Notes on the Fourier Transform

Definition. The continuous domain Fourier Transform (FT) relates a function to its frequency domain equivalent. The FT of a function *g(x)* is defined by the Fourier integral:

$$G(s) = F\{g(x)\} = \int_{-\infty}^{\infty} g(x)e^{-i2pxs}dx$$

for $x, s \in \Re$. There are a variety of existence criteria and the FT doesn't exist for all functions. For example, the function *g(x) = cos(1/x)* has an infinite number of oscillations as $x \to 0$ and the FT integral can't be evaluated. However, if the FT exists, then there is an inverse FT relationship:

$$g(x) = F^{-1}\{G(s)\} = \int_{-\infty}^{\infty} G(s)e^{i2pxs}ds$$

Uniqueness: Given the existence of the inverse FT, it follows that if the FT exists, it must be unique. That is, for a function forms a unique pair with its FT:

$$g(x) \leftrightarrow G(s)$$

Caveat. An exception to the uniqueness property is a class of functions called null functions. An example is the continuous function $f(x) = \{ \begin{matrix} 1, x = 0 \\ 0, x \neq 0 \end{matrix}$. This function and others like it have the same Fourier transform as *f(x)* = 0: *F(s)* = 0. Thus, the uniqueness exists only for a function plus or minus arbitrary null functions. In practice, this caveat is not important and for the purposes of this class we will assume that the FT is unique.

Symmetry Definitions. We first decompose some function *g(x)* in to even and odd components, *e(x)* and *o(x)*, respectively, as follows:

$$e(x) = \tfrac{1}{2}[g(x) + g(-x)]$$

$$o(x) = \tfrac{1}{2}[g(x) - g(-x)]$$

thus,

$$g(x) = e(x) + o(x)$$

and

$$e(x) = e(-x) \text{ and } o(x) = -o(x)$$

A function, *g(x)*, is Hermitian Symmetric (Conjugate Symmetric) if:

$$\mathrm{Re}\{g(x)\} = e(x) \text{ and } \mathrm{Im}\{g(x)\} = o(x)$$

thus,

$$g(x) = e(x) + io(x) = g^*(-x)$$

Symmetry Properties of the FT. There are several related properties:

1. If *g(x)* is real, then *G(s)* is Hermitian symmetric (e.g. *G(s) = G*(-s)*).
2. If *g(x)* is real and even, *G(s)* is real and even.
3. If *g(x)* is real and odd, *G(s)* is imaginary and odd.
4. If *g(x)* is real, *G(s)* can be defined strictly by non-negative frequencies ($s \geq 0$).
5. If *g(x)* is imaginary, then *G(s)* is Anti-Hermitian symmetric (e.g. *G(s) = -G*(-s)*).

Proof of 1.

$$G(s) = \int g(x) e^{-i2psx} dx$$

$$= \int [e(x) + o(x)[\cos 2psx - i \sin 2psx] dx \qquad \text{(cos is even, sin is odd)}$$

$$= \int e(x) \cos 2psx dx + \int o(x) \cos 2psx dx - i \int e(x) \sin 2psx dx - i \int o(x) \sin 2psx dx$$

$$= \int e'(x) dx + \int o'(x) dx - i \int o''(x) dx - i \int e''(x) dx$$

$$= E(s) + 0 - i \cdot 0 - iO(s) \quad \left(\text{cos is even in } s\text{, sin is odd is } s\text{, } \int_{-\infty}^{\infty} odd(x) = 0\right)$$

$$= E(s) - iO(s)$$

$$= E(-s) + iO(-s)$$

$$= G^*(s) \qquad \text{Q.E.D.}$$

Comment. One interesting consequence of the symmetry properties is that if g(x) is real, the only one-half of the Fourier transform is necessary to specify the function – this follows from property 1. above. More specifically, g(x) is strictly determined by G(s) for all non-negative frequencies (s).

Comment on negative frequencies. Consider a real-valued signal – imagine a voltage on a wire or the sound pressure against your eardrum – the Fourier transform of these is completely specified by the positive frequencies (since *G(-s) = G*(s)*). We can argue that we have the concept of a frequency (oscillations/second), but it doesn't really make physical sense to talk about positive or negative frequencies.. We could argue that the having positive and negative frequencies is merely a mathematical convenience. Are there cases where negative frequencies have meaning? Consider the bit in a drill – it can turn clockwise or counter clockwise and different rotational rates. Here positive and negative frequencies have physical meaning (the direction of rotation). As we shall see, the magnetic moment in NMR is a case where the sign indicates the direction of precession.

Convolution Definition. The convolution operator is defined as:

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(x) h(x - x) dx$$

The convolution operator commutes:

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(x) h(x - x) dx = \int_{-\infty}^{\infty} g(x - x) h(x) dx = h(x) * g(x)$$

[The delta function material is only for completeness – not necessary for this class**]**

The delta function, $d(x)$. The delta function is a mathematical construct that is infinitely high in amplitude, infinitely short in duration and has unity area:

$$d(x) = \{ {}^{\infty,\, x=0}_{0,\, x \neq 0} \quad \text{and} \quad \int d(x)dx = 1$$

Most properties of $d(x)$ are defined only in a limiting case (e.g. as a sequence of functions $g_n(x) \to d(x)$) or under an integral. Some properties of $d(x)$:

$$\int d(x)g(x)dx = g(0), \text{ with } g(x) \text{ continuous at } x = 0$$

$$\int d(x-a)g(x)dx = g(a), \text{ with } g(x) \text{ continuous at } x = a$$

Delta function properties. First two are technically only defined under the integral, but we'll still talk about them.

Similarity (stretching)	$d(ax) = \frac{1}{\lvert a \rvert} d(x)$
Product/Sifting	$g(x)d(x-a) = g(a)d(x-a)$
Sifting	$\int g(x)d(x-a)dx = g(a)$
Convolution	$g(x) * d(x) = d(x) * g(x) = g(x)$ $g(x) * d(x-a) = d(x-a) * g(x) = g(x-a)$

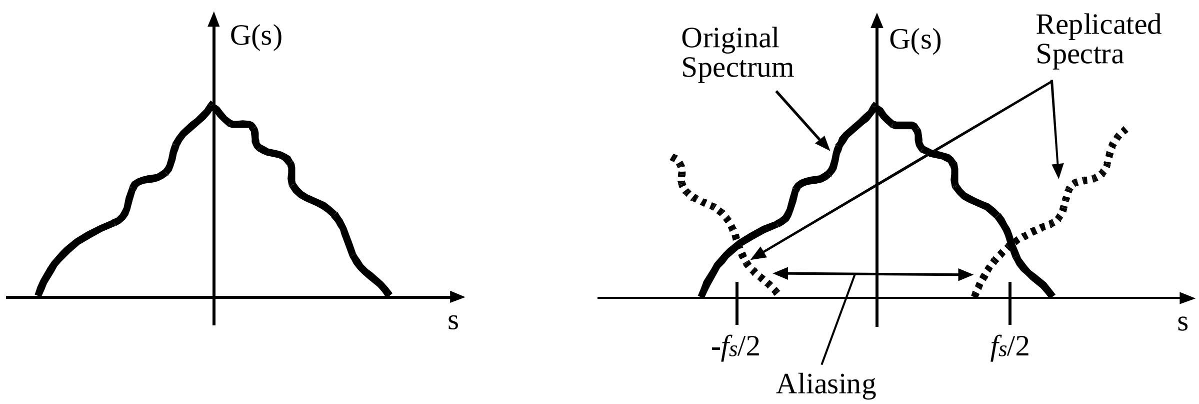
Fourier Transform Theorems. There are many Fourier transform properties and theorems. This is a partial list. Assume that $F\{g(x)\} = G(s)$, $F\{h(x)\} = H(s)$ and that *a* and *b* are constants:

Linearity	$F\{ag(x) + bh(x)\} = aG(s) + bH(s)$
Similarity (stretching)	$F\{g(ax)\} = \frac{1}{\lvert a \rvert} G(\frac{s}{a})$
Shift	$F\{g(x-a)\} = G(s)e^{-i2pas}$
Convolution	$F\{g(x) * h(x)\} = G(s)H(s)$
Product	$F\{g(x)h(x)\} = G(s) * H(s)$
Complex Modulation	$F\{g(x)e^{i2ps_0x}\} = G(s-s_0)$
Modulation	$F\{g(x)\cos(2ps_0x)\} = \frac{1}{2}[G(s-s_0) + G(s+s_0)]$ $F\{g(x)\sin(2ps_0x)\} = \frac{i}{2}[G(s-s_0) - G(s+s_0)]$
Rayleigh's Power	$\int \lvert g(x) \rvert^2 dx = \int \lvert G(s) \rvert^2 ds$
Axis Reversal	$F\{g(-x)\} = G^*(s)$
Complex Conjugation	$F\{g^*(x)\} = G^*(-s)$
Autocorrelation	$F\{g(x) * g(-x)\} = G(s)G^*(s) = \lvert G(s) \rvert^2$
Reverse Relationships	$F\{G(x)\} = g(-s)$

Sampling Theory. When manipulating real objects in a computer, we must first sample the continuous domain object into a discretized version that the computer can handle. There are numerous ways to think about the sampling, but we will consider the effect on the frequency domain (s) of sampling uniformly in x. Consider a signal $g(x)$, with FT $G(s)$, which is sampled with spacing Δx, e.g.: $g_s(n) = g(n\,\Delta x)$, where g_s is a discrete function. The FT of g_s is

$$G_s(s) = \sum_m G(s - mf_s)$$

evaluated from $-f_s/2 \le s \le f_s/2$, where $f_s = 1/\Delta x$, the sampling frequency. What this relationship says is that the sampled spectrum is the original spectrum replicated with spacing f_s and that only frequencies less than $f_s/2$ can be represented in the discrete domain signal. Any components that lie outside of this spectral region ($-f_s/2 \le s \le f_s/2$) results in "aliasing" – the mis-assignment of spectral information.



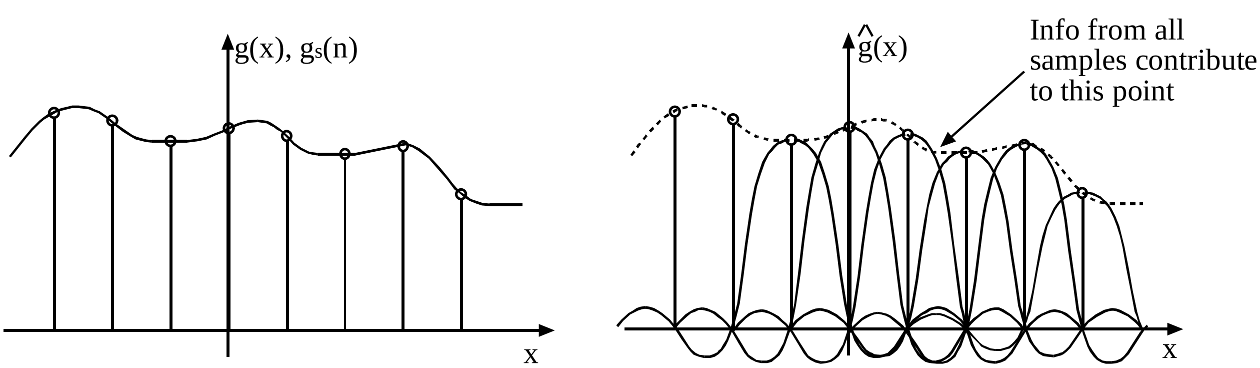
The Whittaker-Shannon sampling theorem states that a band limited function with maximum frequency s_{max} can be fully represented by a discrete time equivalent provided the sampling frequency satisfies the Nyquist sampling criterion:

$$f_s = \frac{1}{\Delta x} \ge 2s_{\max}$$

where f_s is known as the Nyquist frequency. The reconstructed signal is given by:

$$\hat{g}(x) = \sum_{n=-\infty}^{\infty} g(n\Delta x)\,\mathrm{sinc}\left[f_s(x - n\Delta x)\right]$$

If the Nyquist criterion is met, then $\hat{g}(x) = g(x)$.



Some common FT pairs:

$g(x)$	$G(s)$
1	$d(s)$
$d(x)$	1
$\cos(2ps_0x)$ $\sin(2ps_0x)$	$\frac{1}{2}\left[d(s-s_0)+d(s+s_0)\right]$ $\frac{i}{2}\left[d(s-s_0)-d(s+s_0)\right]$
$\mathrm{rect}(x) = \{\begin{matrix} 1 & x < \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \end{matrix}$	$(x) = \frac{px)}{p}$
sinc( )	$s)$
$\mathrm{triangle}(x) = \{\begin{matrix} 1-\lvert x \rvert & x < 1 \\ 0 & x \geq 1 \end{matrix}$	$\mathrm{sinc}^2(s)$
e^{-px^2}	e^{-ps^2}

Units. If x has units of Q, then s will have units of "cycles/Q" or Q^{-1}. Please note that this is not an angular frequency with units of radians/Q, but just plain Q^{-1}. Please also keep in mind that x is the index of variation – for example, we can have $g(x)$ represent a velocity that varies as a function of spatial location x. The function $g(x)$ has units cm/s, but x has units cm.

Examples:

Time Seconds (s)	Temporal Frequency s^{-1}, Hz, cycles/s
Distance cm	Spatial Frequency cm^{-1}, cycles/cm

<u>Notes on the 2D Fourier Transform</u>

Definition. The 2D Fourier Transform (FT) relates a function to its frequency domain equivalent. The FT of a function *g(x,y)* is defined by the 2D Fourier integral:

$$G(u,v) = F\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-i2p(xu+vy)}dxdy$$

There is also an inverse FT relationship:

$$g(x,y) = F^{-1}\{G(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u.v)e^{i2p(xu+vy)}dudv$$

Uniqueness: Given the existence of the inverse FT, it follows that if the FT exists, it must be unique. That is, for a function forms a unique pair with its FT:

$$g(x,y) \leftrightarrow G(u,v)$$

Symmetry Properties of the FT. If *g(x,y)* is real, then *G(u,v)* is Hermitian Symmetric, that is, *G(u,v) = G*(-u,-v)*. If *g(x,y)* is real and even, that is, *g(x,y)=g(-x,-y)*, then *G(u,v)* is also real and even.

The delta function, $d(x,y)$. The delta function in two is equal the to product of two 1D delta functions $d(x,y) = d(x)d(y)$. In a manner similar to the 1D delta function, the 2D delta function has the following definition:

$$d(x,y) = \begin{cases} \infty, x = 0 \text{ and } y = 0 \\ 0, \text{ otherwise} \end{cases} \text{ and } \int\int d(x,y)dxdy = 1$$

Most properties of $d(x,y)$ can be derived from the 1D delta function. There is also a polar coordinate version of the 2D delta function: $d(x,y) = d(r)/pr$.

Fourier Transform Theorems. Let *a* and *b* are non-zero constants.

Linearity

$$af(x,y) + bg(x,y) \leftrightarrow aG(u,v) + bH(u,v)$$

Magnification

$$g(ax,by) \leftrightarrow \frac{1}{|ab|}G(\tfrac{u}{a},\tfrac{v}{b})$$

Shift

$$g(x-a,y-b) \leftrightarrow G(u,v)e^{-i2p(ua+vb)}$$

Convolution

$$g(x,y)**h(x,y) = \int\int g(x,h)h(x-x,y-h)dxdh \leftrightarrow G(u,v)H(u,v)$$

$$g(x,y)h(x,y) \leftrightarrow G(u,v)**H(u,v)$$

Separability

$$g(x,y) = g_X(x)g_Y(y) \leftrightarrow G_X(u)G_Y(v) = F_{1D,x}\{g_X(x)\}F_{1D,y}\{g_Y(y)\}$$

Sampling Theory in 2D. We now sample a 2D object and will consider the effect on the frequency domain (u,v) of sampling uniformly in x and y. Consider a signal $g(x,y)$, with FT $G(u,v)$, which is sampled with spacing Δx and Δy, e.g.: $g_s(n,m) = g(n\,\Delta x,\, m\,\Delta y)$, where g_s is a discrete function. The FT of g_s is

$$G_s(u,v) = \sum_{n,m=-\infty}^{\infty} G(u - \tfrac{n}{\Delta x}, v - \tfrac{m}{\Delta y})$$

evaluated for $-\frac{1}{2\Delta x} \le u \le \frac{1}{2\Delta x}; -\frac{1}{2\Delta y} \le v \le \frac{1}{2\Delta y}$. What this relationship says is that the sampled spectrum is the original spectrum replicated with spacing ${}^{1}\!/_{\Delta x}$ and ${}^{1}\!/_{\Delta y}$ (the sampling frequencies) and that only frequencies less than one-half of these frequencies can be represented in the discrete domain signal. Any components that lie outside of this spectral region will result in "aliasing" – the mis-assignment of spatial frequency information.

The Whittaker-Shannon sampling theorem in 2D states that a band limited function with maximum frequencies $s_{max,x}$ and $s_{max,y}$ can be fully represented by a discrete time equivalent provided the sampling frequency satisfies the Nyquist sampling criterion:

$$\frac{1}{\Delta x} \ge 2s_{\max,x} \text{ and } \frac{1}{\Delta y} \ge 2s_{\max,y}$$

Under these circumstances, there is no spectral overlap (or aliasing) the original spectrum and by uniqueness of the FT, the original signal can be reconstructed. The reconstructed image is given by:

$$\hat{g}(x,y) = \sum_{n,m=-\infty}^{\infty} \mathrm{sinc}(\tfrac{x-n\Delta x}{\Delta x})\mathrm{sinc}(\tfrac{y-m\Delta y}{\Delta y}) g(n\Delta x, m\Delta y)$$

If the Nyquist criterion is met, then $\hat{g}(x,y) = g(x,y)$. This is "sinc" interpolation in 2D.

Some common 2D FT pairs:

$$1 \leftrightarrow d(u,v)$$

$$d(x,y) \leftrightarrow 1$$

$$d(x-a, y-b) \leftrightarrow e^{-i2p(ua+vb)}$$

$$e^{-pr^2} = e^{-px^2}e^{-py^2} \leftrightarrow e^{-pr^2} = e^{-pu^2}e^{-pv^2}$$

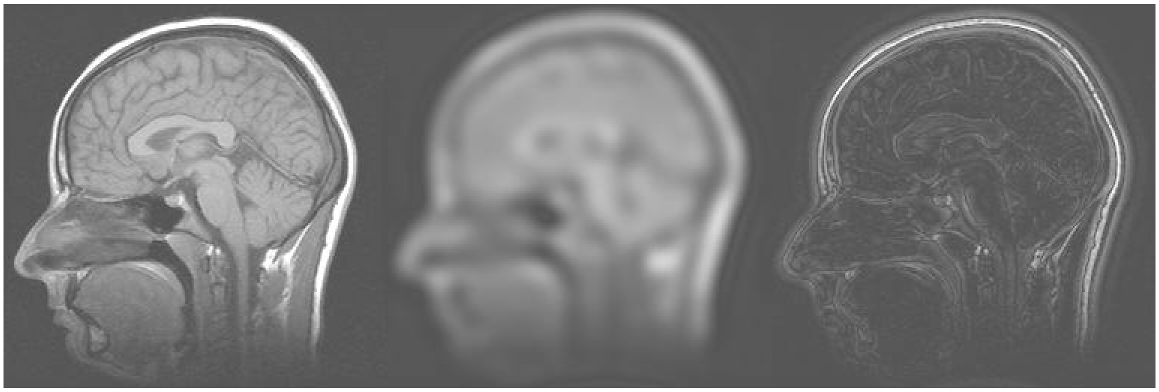
$$\cos(2px) \leftrightarrow \tfrac{1}{2}\left[d(u-1) + d(u+1)\right]d(v)$$

$$\mathrm{rect}(y) \leftrightarrow d(u)\mathrm{sinc}(v)$$

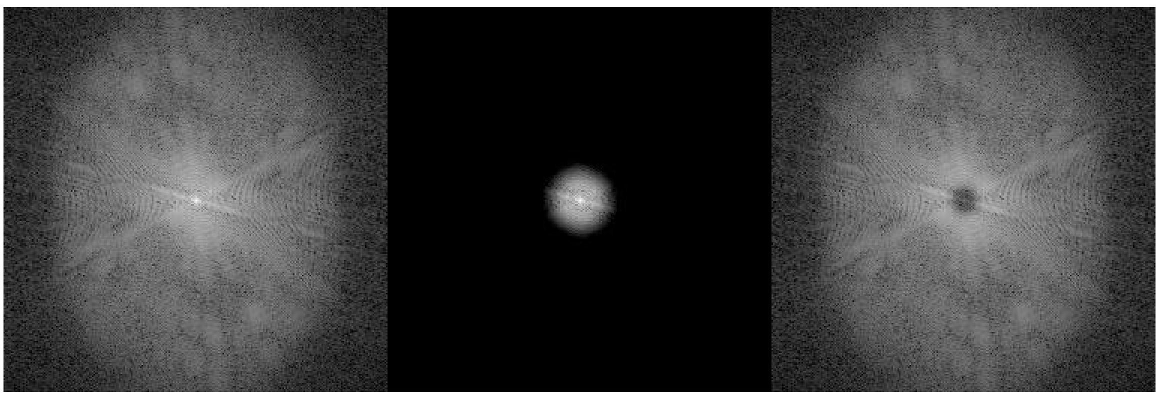
$$\mathrm{rect}(ax)\mathrm{rect}(by) \leftrightarrow \tfrac{1}{|ab|}\mathrm{sinc}(\tfrac{u}{a})\mathrm{sinc}(\tfrac{v}{b})$$

$$\mathrm{circ}(r) = \{\begin{matrix} 1, r \le 1 \\ 0, r > 1 \end{matrix} \leftrightarrow \frac{J_1(2pr)}{r} = \mathrm{jinc}(r))$$

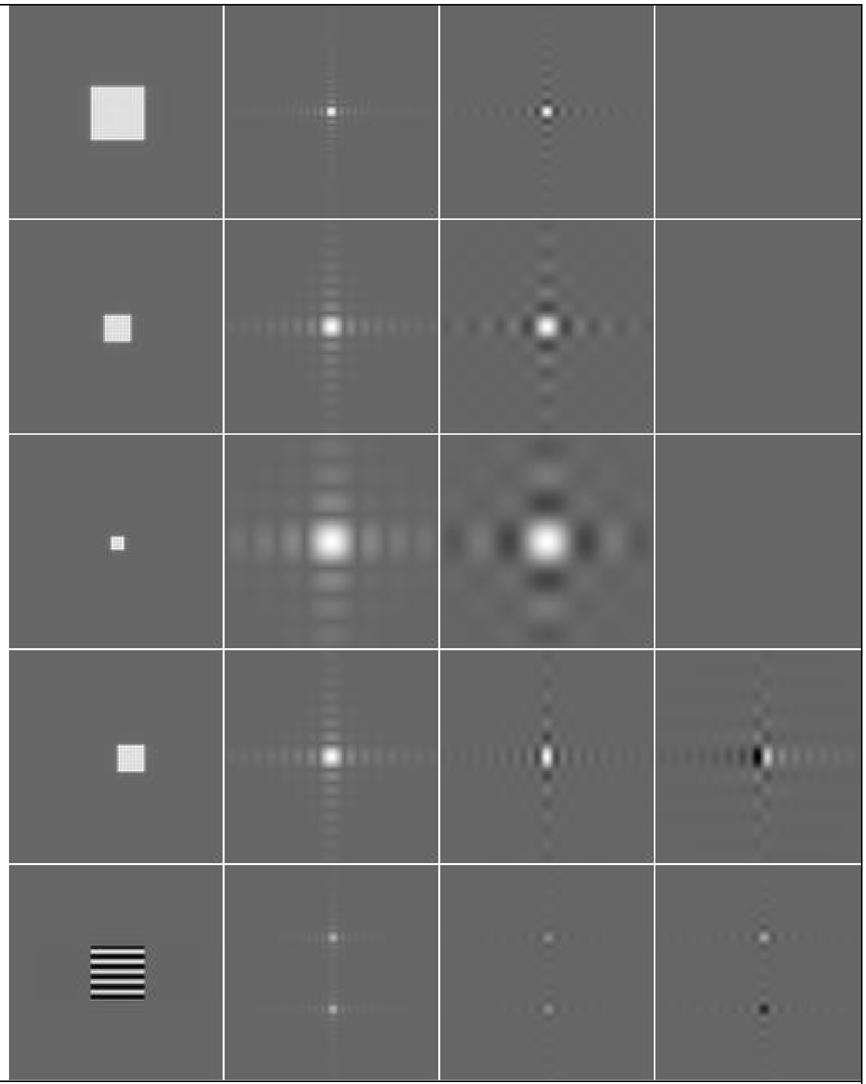
Examples of Fourier Transforms:



2D data in image domain	Low spatial freq data (image domain)	High spatial freq data (image domain)



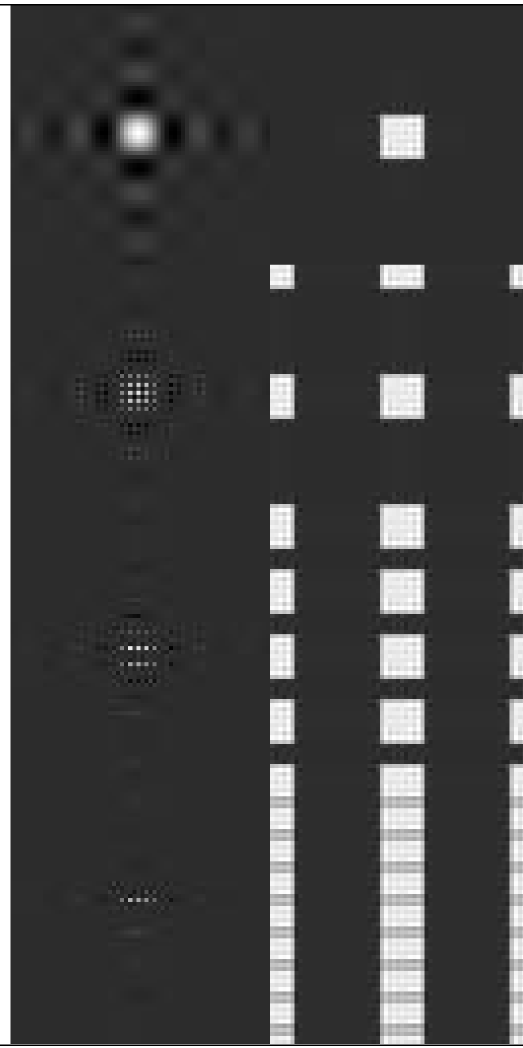
2D data in Fourier domain	Low spatial freq data (Fourier domain)	High spatial freq data (Fourier domain)

g(x,y) = rect(x)rec(y) G(u,v) = sinc(u)sinc(v)	
scaling (magnification) property	
scaling (magnification) property	
shifting property	
modulation	
	Image Abs(Fourier) Real(Fourier) Imag(Fourier)

<table>
<tr>
<td>$g(x,y) = \mathrm{sinc}(x)\mathrm{sinc}(y)$
$G(u,v) = \mathrm{rect}(u)\mathrm{rect}(v)$

sampling pattern with $\Delta x = \Delta y$
In the Fourier transform we have the replication pattern with spacing
$1/\Delta x = 1/\Delta y$

sampling pattern with $\Delta x < \Delta y$
In the Fourier transform we have the replication pattern with spacing
$1/\Delta x > 1/\Delta y$

sampling pattern with $\Delta x << \Delta y$
This has aliasing in the y (v) direction</td>
<td></td>
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<td>Image Data Fourier Data</td>
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