \ddots & \\ & & & E[X_N^2] \end{bmatrix}$$

We want to show this is I.

Consider

$$E[X_1^2] = \int_{\mathbb{R}^N} x_1^2 \, f_{0,I}(\underline{x}) \, d\underline{x}$$

$$= \int x_1^2 \, (2\pi)^{-N/2} e^{-\frac{1}{2}\underline{x}^T \underline{x}} \, d\underline{x}$$

$$= \int x_1^2 (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_1^2} dx_1 \cdot \int (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_2^2} dx_2$$

$$\cdots \int (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_N^2} dx_n$$

$$= \left(\text{Variance of } N(0,1)\right) \cdot 1 \cdots 1$$

$$= 1.$$

Similarly, $E[X_i^2] = 1 \quad \forall i$.

Now consider

$$E[X_1 X_2] = \int x_1 x_2 (2\pi)^{-N/2} e^{-\frac{1}{2}(x_1^2 + \cdots + x_N^2)} \, d\underline{x}$$

$$= \int x_1 (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_1^2} dx_1 \cdot \int x_2 (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_2^2} dx_2$$

$$\cdot \int (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_3^2} dx_3 \cdots$$

$$= 0$$

Similarly, $E[X_i X_j] = 0 \quad \forall i \neq j$

Now consider general $X \sim N(\mu, \Sigma)$.

Fact: If $X \sim N(\mu, \Sigma)$ and $Y = AX + b$,
then $Y \sim N(A\mu + b, A\Sigma A^T)$.

Proof: Use characteristic function, $\Phi(\omega) = e^{i\omega^T\mu - \frac{1}{2}\omega^T \Sigma \omega}$

Set $Y = \Lambda^{-\frac{1}{2}} U^T (X - \mu)$, where $\Sigma = U \Lambda U^T$.

Then $Y \sim N(0, I)$, so

$$E[XX^T] = E[U\Lambda^{\frac{1}{2}} Y \cdot Y^T \Lambda^{\frac{1}{2}} U^T]$$

$$= U\Lambda^{\frac{1}{2}} E[YY^T] \cdot \Lambda^{\frac{1}{2}} U^T = U\Lambda^{\frac{1}{2}} I \cdot \Lambda^{\frac{1}{2}} U^T$$

$$= U \Lambda U^T = \Sigma. \qquad \square$$

Random Number Generation

Suppose you wish to generate a realization of $X \sim N(m, R)$, but you only have access to a univariate, standard normal RNG, such as Matlab's randn.

① Generate $Y \sim N(0, I)$:

```
>> Y = randn(N,1);
```

② Compute spectral decomp. of R:

```
>> [U,L] = eig(R);
```

③ Transform $X = UL^{\frac{1}{2}}Y + m$:

```
>> X = U*sqrt(L)*Y + m;
```

Then $X \sim N\left(UL^{\frac{1}{2}} \cdot 0 + \mu,\ UL^{\frac{1}{2}} \cdot I \cdot L^{\frac{1}{2}} U^T\right)$

$$= N(m, R)$$

EIGENFILTERS

$$f[n] \longrightarrow \underset{\uparrow\, v[n]}{\oplus} \longrightarrow \boxed{\underline{h}} \longrightarrow y[n]$$

$f[n]$: 0 mean, wss rand process (signal)

$v[n]$: 0 mean, white noise process, $E[|v[n]|^2] = \sigma^2$

$\underline{h}$: FIR filter, length m

Denote

$$\underline{f}[n] = \begin{bmatrix} f[n] \\ \vdots \\ f[n-m+1] \end{bmatrix}, \quad \underline{v}[n] = \begin{bmatrix} v[n] \\ \vdots \\ v[n-m+1] \end{bmatrix}$$

Then $y[n] = \underline{h}^H(\underline{f}[n] + \underline{v}[n])$

Output power due to signal:

covariance matrix

$$E[|\underline{h}^H \underline{f}[n]|^2] = E[\underline{h}^H \underline{f}[n] \cdot \underline{f}[n] \cdot \underline{h}] = \underline{h}^H R\, \underline{h}$$

Output power due to noise:

$$E[|\underline{h}^H \underline{v}[n]|^2] = \sigma^2 \underline{h}^H \underline{h}.$$

Maximize ratio: $$\max_{\underline{h} \neq 0} \frac{\underline{h}^H R\, \underline{h}}{\sigma^2 \underline{h}^H \underline{h}} = \frac{\lambda_1}{\sigma^2}$$

$\Rightarrow \underline{h}$ = eigenvector of R corresp. to largest eigenvalue.

Next time could
have a packet on
the Rayleigh quotient,
~~pe~~ PCA, and include
this as an example.

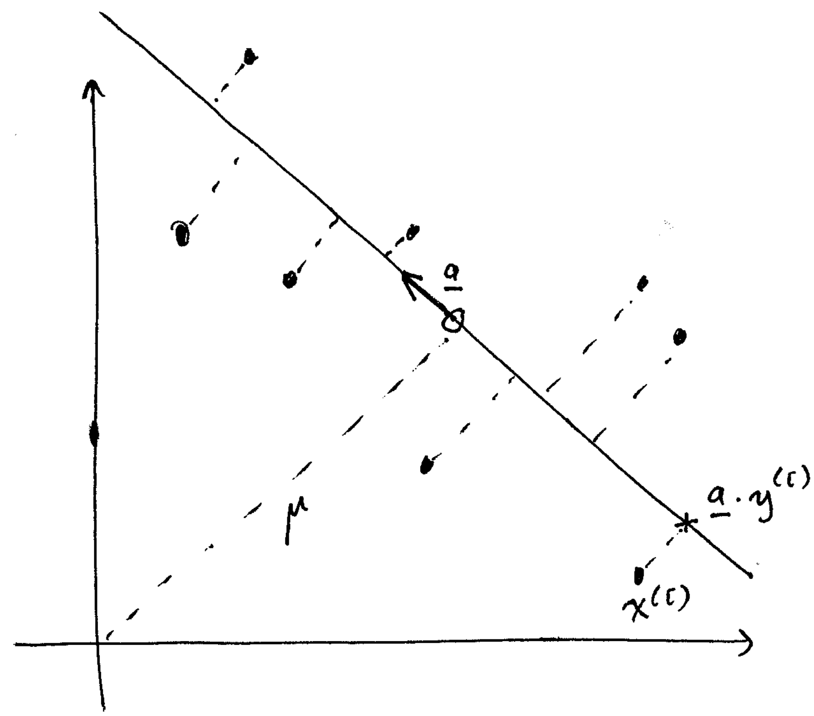
PRINCIPAL COMPONENT ANALYSIS (PCA)

Motivation 1: Low rank approximation

Given $x^{(1)}, \dots, x^{(n)} \in \mathbb{R}^N$, $p < N$. Seek $A \in \mathbb{R}^{N \times p}$, $\mu \in \mathbb{R}^N$
$y^{(1)}, \dots, y^{(n)} \in \mathbb{R}^p$ s.t.

$$x^{(i)} \approx \mu + A \cdot y^{(i)}, \quad i = 1, \dots, n.$$

$N = 2$
$p = 1$



$A = [a]$ (1×1)

Mathematically, we want to minimize

$$\sum_{i=1}^{n} \| x^{(i)} - \mu - A y^{(i)} \|^2$$

w.r.t. $\mu \in \mathbb{R}^N$, $y^{(i)} \in \mathbb{R}^p$, and $A \in \mathbb{R}^{N \times p}$, $A^T A = I$.

no loss of generality, and gives interp. in terms of $\perp$ coordinates.

Motivation 2 : Max Variance Subspace

Given $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^N$, $p < N$. Construct $y^{(1)}, \ldots, y^{(n)} \in \mathbb{R}^p$ as follows:

$\underline{k=1}$: Find $\underline{a}_1 \in \mathbb{R}^N$, $\|\underline{a}_1\| = 1$ s.t.

$$y_1^{(i)} := \underline{a}_1^T (x^{(i)} - \overline{x})$$

has maximum (sample) variance

$\underline{1 < k \leq p}$ Having found $\mu, \underline{a}_1, \ldots, \underline{a}_{k-1}$, find $\underline{a}_k \in \mathbb{R}^N$, $\|\underline{a}_k\| = 1$, $\underline{a}_k \perp \underline{a}_\ell$, $\ell < k$

s.t. $$y_k^{(i)} := \underline{a}_k^T (x^{(i)} - \overline{x})$$

has maximum (sample) variance

It turns out that both of the problems have the same solution: Define the <u>sample mean</u>

$$\bar{x} := \frac{1}{n}\sum_{i=1}^{n} x^{(i)}$$

and <u>sample covariance matrix</u>

$$S := \frac{1}{n}\sum_{i=1}^{n} (x^{(i)} - \bar{x})(x^{(i)} - \bar{x})^T.$$

Let $\quad S = U \Lambda U$

$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0.$

(why is $S \geq 0$?)

<u>Theorem</u> The solutions to both the low-rank approx and max variance approx is given by

$$\underline{a}_i = \underline{u}_i, \quad \underline{\mu} = \bar{x}, \quad y^{(i)} = A^T(\underline{x}^{(i)} - \bar{x})$$

For $x \in \mathbb{R}^N$, we say $y_k = \underline{a}_k^T(\underline{x} - \bar{x})$ is the kth <u>principal component</u> of x.

We will verify this for the max variance formulation.

$$\mathrm{var}\left(\{y_k^{(i)}\}_{i=1}^{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(y_k^{(i)} - \overline{y}_k\right)^2$$

where

$$y_k^{(i)} = a_k^T\left(x^{(i)} - \overline{x}\right)$$

$$\overline{y}_k = \frac{1}{n}\sum y_k^{(i)} = \frac{1}{n}\sum\left(a_k^T\left(x^{(i)} - \overline{x}\right)\right)$$

$$= a_k^T\left(\overline{x} - \overline{x}\right) = 0$$

So

$$\mathrm{var}\left(\{y_k^{(i)}\}\right) = \frac{1}{n}\sum_{i=1}^{n} a_k^T\left(x^{(i)} - \overline{x}\right)\left(x^{(i)} - \overline{x}\right)a_k$$

$$= a_k^T S\, a_k$$

Since $S \succeq 0$, the result follows from the following more general result.

$$\max_{a \neq 0} \frac{a^H W a}{a^H a} = \max_{\|a\|=1} \frac{a^H W a}{a^H a}$$

Rayleigh Quotients

Theorem Let $W \geq 0$, $W \in \mathbb{C}^{N \times N}$. Let $\lambda_1 \geq \cdots \geq \lambda_N \geq 0$ be eigenvalues of W, $\underline{u}_1, \ldots, \underline{u}_N$ corresp. eigenvectors. Then

$$\max_{a : \|a\|=1} a^H W a = \lambda_1$$

and the max is achieved by $\underline{u}_1$. In addition:

$$\max_{\substack{a : \|a\|=1, \\ a \perp u_\ell, \ell < k}} a^H W a = \lambda_k$$

and the max is achieved by $\underline{u}_k$.

Proof: Write $W = U \Lambda U^H$. Let $a \in \mathbb{C}^N$, $\|a\|=1$.

Then $a^H W a = a^H U \Lambda U^H a = (U^H a) \Lambda (U^H a)$.

Set $b = U^H a$. Then $\|b\|=1$, and

$$\max_{a : \|a\|=1} a^H W a = \max_{b : \|b\|=1} b^H \Lambda b.$$

Now $b^H \Lambda b = \sum_{j=1}^{N} \lambda_j |b_j|^2$. This is maximized when $b = [1 \; 0 \cdots 0]^T \Rightarrow a = Ub = \underline{u}_1$.

Uniqueness? If $\lambda_1 > \lambda_2$, then maximizer is ~~(complex)~~ unit scalar multiple of $\underline{u}_1$. Else, any ~~sc~~ vector in eigenspace of λ_1.

Similarly, if $a \perp \{\underline{u}_1, \dots, \underline{u}_{k-1}\}$, then

$$b = \mathcal{U}^H a = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_k \\ \vdots \\ b_N \end{bmatrix}, \quad \text{so} \quad b^H \Lambda b = \sum \lambda_j |b_j|^2$$

is maximized when $b_k = 1 \Rightarrow a = \underline{u}_k$.

Similar comments on uniqueness. $\blacksquare$

The quantity $\dfrac{\lambda_1 + \cdots + \lambda_p}{\lambda_1 + \cdots + \lambda_N}$ is the % total variation explained by the first p PCs.

used to choose p.

$\boxed{\text{Population Versions}}$

Now suppose we have a RV $X \in \mathbb{R}^N$ as opposed to a random sample. The previous results generalize to this setting. Here the problems are (assume X is zero mean, $E[XX^T] = R$)

- $$\min_{\substack{A^TA = I_{r \times r} \\ Y = KX,\ K \in \mathbb{R}^{p \times N}}} E[\|X - AY\|^2]$$
- $$\max_{\substack{\|a\|=1 \\ a \perp a_1, \dots, a_{k-1}}} a^T R a$$

Solution: A = first p eigenvectors of R, $K = A^T$

The proofs are essentially the same.

$\boxed{\text{Karhunen-Loeve Transform}}$

Let $X \in \mathbb{R}^n$ be a zero mean RV with $E[XX^T] = R$.

Write $R = U \Lambda U^T$, U orthogonal, $\lambda_1 \geq \cdots \geq \lambda_N \geq 0$.

Set $Y = U^T X$. Then

$$E[YY^T] = U^T R U = \Lambda \qquad \text{(uncorrelated)}$$

$\Rightarrow$ U^H is a "whitening filter,"

called the KLT. The first p variables

of Y give the best p-dim linear approximation to X

in the sense of minimizing $E[\|X - A\begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix}\|^2$

for some A. The KLT is often used as a

theoretical tool in communication / compression theory.

THE SINGULAR VALUE DECOMPOSITION

<u>Theorem</u> Let $A \in \mathbb{C}^{m \times n}$. There exist unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, $p = \min(m,n)$, s.t.

$$A = U \Sigma V^H$$

where $\Sigma \in \mathbb{R}^{m \times n}$, $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_p)$.

If $A \in \mathbb{R}^{m \times n}$, then U, V are real, orthogonal matrices.

$m \leq n$

$$\begin{bmatrix} & A & \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_p & \end{bmatrix} \begin{bmatrix} \\ & V^H & \\ \\ \end{bmatrix}$$

$m \geq n$

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} = \begin{bmatrix} \\ & U & \\ \\ \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_p \\ \\ \end{bmatrix} \begin{bmatrix} V^H \end{bmatrix}$$

The SVD exists for arbitrary matrices.

$\sigma_1, \dots, \sigma_p$ are called the <u>singular values</u> of A

$\mathcal{U} = [\underline{u}_1 \ \cdots \ \underline{u}_m] \to$ <u>left singular vectors</u>

$V = [\underline{v}_1 \ \cdots \ \underline{v}_n] \to$ <u>right " "</u>

<u>Lemma</u> Let $A \in \mathbb{C}^{m \times n}$. Then $A^H A$ and AA^H are PSD, and they have the same positive eigenvalues, with the same multiplicities.

Pf: $\underline{x}^H A^H A \underline{x} = \| A \underline{x} \|^2 \geq 0$

$\underline{x}^H A A^H \underline{x} = \| A^H \underline{x} \|^2 \geq 0$

Suppose $A^H A \underline{v} = \lambda \underline{v}$, $\lambda > 0$, $v \neq \underline{0}$. Set $\underline{u} = A\underline{v}$. Then $AA^H u = AA^H A v = A \lambda v$
$= \lambda A v = \lambda u$. Similarly, suppose $AA^H u = \lambda u$.
Set $v = A^H u$. Then $A^H A v = A^H A A^H u = A^H \lambda u$
$= \lambda A^H u = \lambda v$. Where did we use $\lambda > 0$?
If $\lambda = 0$, then $u = Av = \lambda v = \underline{0}$, so $\underline{u}$ is not an eigenvector.

Multiplicities: Since $A^H A$ and AA^H are diagonalizable, geom. mult = alg. mult. Also, from construction, if $v_1 \perp v_2$ are in the same eigenspace, then $\langle Av_1, Av_2 \rangle = 0$, so eigenspaces maps to eigenspaces.

Lemma Let $A \in \mathbb{C}^{m \times n}$. Then ① $R(A)^\perp = N(A^H)$

② $R(A^H)^\perp = N(A)$.

Proof of ①: $\underline{N(A^H) \subseteq R(A)^\perp}$ Let $x \in N(A^H)$

and $y \in R(A)$, so that $y = Az$ for some $z \in \mathbb{C}^n$.

Then $\langle x, y \rangle = \langle x, Az \rangle = \langle A^H x, z \rangle = \langle 0, z \rangle = 0$.

$\underline{R(A)^\perp \subseteq N(A^H)}$: Suppose $x \perp y \quad \forall y \in R(A)$.

Then $x \perp Az \quad \forall z \in \mathbb{C}^n$. That is

$$0 = \langle x, Az \rangle = \langle A^H x, z \rangle \quad \forall z \in \mathbb{C}^n$$

$$\Rightarrow A^H x = \underline{0} \Rightarrow x \in N(A^H).$$

Proof of ②: Apply ① with $A \to A^H$.

Proof of SVD Theorem

Let $A^H A = V \Lambda V^H$ be the spectral decomposition of $A^H A$, with positive eigenvalues $\lambda_1 \geq \cdots \geq \lambda_r$, $r \leq p$.

and $V = [\underline{v}_1 \cdots \underline{v}_n]$. For $i = 1, \ldots, r$, set

$$\underline{u}_i := \frac{1}{\sqrt{\lambda_i}} A \underline{v}_i$$

Notice

$$\langle \underline{u}_i, \underline{u}_j \rangle = \frac{1}{\sqrt{\lambda_i \lambda_j}} (A \underline{v}_i)^H (A \underline{v}_j) = \underline{v}_i^H A^H A \underline{v}_j$$

$$\frac{\lambda_j}{\sqrt{\lambda_i \lambda_j}} \underline{v}_i^H \underline{v}_j = \delta_{ij} \quad , \quad 1 \le i, j \le r.$$

Let $\underline{u}_{r+1}, \ldots, \underline{u}_m \in \mathbb{C}^m$ be such that

$\mathcal{U} = [\underline{u}_1 \ \ldots \ \underline{u}_m]$ is unitary. Then

$$A A^H = \sum_{i=1}^{r} \lambda_i \underline{u}_i \underline{u}_i^H + \sum_{i=r+1}^{m} 0 \cdot u_i u_i^H$$

$$= \mathcal{U} \cdot \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \mathcal{U}^H$$

We will show

$$\mathcal{U}^H A V = \operatorname{diag}(\sigma_1, \ldots, \sigma_p) \quad , \quad \sigma_i = \sqrt{\lambda_i}$$

The (i,j) element of $\mathcal{U}^H A V$ is

$$\underline{e}_i^H (\mathcal{U}^H A V) \underline{e}_j = (\mathcal{U} \underline{e}_i)^H A V \underline{e}_j = \underline{u}_i^H A \underline{v}_j.$$

<u>Case 1 : $i \le r$</u> :

$$\underline{u}_i^H A \underline{v}_j = \frac{1}{\sqrt{\lambda_i}} (A \underline{v}_i)^H A \underline{v}_j = \frac{1}{\sqrt{\lambda_i}} \underline{v}_i^H A^H A \underline{v}_j$$

$$= \frac{\lambda_j}{\sqrt{\lambda_i}} v_i^H v_j = \sqrt{\lambda_i} \, \delta_{ij}$$

✓

Here, use 2^{nd} lemma
to show $\underline{u}_{r+1}, \ldots, \underline{u}_m$
$\in \mathcal{E}_0 = N(AA^H)$ to
improve this argument

Case 2 : $i > r$:

$$A \cdot A^H \underline{u}_i = \underline{0} \implies A^H \underline{u}_i \in N(A)$$

Obviously, $\quad A^H \underline{u}_i \in R(A^H)$

So $\quad A^H \underline{u}_i \in N(A) \cap R(A^H) = \{\underline{0}\}$

because $N(A)$, $R(A^H)$ are orthogonal compliments.

Thus $\quad A^H \underline{u}_i = \underline{0}$. Therefore

$$\underline{u}_i^H A \underline{v}_j = (A^H \underline{u}_i)^H \underline{v}_j = 0 . \qquad \blacksquare$$

For the case of real A, use the same argument, but apply the spectral theorem for real symmetric matrices.

Observations /

- right singular vectors = eigenvectors of $A^H A$
- left " " = " " $A A^H$
- positive singular values = square roots of positive eigenvalues of $A^H A / A A^H$.

This suggests a method for computing the SVD, but not a very efficient one.

This is now redundant,
based on 1st
sticky above.

SVD and rank

Claim: $N(A) = N(A^H A)$.

($\subset$) $\underline{x} \in N(A) \Rightarrow A\underline{x} = \underline{0} \Rightarrow A^H A \underline{x} = 0 \Rightarrow \underline{x} \in N(A^H A)$

($\supset$) $\underline{x} \in N(A^H A) \Rightarrow A^H A \underline{x} = 0 \Rightarrow \underline{x}^H A^H A \underline{x} = 0$

$\Rightarrow \| A\underline{x} \|^2 = 0 \Rightarrow A\underline{x} = \underline{0} \Rightarrow \underline{x} \in N(A)$.

Therefore

$$
\begin{aligned}
\text{rank}(A) &= n - \dim(N(A)) \\
&= n - \dim(N(A^H A)) \\
&= \text{rank}(A^H A) \\
&= r \quad (\#\text{ of nonzero eigenvalues}) \\
&= \#\text{ of nonzero singular values.}
\end{aligned}
$$

Sum of rank one matrices

For any $\underline{x}$, $A\underline{x} = U \Sigma V^H \underline{x} = U \Sigma \begin{bmatrix} \underline{v}_1^H \underline{x} \\ \vdots \\ \underline{v}_n^H \underline{x} \end{bmatrix}$

$$
= U \cdot \begin{bmatrix} \sigma_1 \underline{v}_1^H \underline{x} \\ \vdots \\ \sigma_r \underline{v}_r^H \underline{x} \\ 0 \end{bmatrix} = \sum_{i=1}^{r} (\sigma_i \underline{v}_i^H \underline{x}) \cdot \underline{u}_i
$$

$$
= \left(\sum_{i=1}^{n} \sigma_i \underline{u}_i \underline{v}_i^H \right) x \Rightarrow A = \sum_{i=1}^{r} \sigma_i \underline{u}_i \underline{v}_i^H .
$$

MINIMUM NORM SOLUTION FOR UNDERDETERMINED L.S.

Recall the least squares problem:

$$\min_{\theta \in K^p} \| \underline{x} - A \underline{\theta} \|^2$$

where $\underline{x} \in K^N$, $A \in K^{N \times p}$, $K = \mathbb{R}$ or $\mathbb{C}$.

If $\text{rank}(A) = p$, the solution is $\hat{\underline{\theta}} = (A^H A)^{-1} A^H \underline{x}$.

What if $\text{rank}(A) < p$? We could just eliminate redundant columns. But suppose we don't want to do that. Then there are infinitely many minimizers $\{\theta = \theta_0 + v \mid v \in N(A)\}$, where θ_0 is a solution.

$\dim > 0$ if $\text{rank}(A) < p$.

Let's find the solution with minimum norm.

<u>Diagonal Case</u>

First suppose $A = \Sigma$ where

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \qquad N \times p$$

$\sigma_1 \geq \cdots \geq \sigma_r > 0$. Notice $= \Pi_A \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Then a least squares solution is a solution to

$$\begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_p \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then one solution is $\underline{\theta}_0 = \left[\frac{x_1}{\sigma_1} \; \cdots \; \frac{x_r}{\sigma_r} \; 0 \cdots 0 \right]^T$.

An arbitrary solution has the form $\underline{\theta} = \underline{\theta}_0 + \underline{\theta}_1$,

where $\underline{\theta}_1 \in N(A)$, i.e. $\underline{\theta}_1 = [0 \cdots 0 \;\; \theta_{r+1} \cdots \theta_p]^T$

Now $\|\underline{\theta}\|^2 = \|\underline{\theta}_0\|^2 + \|\underline{\theta}_1\|^2 \geq \|\underline{\theta}_0\|^2$,

so the minimum norm solution is

$$\underline{\theta}_0 = \underbrace{\begin{pmatrix} \frac{1}{\sigma_1} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{\sigma_r} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}}_{p \times N} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \Sigma^+ \underline{x}$$

General Case

Let $A = U \Sigma V^H$ (SVD). Then

$$\min_{\underline{\theta}} \| \underline{x} - A \underline{\theta} \| = \min_{\underline{\theta}} \| \underline{x} - U \Sigma V^H \underline{\theta} \|$$

$$= \min_{\underline{\theta}} \| U^H \underline{x} - \Sigma V^H \underline{\theta} \|$$

Now set $\underline{\phi} = V^H \underline{\theta}$ (change of variables),

$\underline{y} = U^H \underline{x}$. Then

$$\underset{\substack{\| \\ V^H \hat{\underline{\theta}}}}{\hat{\underline{\phi}}} = \Sigma^+ \underline{y} = \Sigma^+ U^H \underline{x}$$

$$\Rightarrow \quad \hat{\underline{\theta}} = V \Sigma^+ U^H \underline{x}$$

$$(p \times 1) = (p \times p)(p \times N)(N \times N)(N \times 1)$$

This holds for all A, even if $\mathrm{rank}(A) = p$ or if A is invertible.

MATRIX APPROXIMATION

Consider the vector space $\mathbb{C}^{m\times n}$ regarded as $m\times n$ matrices w/ complex entries. This space can be equipped w/ a norm in several ways.

An example of an "operator norm" we will consider the l_2 norm

$$\|A\|_2 := \sup_{\|x\|_2=1} \|Ax\|_2$$

<u>Fact</u>: $\|A\|_2 =$ largest singular value of A.

Given a matrix A with rank r, we could ask what matrix B of rank $k<r$ minimizes $\|A-B\|_2$.

<u>Theorem</u> Let $A \in \mathbb{C}^{m\times n}$, $\text{rank}(A)=r$, and $k<r$. Let $A = U\Sigma V^H$ be the SVD of A. Set

$$A_k := \sum_{i=1}^{k} \sigma_i u_i v_i^H = U\Sigma_k V^H$$

where $\Sigma_k = \text{diag}(\sigma_1, \ldots, \sigma_k)$. Then

$$\|A - A_k\|_2 = \min_{B:\text{rank}(B)=k} \|A-B\|_2 = \sigma_{k+1}$$

Proof: $A - A_k = U \cdot (\Sigma - \Sigma_k) V^H$ so

$\| A - A_k \|_2 =$ largest sing. val. $= \sigma_{k+1}$.

Let $B \in \mathbb{C}^{m \times n}$, $\operatorname{rank}(B) = k$. Let $\underline{z} \in \mathbb{C}^n$ s.t.

$\|\underline{z}\|_2 = 1$ and

$$\underline{z} \in \underbrace{N(B)}_{\dim = n-k} \cap \underbrace{\operatorname{span}\{v_1, \dots, v_{k+1}\}}_{\dim = k+1} \neq \{0\}$$

Then $\|A - B\|_2^2 \geq \|(A-B)z\|_2^2 = \|Az\|^2$

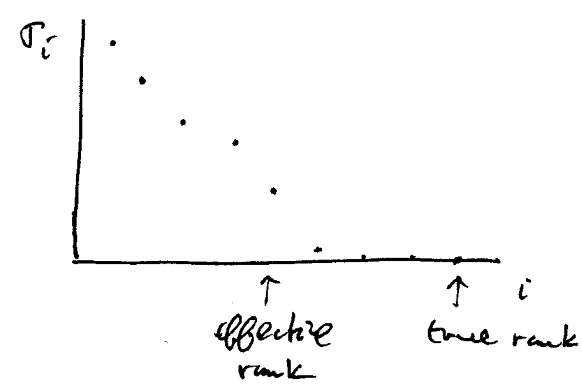
$$= \Big\| \sum_{i=1}^{k+1} \sigma_i \, (\underline{v}_i^H \underline{z}) \, \underline{u}_i \Big\|^2$$

$$= \sum_{i=1}^{k+1} \sigma_i^2 \cdot (\underline{v}_i^H \underline{z})^2 \geq \sigma_{k+1}^2 \qquad \blacksquare$$

Question: Is B unique? Yes, if $\sigma_k > \sigma_{k+1}$.

Effective Rank

Plot of singular values:

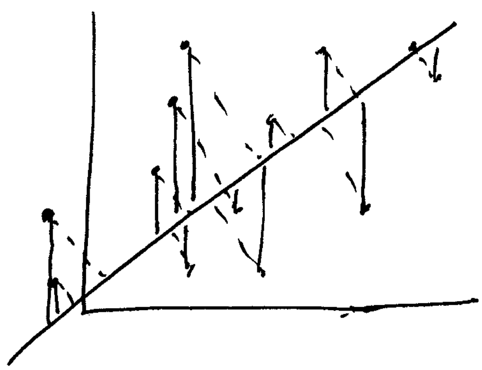


Numerically stable pseudoinverse: first set small singular values to zero.

TOTAL LEAST SQUARES

Given $(x_1, y_1), \ldots, (x_m, y_m)$ find a, b s.t.

$y_i \approx a x_i + b$

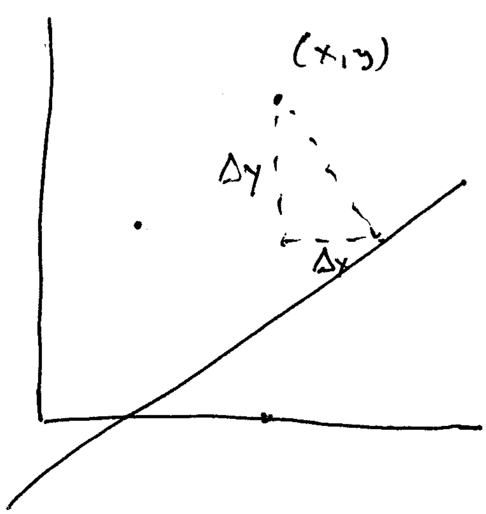


LSQ: vertical errors

TLS: distance to line

Motivation

- Stability for large slope
- Uncertainty in x_i's.



$\text{distance}^2 = (\Delta x)^2 + (\Delta y)^2$

So minimize distance $\Longleftrightarrow$ minimize total error in x and y

Homogeneous Case (see book for inhomogeneous case)

$(\underline{x}_1, y_1), \ldots, (x_m, y_m)$, $\underline{x}_i \in \mathbb{C}^n$, $\underline{a} \in \mathbb{C}^n$

Seek $\underline{a}, \Delta \underline{x}_1, \ldots, \Delta \underline{x}_m, \Delta y_1, \ldots, \Delta y_m$ s.t.

$$\underline{a}^H (\underline{x}_i + \Delta \underline{x}_i) = y_i + \Delta y_i$$

and $\sum_{i=1}^{m} \|\Delta \underline{x}_i\|^2 + |y_i|^2$ is minimal.

In matrix notation:

$$\underline{X} = \begin{bmatrix} x_1^H \\ \vdots \\ x_m^H \end{bmatrix}, \quad \Delta \underline{X} = \begin{bmatrix} \Delta x_1^H \\ \vdots \\ \Delta x_m^H \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad \Delta \underline{y} = \begin{bmatrix} \Delta y_1 \\ \vdots \\ \Delta y_m \end{bmatrix}$$

Want: $(\underline{X} + \Delta \underline{X}) \cdot \underline{a} = (\underline{y} + \Delta \underline{y})$

or $\left[\underline{X} + \Delta \underline{X} \mid \underline{y} + \Delta \underline{y} \right] \begin{bmatrix} \underline{a} \\ -1 \end{bmatrix} = \underline{0}$

or $\Big(\overbrace{[\underline{X} \mid \underline{y}]}^{C =} + \overbrace{[\Delta \underline{X} \mid \Delta \underline{y}]}^{\Delta =} \Big) \begin{bmatrix} \underline{a} \\ -1 \end{bmatrix} = 0$

Need $C + \Delta$ to be rank deficient

and Δ to be as small as possible

~~$\Rightarrow \Delta = \sigma_{k+1} \underline{u}_{k+1} \underline{v}$~~

Let

$$C = \sum_{i=1}^{n+1} \sigma_i \underline{u}_i \underline{v}_i^H$$

be the SVD of C. Then

$$\Delta = -\sigma_{n+1} \underline{u}_{n+1} \underline{v}_{n+1}^H$$

causes $C+\Delta$ to be the optimal rank deficient approx to C.

If $\sigma_n > \sigma_{n+1} \Rightarrow$
Note $\mathcal{N}(C+\Delta) = \text{span}\{\underline{v}_{n+1}\}$, so

$$\begin{bmatrix} \underline{a} \\ -1 \end{bmatrix} = \alpha \cdot \underline{v}_{n+1} \quad \text{for some } \alpha.$$

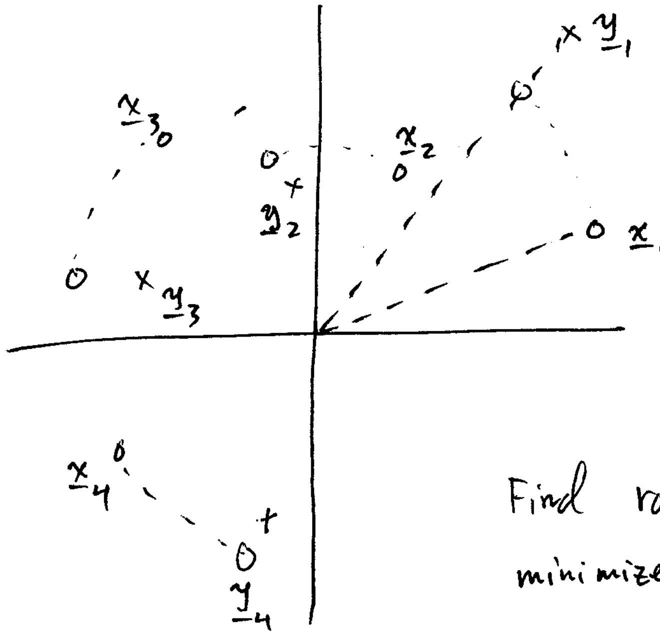
If $\underline{v}_{n+1}$'s last component is 0, no sol'n.

Else, normalize so that last entry is -1.

If $\sigma_n = \sigma_{n+1}$, see book.

Also see book for proof of minimum total distance property.

THE ORTHOGONAL PROCRUSTES PROBLEM



Find rotation which minimizes sum of squared errors.

General problem: Given $\underline{x}_1, \dots, \underline{x}_m, \underline{y}_1, \dots, \underline{y}_m \in \mathbb{C}^n$, find $Q \in \mathbb{C}^{n \times n}$ s.t. $Q^H Q = I$ and

$$\sum_{i=1}^{m} \| \underline{y}_i - Q \underline{x}_i \|^2$$

is minimal.

Denote

$$A = \begin{bmatrix} \underline{y}_1 & \cdots & \underline{y}_m \end{bmatrix}, \quad B = \begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_m \end{bmatrix}$$

and $C = A - QB$. Then

$$\sum \| \underline{y}_i - Q \underline{x}_i \|^2 = \sum c_{ij}^2$$

$$= \|C\|_F^2 = tr(C C^H)$$

$$= tr(A A^H) + tr(B B^H) - tr(QBA^H) - tr(AB^H Q^H)$$

$\Rightarrow$ need to maximize $tr(QBA^H) + tr(AB^H Q^H)$

<u>Theorem</u> The maximum is achieved by

$$Q = VU^H$$

where $BA^H = U \Sigma V^H$

Proof:
$$BA^H = U\Sigma V^H$$
$$AB^H = V\Sigma^T U^H$$

Denote $Z = V^H Q U$, so

$$Q = V Z U^H$$
$$Q^H = U Z^H V^H$$

(change of variables - Z is still unitary)

Then $\operatorname{tr}(QBA^H) + \operatorname{tr}(AB^H Q^H)$

$$= \operatorname{tr}(VZU^H \cdot U\Sigma V^H) + \operatorname{tr}(V\Sigma^T U^H U Z^H V^H)$$

$$= \operatorname{tr}(Z\Sigma) + \operatorname{tr}(\Sigma^T Z^H)$$

$$= \operatorname{tr}(Z\Sigma) + \operatorname{tr}(Z^H \Sigma)$$

$$= \operatorname{tr}((Z+Z^H)\Sigma)$$

<u>Spectral Thm for Normal Operators</u> (interlude)

If $D^H D = D D^H$, then $\exists$ W unitary, Γ diagonal (w/ possibly complex entries) s.t.

$$D = W \Gamma W^H$$

If $D^H D = D D^H$, we say D is <u>normal</u>.

Examples: Hermitian, skew-Hermetian $(D = -D^H)$, unitary

Thus, let us write $Z = W \Gamma W^H$. Then

$$Z^H = W \bar{\Gamma} W^H$$

and

$$tr((Z + Z^H)\Sigma) = tr(W(\Gamma + \bar{\Gamma})W^H \Sigma)$$

$$= tr((\Gamma + \bar{\Gamma})\Sigma)$$

$$= \sum (\gamma_i + \bar{\gamma}_i)\sigma_i$$

Now the eigenvalues of unitary matrices have magnitude 1.

To see this, if $Zv = \lambda v$, then

$$\|v\|^2 = \|Zv\|^2 = \|\lambda v\|^2 = |\lambda|^2 \|v\|^2.$$

Thus $\sum (\gamma_i + \bar{\gamma}_i)\sigma_i \leq 2\sum_{i=1}^{n} \sigma_i$.

Equality is achieved iff $\gamma_i = 1 \ \forall i \Rightarrow Z = I$

$\Rightarrow Q = VU^H$.

Incorrect — see
next page

Proof that $\operatorname{tr}\left((Z+Z^H)\Sigma\right) \le 2\sum_{i=1}^{n}\sigma_i$

By the spectral theorem for normal matrices, $Z = W\Gamma W^H$ where $WW^H = W^H W = I$ and $\Gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$, $\gamma_i \in \mathbb{C}$.

Denote

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \\ w_{n1} & w_{n2} & & w_{nn} \end{bmatrix} = \begin{bmatrix} \underline{w}_1 & \cdots & \underline{w}_n \end{bmatrix} = \begin{bmatrix} \underline{\tilde{w}}_1^H \\ \vdots \\ \underline{\tilde{w}}_n^H \end{bmatrix}$$

Note that $\|\underline{w}_i\| = \|\underline{\tilde{w}}_i\| = 1 \;\; \forall i$ since W is unitary.

Then $\operatorname{tr}\left((Z+Z^H)\Sigma\right) = \operatorname{tr}\left(W(\Gamma+\bar{\Gamma})W^H\Sigma\right)$

$$= \operatorname{tr}\left(\sum_{j=1}^{n}(\gamma_i+\bar{\gamma}_i)\,\underline{w}_i\underline{w}_i^H\,\Sigma\right)$$

$$= \sum_{j=1}^{n}(\gamma_j+\bar{\gamma}_j)\operatorname{tr}\left(\underline{w}_j\underline{w}_j^H\Sigma\right) = \sum_{j=1}^{n}(\gamma_j+\bar{\gamma}_j)\operatorname{tr}\left(\underline{w}_j^H\Sigma\,\underline{w}_j\right)$$

$$= \sum_{j=1}^{n}(\gamma_j+\bar{\gamma}_j)\sum_{i=1}^{n}\sigma_i|w_{ij}|^2$$

$$= \sum_{i=1}^{n}\sigma_i\sum_{j=1}^{n}(\gamma_j+\bar{\gamma}_j)|w_{ij}|^2$$

$$\le 2\sum_{i=1}^{n}\sigma_i\sum_{j=1}^{n}|w_{ij}|^2 = 2\sum\sigma_i\cdot\|\underline{\tilde{w}}_i\|^2$$

$$= 2\sum_{i=1}^{n}\sigma_i$$

since $|\gamma_i| = 1 \;\Rightarrow\; \gamma_i+\bar{\gamma}_i = 2\,\mathrm{Re}\{\gamma_i\} \le 2$