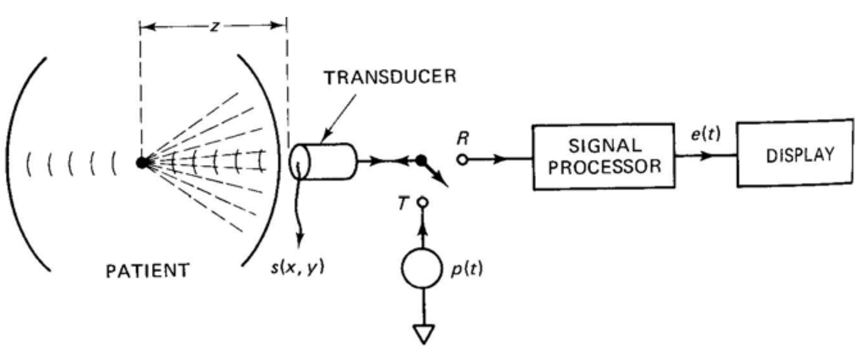
Ultrasound Notes, Part 0 – Roadmap

- Introduction
- Physics: reflections, wave equations, pulse functions, attenuation, dispersion.
- The ultrasound signal: bandpass signals, reflections, speckle noise.
- 2D imaging: voltage, derivation of the point spread function.
- The lateral point spread function and diffraction: steady state, pulsed analyses.
- Array systems: sampling, grating patterns
- System components
- Doppler – if time permits.

Ultrasound Notes, Part I – The Basics

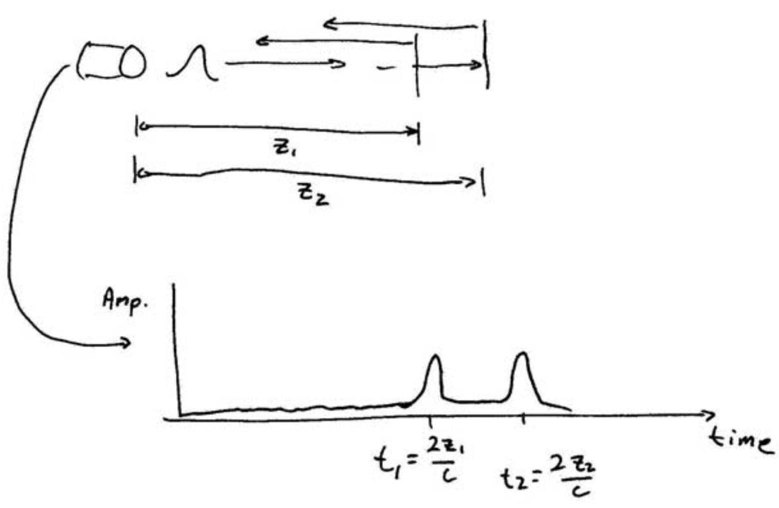
- Ultrasound imaging is a direct, non-reconstructive form of imaging where image formation is obtained by localizing an ultrasonic wave to a small volume in 3D space.
- Two dimension of localization are performed by diffraction (focusing), as in optics.
- One dimension is performed by pulsing, as in RADAR.
- The ultrasonic wave is produced by electrical excitation of a piezoelectric transducer, which is usually transmitted to the body through an impedance matching gel.
- Ultrasonic waves are detected by the same transducer and converted into an electrical signal for processing and display.
- A simple system is shown here (from Macovski):



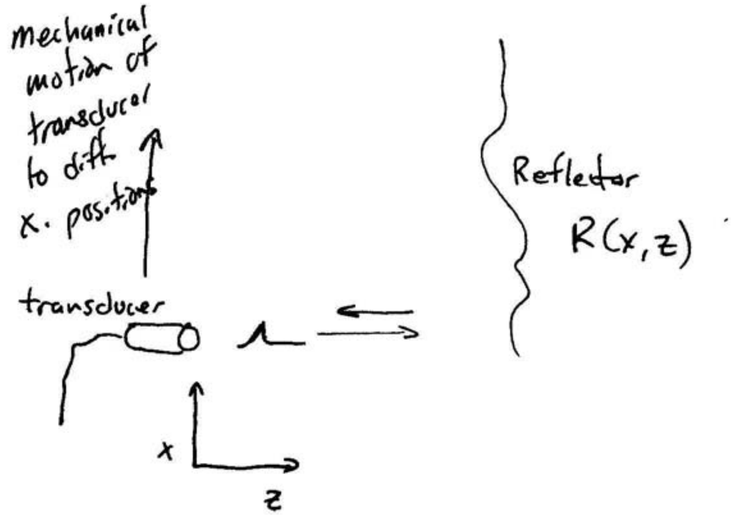
- A pulser excites the transducer with a short pulse, which propagates through the body reflecting off of mechanical inhomogeneities. The time relationship between the transmitted pulse and received signal is given by

$$t_{echo} = \frac{2z}{c}$$

where z is the depth from the transducer and c is the velocity of sound in the body (c ~ 1500 m/s or 1.5 mm/μs)
- Reflected signal at time t are associated with mechanical inhomogeneities at depth $z = tc/2$.
- The lateral extent of the pulse echo is determined by the cross-section of the "beam." This cross-section is depth (z) dependent.
- One line of an image can be obtained by recording the reflected signal as a function of time – the "shape" of the response is determined by the pulse properties, transducer properties and diffraction. If we plot this amplitude as a function of time, we get an "A-mode" (for amplitude) scan:



- 2D and 3D images can be generated by moving the transducer to difference positions (in x and y), thus moving the "beam." Images can be generated by making the amplitude of the response a grayscale level (or brightness) and when all scan lines have been acquired presented in the form of an image. This is known as a "B-mode" (for brightness) scan. A simple 2D system (x,z) is shown here:

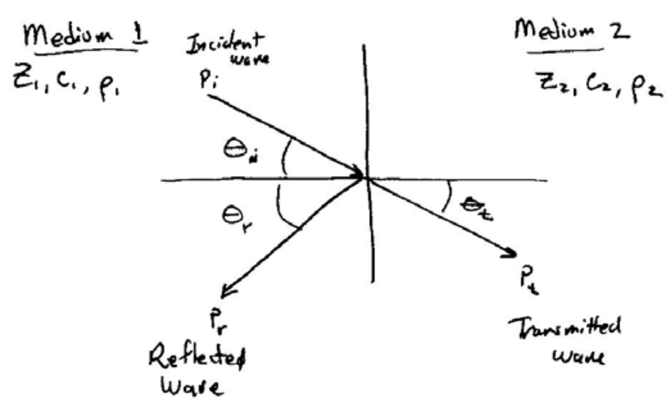


- B-mode imaging can also be done by electronic motion of the beam, as will be discussed in later lectures.
- Finally, a single line can be examined over different time frames and also presented as a grayscale image. This is known as "M-mode" (for motion).

Source of US signals #1: Surface Reflections

In ultrasound, we make images the reflectivity in the body, $R(x,y,z)$.

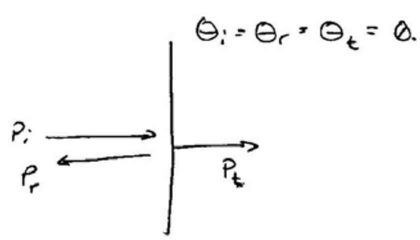
One major source of R is surface reflections, which are generated at the surface of 2 materials.



We define the characteristic acoustical impedance as $Z = \rho c$, where ρ is the density of the material (in kg/m^3) and c is the propagation velocity of acoustic waves (m/s).

Another relationship holds (acoustic Ohm's law): $Z=P/v$, where P is the pressure (1 Pa = N/m^2 = kg/(m s^2)) and v is the particle velocity (m/s). Note the distinction between the microscopic behavior of the constituent particles (v) and the macroscopic behavior of the wavefront (c).

For the following analysis, we'll assume that the incident angle and reflected angles are both zero. (Like a mirror, if the mirror is tilted the reflection will not come back to you – thus in US we typically only see surface reflections from sources that are parallel to the wavefront – this is equivalent to $\theta_i= \theta_r=\theta_t=0$.)



There are a variety of boundary conditions that need to be satisfied. First, the net particle velocity must be the same on either side of the boundary (otherwise, particles will cross the boundary):

$$v_i - v_r = v_t \;\rightarrow\; P_i/Z_1 - P_r/Z_1 = P_t/Z_2$$

Second, the pressures on either side of the boundary must be equal (the boundary will shift to make the pressures match – recall we are working in fluids)

$$P_i + P_r = P_t$$

We can now solve for the reflectivity:

$$R = \frac{P_r}{P_i} = \frac{Z_2 - Z_1}{Z_1 + Z_2} \approx \frac{Z_2 - Z_1}{2Z_0}$$

where Z_0 is average acoustic impedance in the tissue.

Comments:

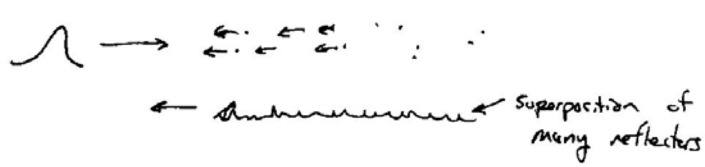
- As we can see $R \in [1,-1]$, $R<0$ indicates that the phase of the reflected signal is reversed and $R>0$ indicates that the phase of the reflected signal is preserved.
- Typically, reflections in the human body (other than tissue/bone or tissue/air interfaces) are very small – on the order of 5% or less ($20\log(|R|) < -25$ dB).
- Intensities in US are a relative measure – not based on any absolute or inherent tissue property. Thus, relative differences in Z are what is important in determining reflected intensities.
- The intensity of the acoustic wave is $I = P^2/2Z$ (energy/time/unit area).

$$I_r/I_i = R^2 \text{ and } I_t/I_i = (1 - R^2)$$

[A (not very interesting) aside: how does this strange $I = P^2/2Z$ relationship come about. Energy density is: $E = \frac{1}{2}mv^2/V = \frac{1}{2}\rho v^2$. (But wait, isn't the motion sinusoidal so E should be ½ of this? No, the other half the energy is stored as potential energy.) Intensity is the propagation of energy through a surface: energy/(time x area) or $I = Ec = \frac{1}{2}\rho v^2 c$, where c is the propagation velocity of the wave (and associated energy). Using $v = P/Z$ and $Z = \rho c$, we get $I = P^2/2Z$. What about units? Units of pressure: Pa = N/m^2 = J/m^3 = kg/(m s^2); units of impedance = kg/(m^2 s) = J s/m^4 = etc. I, therefore has units of J/(m^2 s).]

Source of US signals #2: Volume scattering

On a microscopic level ($< \lambda = c/f$), inhomogeneities in tissue scatter sound waves.



We can define a parameter η = volumetric backscatter coefficient = backscatter cross section per unit volume. Backscatter is:

- weak (20 dB less than surface reflections)
- isotropic (scatters in all directions)
- always present (this is an intrinsic property of a tissue)
- this will be discussed further (below) in the section on Speckle Noise

Wave Equations

Assumptions:

- homogeneous medium
- non-dissipative medium
- longitudinal waves (compression or pressure waves)
  (fluids will not support transverse or shear waves except at surfaces – the difference is similar to the difference between sound propagating through a pound and ripples propagating along the surface)
- the formal differential equation for acoustic waves propagating through a fluid medium is:

$$\nabla^2 P - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}P = 0, \text{ where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Solutions:

- infinite, monochromatic plane wave

$$p(z,t) = e^{i2\pi f(t-z/c)}$$

- general solution for plane waves

$$p(z,t) = \int P(f) e^{i2\pi f(t-z/c)} df$$

where *P(f)* is the distribution of isochromats. We can define a "pulse function", *p(t)*, which is the inverse FT of *P(f)* by looking at the time domain signal at *z*=0:

$$p(t) = p(z=0,t) = \int P(f) e^{i2\pi ft} df = F^{-1}\{P(f)\}$$

and by substitution, we get *p(z,t) = p(0,t-z/c) = p(t-z/c).* Thus we have a pulse function that propagates at velocity *c* → at time *t*, the pulse is at position *z* = *tc*.

- spherical wave

$$p(r,t) = \frac{1}{r} e^{i2\pi f(t-r/c)}, \text{ where } r = \sqrt{x^2 + y^2 + z^2}$$

this describes the behavior of any point scatterer. The amplitude falls off with 1/*r*.

Dispersion

Previous solutions assumed a non-dissipative medium, but soft tissues of the body do dissipate (attenuate) the sound waves and do so in a manner that is frequency dependent, which makes is a dispersive medium.

- we will use equations that are similar to those derived above, but we'll add an attenuation term – $e^{-\alpha l}$, where α is linear attenuation coefficient (units: cm^{-1}) and l is the path length (cm).
- Typical attenuation for f_0=1MHz
  soft tissues of the body – 1-2 dB/cm ($\alpha \approx 0.1$ cm^{-1})
  watery fluids – <0.2 dB/cm

Looking at our propagating/reflecting pulse, the total (roundtrip) path length at receiver is

$$l=2z \rightarrow \text{attenuation} = e^{-\alpha 2z}$$

also, the attenuation is frequency dependent, for example, $\alpha(f) = 0.1\,|f|\ cm^{-1}\ MHz^{-1}$

The received, reflected wave is now:

$$p_z(t-2z/c) = \int R(z)e^{-2z\alpha(f)}P(f)e^{i2\pi f(t-2z/c)}df$$

How does this relate to $p(t) = F^{-1}\{P(f)\}$? [In our notation, we use $p_z(t)$ to represent the pressure function at the receiver for a reflector at depth z and $p(t)$ to represent the transmitted waveform.]

- Assuming $R(z)$=1 and correction for the time delay $t=t-2z/c$.

$$p_z(t) = \int e^{-2z\alpha(f)}P(f)e^{i2\pi ft}df = F^{-1}\left\{e^{-2z\alpha(f)}P(f)\right\}$$
$$= F^{-1}\left\{e^{-2z\alpha(f)}\right\} * p(t) = d_z(t) * p(t)$$

  where $d_z(t)$ is the z dependent dispersion function (depth dependent blurring)

How do we deal with this?

- make $p(t)$ long:

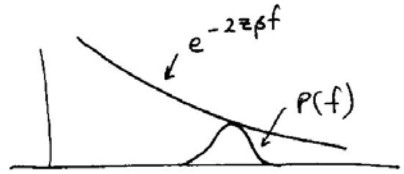
$$P(f) \approx \delta(f - f_0) \rightarrow e^{-2z\alpha(f)} \approx e^{-2z\alpha(f_0)}$$

  which results in:

$$p_z(t) = e^{-2z\alpha(f_0)}p(t)$$

  main problem – poor depth resolution at all depths (z)

- Use a Gaussian frequency spectrum. Suppose $\alpha(f) = \beta|f|$ and $P(f) = e^{-\pi((f-f_0)/W)^2}$,



  then

$$P_z(f) = e^{-2z\beta f} e^{-\pi((f-f_0)/W)^2}$$

$$= e^{-2z\beta f_0 + (Wz\beta)^2} e^{-\pi\left(\left(f-\left(f_0-\frac{W^2 z\beta}{\pi}\right)\right)/W\right)^2}$$

  the width of this pulse has not been degraded and is still ~W, but the center frequency has shifted down slightly (by $W^2 z\beta/\pi$). As a practical matter, pulses used in US often have a Gaussian envelope (in time) and thus will have a Gaussian shaped spectrum, such as that used here.

- Since this commonly used pulse shape is not dramatically affected by dispersion, for most of this class (except a homework problem) we will neglect dispersive effects. Notably, we will assume $\alpha(f) = \alpha(f_0) = \beta f_0$ and $p_z(t) = e^{-2z\alpha(f_0)} p(t)$

Bandpass Signals

The above pulse, *p(t)*, with the Gaussian spectrum is an example of a bandpass pulse that can be represented as an envelope, *a(t)*, modulating a carrier at frequency f_0:

$$p(t) = a(t)e^{i\omega_0 t}, \text{ where } \omega_0 = 2\pi f_0$$

and

$$P(f) = A(f - f_0)$$

$p(t)$ is a complex representation known as the "analytic signal." In general, we will work equations representing the analytic signal with the understanding that we can derive the real pressure signal by taking the real part of $p(t)$:

$$\tilde{p}(t) = \Re\{p(t)\} = a(t)\cos\omega_0 t\,.$$

For the general case of a dispersive medium, we can also define the envelope at the receiver as:

$$a_z(t) = e^{-i\omega_0 t} p_z(t)$$

$$= \int e^{-2z\alpha(f)} A(f - f_0) e^{i2\pi(f-f_0)t} df$$

$$= \int e^{-2z\alpha(f+f_0)} A(f) e^{i2\pi ft} df$$

$$= F^{-1}\left\{e^{-2z\alpha(f+f_0)}\right\} * a(t)$$

$$= d_z'(t) * a(t)$$

For the non-dispersive approximation (and again assuming $R(z) = 1$):

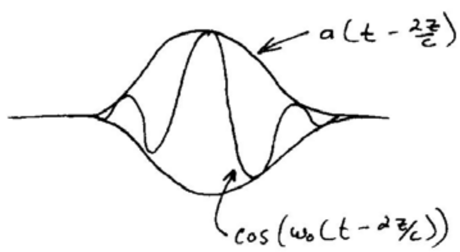
$$a_z(t) = e^{-2z\alpha(f_0)} a(t)$$

Finally, we can describe the time-domain signal at the receiver for a reflector at depth z including the roundtrip propagation delay is:

$$p_z(t - 2z/c) = e^{-2z\alpha(f_0)} a(t - 2z/c) e^{i\omega_0(t-2z/c)}$$

Again, the actual pressure signal at the transducer is the real part:

$$\tilde{p}_z(t - 2z/c) = e^{-2z\alpha(f_0)} a(t - 2z/c)\cos(\omega_0(t - 2z/c))$$



The Signal

Prior expressions for the pressure waveform, e.g., $p_z(t - 2z/c) = e^{-2z\alpha(f_0)} a(t - 2z/c) e^{i\omega_0(t-2z/c)}$, assume a single reflector at some depth z and reflectivity 1. Now we look at what the received voltage is at the transducer for a distribution of reflectors, $R(z)$. The voltage at time t will be related to superposition of the signals from all reflectors the return to the transducer at that point in time (the early parts of the pulse will have traveled farther distances and the later parts of the pulse will have traveled shorter distances). The pressure function at the transducer will then be:

$$v(t) = \mathrm{Re}\left\{ K \int R(z) a\left(t - \frac{2z}{c}\right) e^{-2z\alpha(f_0)} e^{i\omega_0\left(t - \frac{2z}{c}\right)} dz \right\}$$

As a practical matter, the particular methods used in the receiver electronics let us represent the received signal can represent the signal as the absolute value of the sum of analytic signals. More on this in future lectures.

$$v(t) = \left| K \int R(z) a\left(t - \frac{2z}{c}\right) e^{-2z\alpha(f_0)} e^{i\omega_0\left(t - \frac{2z}{c}\right)} dz \right|$$

We have also neglected dispersion of the pulse in these expressions.

Speckle Noise

Speckle noise results from volumetric scatter – that is, scatter from tissue inhomogeneities that are smaller/more closely spaced than the resolution element of the system. In the material presented below, we will derive the distribution of speckle noise.

Ignoring the lateral dimensions (x,y) and attenuation, we examine the expression for the received signal in US:

$$v_c(t) = \left| \int R(z) a\left(t - \frac{2z}{c}\right) e^{i\omega_0\left(t - \frac{2z}{c}\right)} dz \right|$$

(v_c is the attenuation corrected signal). If we look at some point in time (t_0)corresponding to a particular depth location ($z_0 = t_0\, c/2$), then we get:

$$v_c(t_0) = \left| \int R(z_0 - z) a\left(\frac{2z}{c}\right) e^{i\omega_0\left(\frac{2z}{c}\right)} dz \right|$$

Recall that previously, we described a pulse $a(t)$ having width T (or in the z domain, having width Δz. We will now substitute $a(t)$ with rect(t/T):

$$v_c(z_0) = v_c(t_0) = \left| \int R(z_0 - z)\text{rect}\left(\frac{z}{\Delta z}\right) e^{i\omega_0\left(\frac{2z}{c}\right)} dz \right| = \left| \int_{-\Delta z/2}^{\Delta z/2} R'(z) e^{i2kz} dz \right|$$

where *R'* is the reflectivity over the resolution element (and flipped in *z*). We will make some assumptions to discretize this:

- *R'*(*z*) can be represented by many (*N*) discrete reflectors (tissue specific)
- The amplitude of each reflector is *A* (tissue specific)
- The location of each reflector is random

  $R'(z) = \sum_n A\delta(z - z_n)$, where z_n is a random variable.

- The phase of the each reflector is random ($\phi_n = 2kz_n$) and we will assume that it is uniformly distributed between 0 and 2π. (Since Δ*z* is at least 1-2 wavelengths, the phase goes over several cycles, thus all phases should be represented with equal probability.)
- The integral can be represented by a the sum of these reflectors

Thus:

$$v_c(t_0) = \left| \sum_n A e^{i\phi_n} \right| = A \left| \sum_n \cos\phi_n + i \sin\phi_n \right|$$

We now examine the distribution of v_c. First, let's examine the joint statistics of

$$U_N = \sum_n \cos\phi_n \text{ and } V_N = \sum_n \sin\phi_n$$

First, it is relatively easy to see that $E[U_N] = 0$ and $E[V_N] = 0$. Also,

$$E[U_N{}^2] = E\left[\sum_n \cos\phi_n \sum_m \cos\phi_m\right] = E\left[\sum_n \cos^2\phi_n + \sum_{n\neq m} \cos\phi_n \cos\phi_m\right] = N\frac{1}{2} + 0 = \frac{N}{2}$$

where *N* is the number of reflectors in the sum. Similarly:

$$E[V_N{}^2] = \frac{N}{2}$$

and:

$$E[U_N V_N] = E\left[\sum_n \cos\phi_n \sum_m \sin\phi_m\right] = E\left[\sum_n \cos\phi_n \sin\phi_n + \sum_{n\neq m} \cos\phi_n \sin\phi_m\right] = 0$$

and are uncorrelated. Finally, by the central limit theorem, we know that the distributions of U_N and V_N will approach the normal distribution. Thus, of U_N and V_N are independent (uncorrelated jointly normal R.V.'s are independent) are both normally distributed with zero mean and variance $N/2$.

Now,

$$v_c = A\left(U_N{}^2 + V_N{}^2\right)^{1/2},$$

which can be shown to be Rayleigh distributed with parameter $A^2 N/2$, has the following statistics (Rayleigh(σ) ~ $p(x) = \dfrac{x\exp(-x^2/2\sigma^2)}{\sigma^2}$):

$$E[v_c] = \sqrt{\frac{\pi}{2}}\sqrt{\frac{A^2 N}{2}} \text{ and } \sigma_{v_c} = \sqrt{2-\frac{\pi}{2}}\sqrt{\frac{A^2 N}{2}}.$$

Some comments:

- This is often called speckle noise (based on its appearance).
- It comes from constructive/destructive interference superposition of wavefronts.
- The signal intensity $E[v_c]$ and noise variation σ_{v_c} have similar forms, that is, they both are proportional to $A\sqrt{N/2}$.
- Thus as the signal increases, so does the noise.
- For a uniform tissue (with microscopic mechanical inhomogeneities causing volumetric scatter), the signal-to-noise ratio (SNR) for that tissue is "fixed." In fact:

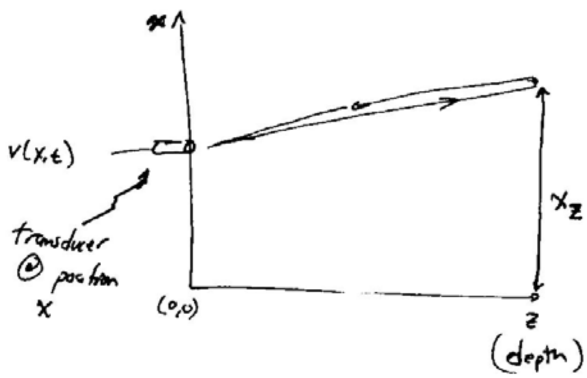
$$\text{SNR} = \frac{E[v_c]}{\sigma_{v_c}} = \frac{\sqrt{\pi/2}}{\sqrt{2-\pi/2}} \approx 1.9$$

  which is a very poor signal to noise ratio.
- Averaging (without moving the transducer position or changing imaging frequencies) cannot be used to improve the speckle SNR – this is because the phase relationship between all of the reflectors does not change.
- Averaging with the transducer in another position or using a different transmitter frequency will lead to different speckle noise because the phase relationship is determined by $e^{i\omega_0\left(\frac{2z}{c}\right)}$. These different noise patterns could be averaged to improve the SNR.

- Viewing on a log scale is often useful since it makes the noise look additive (the noise is multiplicative in the units of v_c.

2D Imaging

We now describe a simple model for a 2D imaging system in which the transducer/receiver is moved to different x positions:



Here the transducer voltage is:

$$v(x,t) = \left| K \iint R(x_z, z) B(x - x_z, z) a\left(t - \frac{2z}{c}\right) e^{i\omega_0\left(t - \frac{2z}{c}\right)} e^{-2z\alpha(f_0)} dx_z dz \right|$$

where x is the location the transducer and x_z is the x-coordinate at depth z in the reflecting object being imaged.

$B(x_z,z)$ is the beam pattern for the transducer positioned at the origin and can be described as:

$$B(x_z,z) = h_T(x_z,z,\omega_0)\, h_R(x_z,z,\omega_0)$$

where h_T is the frequency dependent transmit field pattern (diffraction pattern) and h_R is the frequency dependent receive field pattern.

At this point it is convenient to assume that the pulse ($a(t)$) duration is short, that is:

$$\alpha(f_0)cT << 1$$

where T is the pulse duration. Then

$$a\left(t - \frac{2z}{c}\right) e^{-2z\alpha(f_0)} \approx a\left(t - \frac{2z}{c}\right) e^{-ct\alpha(f_0)}$$

by the sifting property of delta functions. We can now make a gain compensated receive signal by increasing the gain as a function of time:

$$v_c(x,t) = \frac{v(x,t)}{K} e^{ct\alpha(f_0)}$$
$$= \left| \iint R(x_z, z) B(x - x_z, z) a\left(t - \frac{2z}{c}\right) e^{i\omega_0\left(t - \frac{2z}{c}\right)} dx_z\, dz \right|$$

- "Impulse response function" or "point spread function" – we can determine the impulse response function at depth z_0 by setting $R(x_z,z)=\delta(x_z,z-z_0)$:

$$v_c(x,t) = \left| \iint \delta(x_z, z - z_0) B(x - x_z, z) a\left(t - \frac{2z}{c}\right) e^{i\omega_0\left(t - \frac{2z}{c}\right)} dx_z dz \right|$$
$$= \left| B(x, z_0) a\left(t - \frac{2z_0}{c}\right) \right|$$

  setting $t=2z/c$:

$$h(x,z;0,z_0) = I(x,z) = v_c(x, \frac{2z}{c}) = \left| B(x,z_0) a\left(\frac{2(z-z_0)}{c}\right) \right|$$

  which, if the beam pattern were constant as a function of depth, would be space invariant.

- For a real, positive beam pattern and envelope, we simply get:

$$I(x,z) = B(x,z_0) a\left(\frac{2(z-z_0)}{c}\right)$$

  and thus the beam (diffraction) pattern directly controls the spatial resolution in x (and y) and the pulse envelope direction controls the resolution in z.

- One objective in US imaging is to make $B(x - x_z, z) \approx \delta(x - x_z)$ (make the beam very narrow to give good x resolution) and $a\left(t - \frac{2z}{c}\right) \approx \delta\left(t - \frac{2z}{c}\right)$ (make the pulse very short to give good z resolution). If we do so, then we get an image of the reflectivity:

$$v_c(x,t) \approx \left| R\left(x_z, \frac{ct}{2}\right) \right| = \left| R(x_z, z) \right|$$

Issues related to Depth (*z*) Resolution

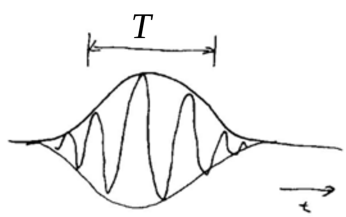
As described above, <u>depth resolution is largely determined by the envelope function *a*(*t*)</u>. Recall, the pulse function is:

$$\tilde{p}(t) = \Re\{p(t)\} = a(t)\cos\omega_0 t$$

The wavelength of a traveling wave is given by:

$$\lambda = \frac{c}{f_0}$$

Thus, for the typical range of US frequencies of 1 to 10 MHz, we have wavelengths that range from 1.5 to 0.15 mm, respectively. Commonly, a(t) is chosen so that it has a duration of 2-3 periods (e.g. $T = 3/f_0$).



The depth impulse response function is approximately $a(2z/c)$ → the width of which is

$$\Delta z = cT/2 \text{ (e.g. } \Delta z = 1.5\, c/f_0 = 1.5\, \lambda).$$

- In US, we now have an SNR resolution trade-off:

  higher f_0 → smaller λ → smaller Δz, but…

  higher f_0 → higher $\alpha(f) = \beta|f|$ → smaller signal for deep tissues

- Recall that attenuation is about 1dB/cm/MHz, so for a 5 cm deep organ (10 cm roundtrip), the attenuation will be 10 dB @ 1 MHz, but 100 dB at 10 MHz.