Ultrasound Notes, Part II - Diffraction

Analysis of the Lateral Response

As discussed previously, the depth (z) response is largely determined by the envelop function, $a(t)$. In the following sections, we will concern ourselves with deriving the beam pattern $B(x - x_z, z)$. Consider the case of a focused US beam. How small of a spot can the beam be focused? Can it be infinitely small? What happens in depth planes other than the focal plane? What parameters control the resolution?

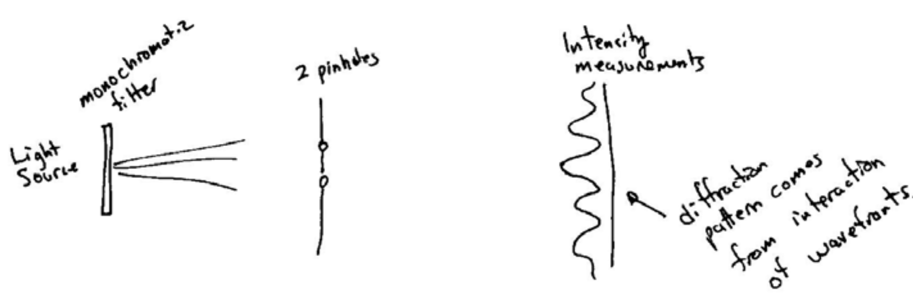
The answers to these questions are given by a ***diffraction*** analysis.

We will examine diffraction several different ways:

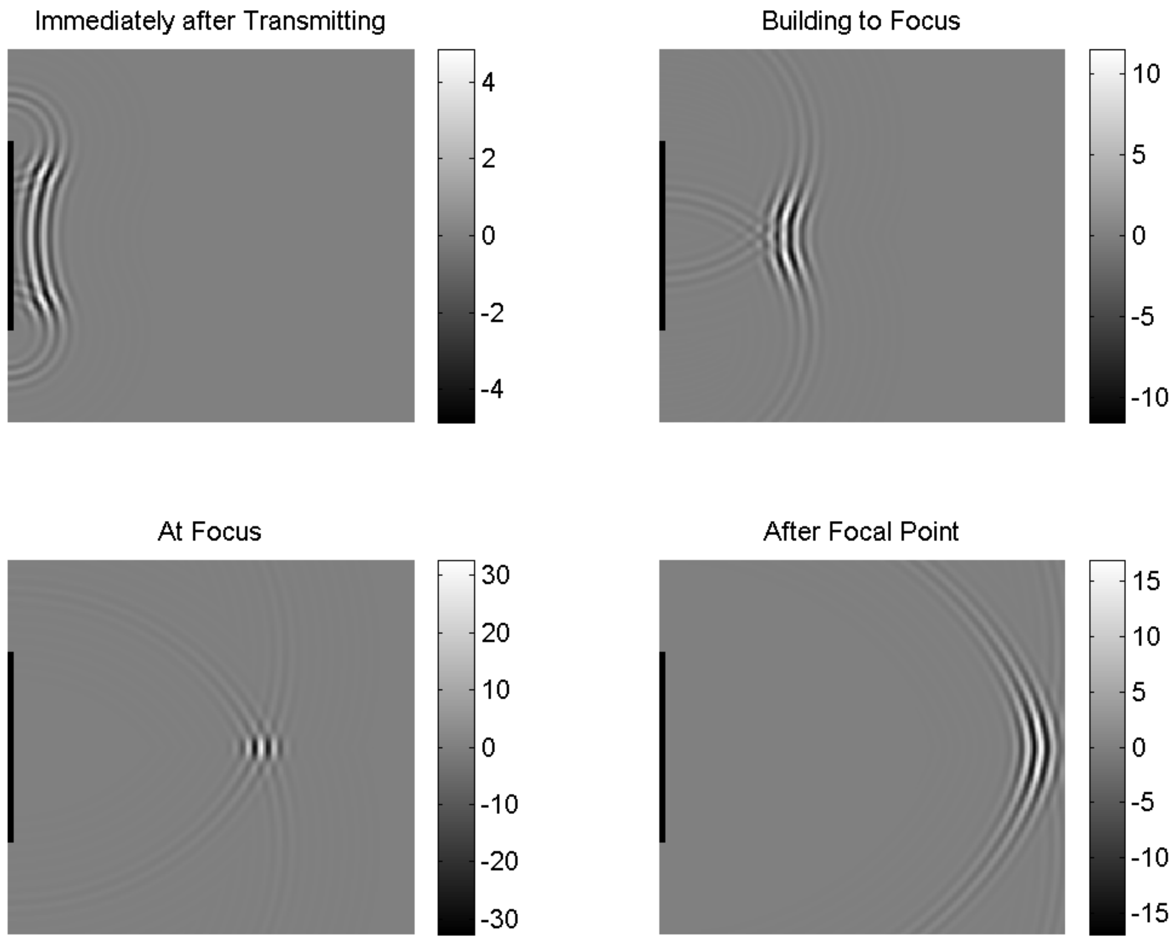
1. First, we consider a "steady-state" analysis, in which we ignore the envelope function. We will do this in both rectangular and polar coordinates.
2. We will then put the envelop function back into the equations and see its effect.
3. Finally, we go back to the "steady-state" analysis, but will consider an array transducer- that is, the transducer will be made of individual elements.

Diffraction

So, what is diffraction? – Historically, has meant optical phenomena that could not be explained by reflection (mirrors) or refraction (lenses). More generally, it has come to mean phenomena that can be caused by an interaction of wavefronts. The classic case is monochromatic light passing through two pinholes:



Diffraction in Ultrasound:



In these images, the transducer is indicated by the black line along the left margin and the transmitted wave is curved to focus at a particular point. Previously, we discussed the depth resolution was determined by the envelope function (in the case a Gaussian). The lateral localization function is more complicated and is determined by diffraction.

Steady-State Diffraction in Ultrasound. Our model has the following characteristics:

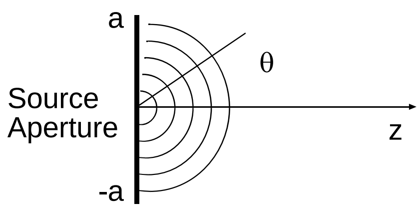
1. Every point on the transducer (aperture) can be modeled using a spherical wave model, modified by a directional dependence term (cos θ) that corresponds to the transducer insertion efficiency.
2. We start our analysis by assuming a steady state model – that is, we will forget, for the moment, that this is a pulsed system and ignore the time propagation of both the pulse and

the wavefronts. By ignoring time, we will, in essence, take a snapshot look at the wave fronts.

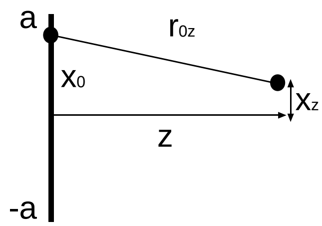
3. In this analysis, we will use the analytic signal (complex) representation for the true pressure wave, that is, we will use e^{ikr} instead of $\cos kr$, where $k = \frac{\omega_0}{c} = \frac{2\pi}{\lambda} \equiv$ wavenumber. The steady-state description of the pressure wave is then:

   $$p(r) = \frac{e^{ikr}}{r}\cos\theta$$

   where θ is the obliquity angle.



4. We will neglect attenuation and perform the analysis in 2D (ignoring the y dimension).



At some depth position z in the object and at a lateral position x_z, the pressure signal will be the superposition of all point sources in the aperture. The superposition of sources of spherical wavefronts is known as the "Huygens-Fresnel" principle. The pressure wave functions is:

$$p(z, x_z) = \int_{-a}^{a} \frac{e^{ikr_{0z}}}{r_{0z}} \cos\theta_{0z} dx_0 \text{, where } r_{0z} = \sqrt{(x_0 - x_z)^2 + z^2}\text{, } \cos\theta_{0z} = \frac{z}{r_{0z}}.$$

The variables x_0 and x_z are coordinates in the source plane and the plane at depth z, respectively. Simplifying a little, and we get:

$$p(z, x_z) = \int_{-a}^{a} \frac{ze^{ikr_{0z}}}{(r_{0z})^2} dx_0 \text{ (equivalent to first part of Macovski, 9.24)}$$

For any reflector at position (z, x_z) (assume R=1), the pattern reflect to the transducer is exactly the same. This symmetrical relationship is known as the Helmholtz reciprocity theorem. The pressure wave at the source will be:

$$p_z(x_0) = p(z, x_z)\frac{ze^{ikr_{0z}}}{(r_{0z})^2} = \int_{-a}^{a}\frac{ze^{ikr'_{0z}}}{(r'_{0z})^2}dx'_0\,\frac{ze^{ikr_{0z}}}{(r_{0z})^2}$$

The received signal can be represented as the integrated complex signal over the transducer:

$$v_z(x_z) = \int_{-a}^{a} p_z(x_0)dx_0 = \left[\int_{-a}^{a}\frac{ze^{ikr_{0z}}}{(r_{0z})^2}dx_0\right]^2 = \left[p(z, x_z)\right]^2$$

(equivalent to first part of Macovski, 9.25)

Letting x be the position of the transmitter, the above equation represent the combined transmit/receive beam pattern for an ultrasound system:

$$B(x - x_z, z) = \left[p(z, x - x_z)\right]^2$$

Fresnel Zone. Whenever the depth of a reflector is substantially larger than the lateral displacement from all points in the source (transducer aperture)::

$$z^2 >> (x_0 - x_z)^2$$

then the Fresnel approximation holds. Specifically, the $e^{ikr_{0z}}$ term can be simplified using:

$$\begin{aligned} r_{0z} &= \sqrt{(x_0 - x_z)^2 + z^2} \\ &= z\sqrt{1 + \frac{(x_0 - x_z)^2}{z^2}} \\ &\cong z\left(1 + \frac{1}{2}\frac{(x_0 - x_z)^2}{z^2}\right) \\ &= z + \frac{1}{2}\frac{(x_0 - x_z)^2}{z} \end{aligned}$$

[Here, we used the Taylor series expansion of $f(u) = \sqrt{1+u}$ expanded around u, keeping the first two terms $f(u) = \sqrt{1+u} \cong 1 + \frac{1}{2}u$, where $u = \frac{(x_0 - x_z)^2}{z^2}$.]

In addition to the usual (above) Fresnel approximation, we will simplify the smoothly varying scalar term (results are less sensitive to this approximations to this term):

$$\frac{z}{(r_{0z})^2} \cong \frac{1}{z}$$

Applying both approximations, we get:

$$p(z, x_z) = \frac{e^{ikz}}{z} \int_{-a}^{a} e^{ik\frac{(x_0 - x_z)^2}{2z}} dx_0$$

Most ultrasound systems are assumed to operate in the Fresnel zone (where this approximation holds). Previously, $p(z, x_z)$ was the superposition of <u>spherical</u> wavefronts. With the Fresnel approximation, this is replaced with <u>quadratic</u> wavefronts.

The make our solution more general, we now replace the source function with a potentially complex driving function:

$s(x_0) = |s(x_0)| e^{\phi(x_0)}$, which we will assume is bounded to [-*a*, *a*].

Our new pressure function for position (z, x_z) is:

$$p(z, x_z) = \frac{e^{ikz}}{z} \int_{-\infty}^{\infty} s(x_0) e^{ik\frac{(x_z - x_0)^2}{2z}} dx_0$$
$$= \frac{e^{ikz}}{z} \left[s(x_z) * e^{ik\frac{x_z^2}{2z}} \right]$$

This says that the pressure pattern can be represented by the convolution of the driving function with a z-dependent, quadratic phase function (equivalent to Macovski, 9.31). This form is not particularly useful, but we can rewrite the pressure function so are more useful expression:

$$p(z, x_z) = \frac{e^{ikz} e^{ik\frac{x_z^2}{2z}}}{z} \int_{-\infty}^{\infty} s(x_0) e^{-ik\frac{x_0 x_z}{z}} e^{ik\frac{x_0^2}{2z}} dx_0$$

Fraunhoffer Zone. Even farther away from the transducer, another approximation can be made. For the "Fraunhoffer approximation," we require that:

$$\frac{kx_0^2}{2z} << \pi \text{ for all } x_0 \in [-a,a] \text{, or}$$

$$\frac{ka^2}{2z} << \pi \text{ or } \frac{2\pi a^2}{2\lambda z} << \pi \text{ or } z >> \frac{a^2}{\lambda} \text{, usually we use } z > 4\frac{a^2}{\lambda}$$

Under these assumptions,

$$e^{ik\frac{x_0^2}{2z}} \approx 1$$

$$p(z,x_z) = \frac{e^{ikz} e^{ik\frac{x_z^2}{2z}}}{z} \int s(x_0) e^{-ik\frac{x_0 x_z}{z}} dx_0$$

With a simple substitution, $u = x_z / \lambda z$, we get:

$$p(z,x_z) = \frac{e^{ikz} e^{ik\frac{x_z^2}{2z}}}{z} \int s(x_0) e^{-i2\pi x_0 u} dx_0$$

$$= \frac{e^{ikz} e^{ik\frac{x_z^2}{2z}}}{z} \Im\{s(x_0)\}\Big|_{u=\frac{x_z}{\lambda z}}$$

(equivalent to Macovski, 9.39)

That is, the field pattern in the Fraunhoffer zone (often call "far field"), is Fourier transform of the aperture (transducer) function. In this zone, the wavefronts arriving at (z, x_z) from any point on the transducer will no longer look spherical or quadratic, but will look <u>planar</u> (linear in 2D case).

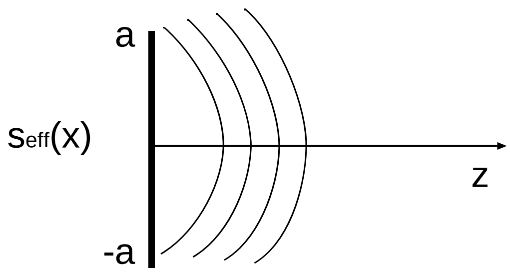
Is the Fraunhoffer zone common? Let's look at an example: For a = 1 cm and f_0 = 1.5 MHz, we'll get λ = 1 mm and the Fraunhoffer zone will be for z > 40 cm (outside the body).

Fresnel Again. While most ultrasound systems do not function in the Fraunhoffer zone, but it is instructive to examine the equations governing the Fraunhoffer zone because by a simple modification, the Fresnel zone can be analyzed using this approximation. For this, let's define an effective aperture (transducer) function:

$$s_{eff}(x_0) = s(x_0)e^{ik\frac{x_0^2}{2z}}$$

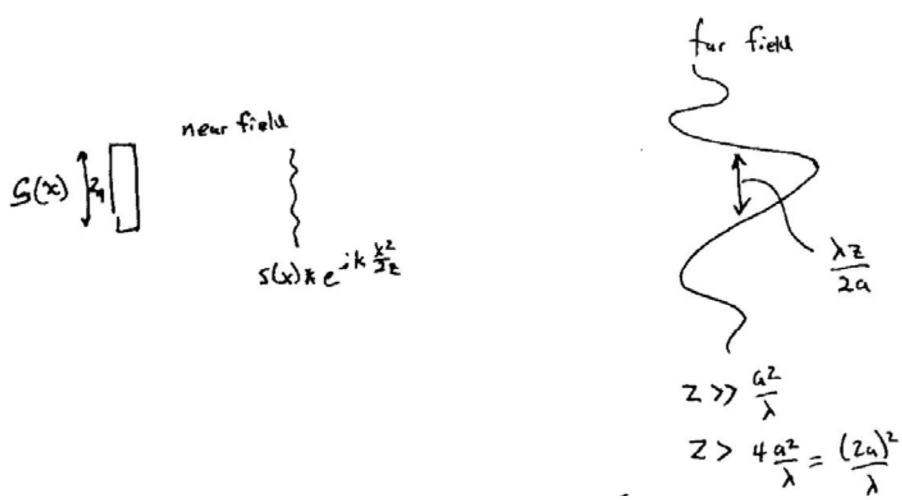
Keep in mind, that the effective source function is *z*-dependent. The Fresnel approximation can now be represented as:

$$p(z, x_z) = \frac{e^{ikz}e^{ik\frac{x_z^2}{2z}}}{z}\Im\{s_{eff}(x_0)\}\Big|_{u=\frac{x_z}{\lambda z}}$$

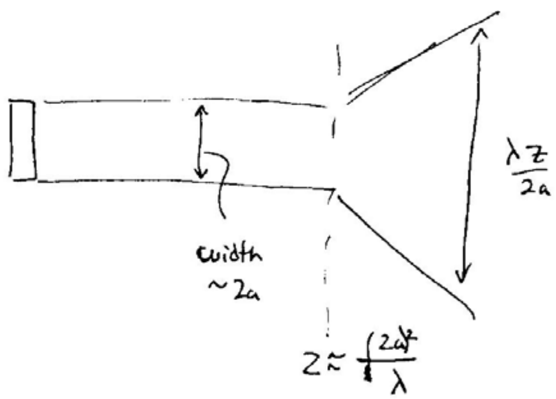


Beam patterns

The beam pattern changes as one moves from near field to far field (Fresnel to Fraunhoffer zones).



The approximate widths of these regions is given here:



- In US we work mainly in the Fresnel zone, though in some cases also the Fraunhoffer zone. For example, for a 2 cm transducer (a=1 cm), f_0=1.5 MHz, λ = 1mm, the far field starts roughly at $z = (2a)^2/\lambda = 40$ cm. Also, consider that at z = 40 cm, $\lambda z/2a$ = 2 cm.
- For RADAR and especially, radio telescopes → Fraunhoffer zone.

Complex Transducer Functions

Topic 1: Focussing

Let's consider a true complex driving function in the Fresnel zone:

$$s(x_0) = \left|s(x_0)\right| e^{-ik\frac{x_0^2}{2z_0}}$$

then for $z = z_0$:

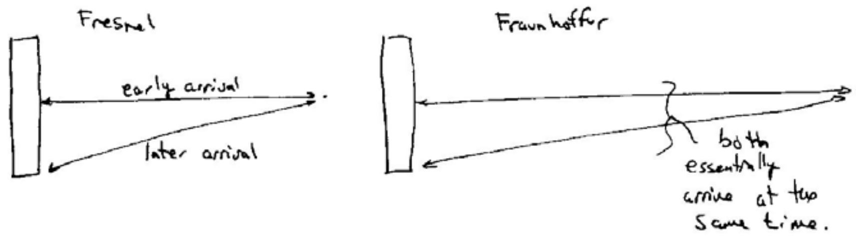
$$s_{eff}(x_0) = \left|s(x_0)\right|$$

and

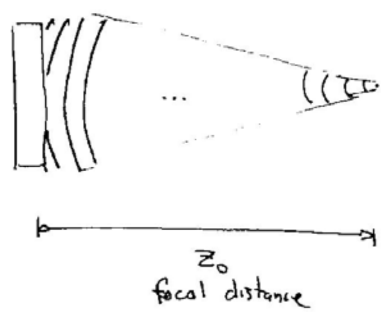
$$p(z_0, x_z) = \frac{e^{ikz_0} e^{ik\frac{x_z^2}{2z_0}}}{z_0} \Im\left\{\left|s(x_0)\right|\right\}\Big|_{u=\frac{x_z}{\lambda z_0}}$$

This is the same result as the Fraunhoffer zone!

One main difference between the Fresnel and Fraunhoffer zones is that in the Fresnel zone wavefonts from different parts of the transducer arrive at a point in front of the transducer at different times, whereas in the Fraunhoffer zone they are assumed to arrive at the same time.



A complex, focusing transducer function will lead to all wavefronts arriving at the same time at a point at depth z_0, in a manner similar to the Fraunhoffer zone. In essence, we are attempting to focus our wavefronts to a point:



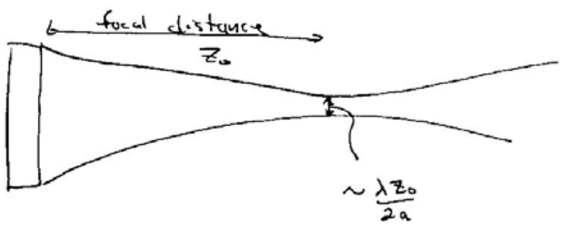
- By pre-encoding the phase of the wavefront, we can eliminate the quadratic phase at depth z_0 in the object (but only for depth z_0). That is to say, we can focus only a single depth plane.
- How well can we focus? To a point? No, the size is diffraction limited by the shape of

$$\Im\{|s(x_0)|\}\Big|_{u=\frac{x_z}{\lambda z_0}}$$

  Consider the preceding example of a 2 cm transducer (a=1 cm), f_0=1.5 MHz, λ = 1mm, and a far field $z > (2a)^2/\lambda = 40$ cm. Now let z_0=5 cm. The width of the response is roughly $\frac{\lambda z_0}{2a} = 2.5\,\text{mm}$. This is a much smaller beam cross section than the roughly $2a$ = 2 cm that we had for the uniform wavefront case in the Fresnel zone.
- For depth planes away from z_0, we have:

$$s_{eff}(x_0) = |s(x_0)|e^{-ik\frac{x_0^2}{2z_0}}e^{-ik\frac{x_0^2}{2z}} = |s(x_0)|e^{-ik\frac{x_0^2}{2}\left(\frac{1}{z}-\frac{1}{z_0}\right)}$$

  and we will have a new beam pattern that might look something like this:



- Any drawbacks? Yes – worse performance in the far field.
- A fixed focus system can be done using mechanical material properties. For example, focusing can be accomplished by using materials with fast propagation velocities at the transducer edges and slower velocities at the center. This will lead the edges having the phase be advanced relative to the center.
- For a fixed (mechanical) focus system, the beam pattern is the same for transmit and receive:

$$B(x_z, z) = \left[p(z, x_z)\right]^2$$

- In an array system, focusing can be done electronically. Here it is possible to have different aperture functions for transmit and receive, e.g. $s_{T,eff}(x_0)$ and $s_{R,eff}(x_0)$. Then:

$$h_T(\omega, z, x_z) = \frac{e^{ikz} e^{ik\frac{x_z^2}{2z}}}{z} \Im\left\{s_{T,eff}(x_0)\right\}\Big|_{u=\frac{x_z}{\lambda z}}$$

and similarly for h_R.

Topic 2: Beam Steering

Suppose we wish to shift the focal point laterally in x_z. Let's choose some depth plane (z_0) and a lateral shift (x_{z0}) and introduce another new complex driving function with a linear phase variation across the phase of the transducer:

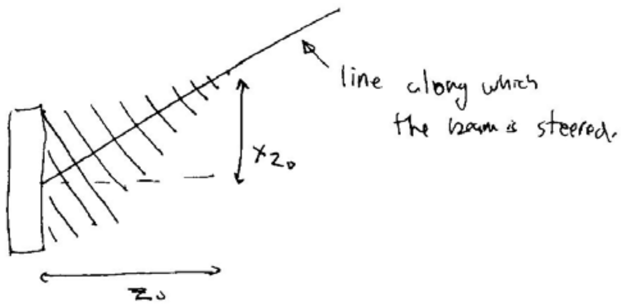
$$s'(x_0) = s_{eff}(x_0) e^{i2\pi \frac{x_{z0}}{z_0 \lambda} x_0}$$

then

$$p(z,x_z) = \frac{e^{ikz}e^{ik\frac{x_z^2}{2z}}}{z}\Im\left\{s_{eff}(x_0)e^{i2\pi\frac{x_{z0}}{z_0\lambda}x_0}\right\}\Bigg|_{u=\frac{x_z}{\lambda z}}$$

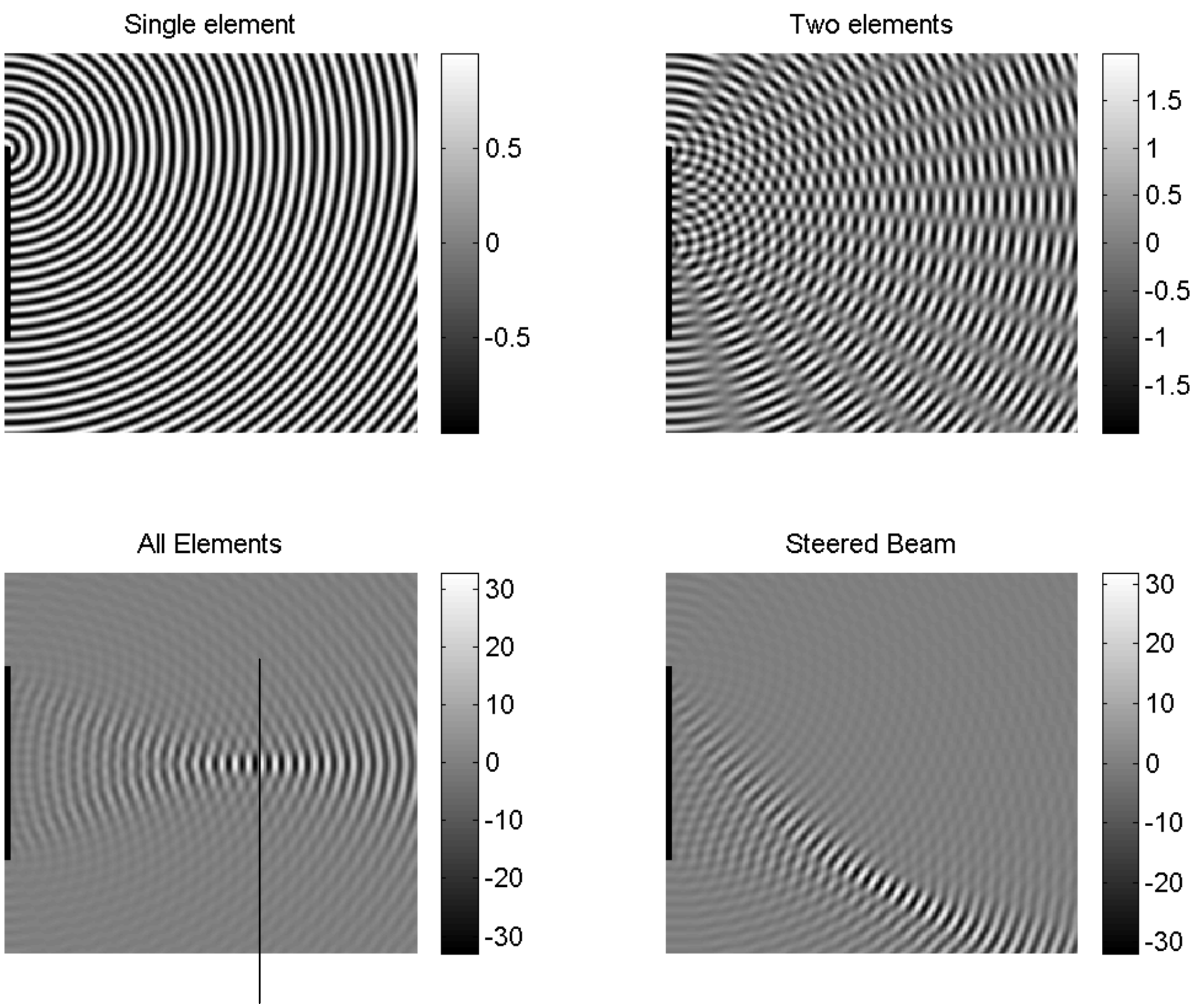
$$= \frac{e^{ikz}e^{ik\frac{x_z^2}{2z}}}{z}\Im\left\{s_{eff}(x_0)\right\}\Big|_{u=\frac{x_z}{\lambda z}-\frac{x_{z0}}{z_0\lambda}}$$

- At $z = z_0$, the response pattern is shifted by x_{z0}.
- For arbitrary z, the response is shifted by $g(x_z) = \frac{z}{z_0}x_{z_0}$, a line through the center of the transducer:



Steady-state diffraction

The steady-state approximation results from fixing a point in time and allowing the pressure wave to exist over all space. In the plots below, we can see that we are building the diffraction pattern from an assumption of spherical waves emanating from points on the transducer. As we add more elements, the beam pattern takes shape. By adding phase to these patterns, we can steer the beam in one direction or another.

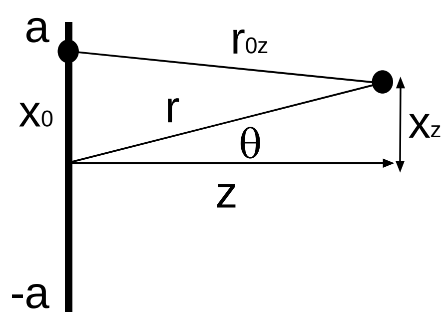


Along this line (at the focal depth), we have a Fourier transform relationship between the

Diffraction in Polar Coordinates. We start out with the same basic equation for a steady-state spherical wave:

$$p(r) = \frac{e^{ikr}}{r}\cos\theta$$

where $k = \frac{\omega_0}{c} = \frac{2\pi}{\lambda} \equiv$ wavenumber, and θ is the obliquity angle. Again, we will neglect attenuation and perform the analysis in 2D (ignoring the *y* dimension).



At some depth position *z* in the object and at a lateral position x_z, the pressure signal will be the superposition of all point sources in the aperture. The pressure wave functions is:

$$\begin{aligned} p(r,\theta) &= p(z,x_z) \\ &= \int s(x_0)\frac{e^{ikr_{0z}}}{r_{0z}}\cos\theta_{0z}\,dx_0 \end{aligned} \quad \text{where} \quad \begin{aligned} r_{0z} &= \sqrt{(x_0 - x_z)^2 + z^2} \\ r &= \sqrt{x_z^{\,2} + z^2} \end{aligned}$$

Fresnel Zone. Whenever the depth of a reflector is substantially larger than the lateral displacement from all points in the source (transducer aperture)::

$$r^2 >> x_0^2 \text{ or } r^2 >> a^2$$

then the Fresnel approximation holds. Specifically, the $e^{ikr_{0z}}$ term can be simplified using:

$$\begin{aligned}
r_{0z} &= \sqrt{(x_0 - x_z)^2 + z^2} \\
&= \sqrt{(x_0 - r\sin\theta)^2 + (r\cos\theta)^2} \\
&= \sqrt{x_0^2 - 2x_0 r\sin\theta + r^2} \\
&= r\sqrt{1 - 2\frac{x_0}{r}\sin\theta + \left(\frac{x_0}{r}\right)^2}
\end{aligned}$$

We will use the Taylor series expansion of $f(u) = \sqrt{1+u}$ expanded around u, keeping the first three terms:

$$f(u) = \sqrt{1+u} \cong 1 + \frac{1}{2}u - \frac{1}{8}u^2$$

where $u = -2\frac{x_0}{r}\sin\theta + \left(\frac{x_0}{r}\right)^2$.

$$\begin{aligned}
f(u) &\cong 1 + \frac{1}{2}\left(-2\frac{x_0}{r}\sin\theta + \left(\frac{x_0}{r}\right)^2\right) - \frac{1}{8}\left(4\left(\frac{x_0}{r}\right)^2 \sin^2\theta + O\left(\left(\frac{x_0}{r}\right)^3\right)\right) \\
&\cong 1 - \frac{x_0}{r}\sin\theta + \frac{1}{2}\left(\frac{x_0}{r}\right)^2 - \frac{1}{2}\left(\frac{x_0}{r}\right)^2 \sin^2\theta \\
&= 1 - \frac{x_0}{r}\sin\theta + \frac{1}{2}\left(\frac{x_0}{r}\right)^2 (1 - \sin^2\theta) \\
&= 1 - \frac{x_0}{r}\sin\theta + \frac{1}{2}\left(\frac{x_0}{r}\right)^2 \cos^2\theta
\end{aligned}$$

by keeping only the quadratic terms. Thus:

$$r_{0z} \cong r - x_0 \sin\theta + \frac{x_0^2}{2r}\cos^2\theta$$

In addition to the usual (above) Fresnel approximation, we will simply the smoothly varying scalar terms (results are less sensitive to this approximations to this term):

$$\begin{aligned}
\frac{1}{r_{0z}} &\cong \frac{1}{r} \\
\cos\theta_{0z} &\cong \cos\theta
\end{aligned}$$

Applying both approximations, we get:

$$p(r,\theta) = \frac{\cos\theta \cdot e^{ikr}}{r} \int s(x_0) e^{-ikx_0 \sin\theta} e^{ik\frac{(x_0 \cos\theta)^2}{2r}} dx_0$$

In the above expression, we can think of the three exponential terms in the following way:

- e^{ikr} - This is the propagation term; phase accumulation with distance from the transducer.
- $e^{-ikx_0 \sin\theta}$ - This is the lateral deflection or beam steering term.
- $e^{ik\frac{(x_0 \cos\theta)^2}{2r}}$ - This is the wavefront curvature (quadratic) or focusing term.

Fraunhoffer Zone. In the far field, the wavefront curvature term can be neglected if:

$$k\frac{(x_0 \cos\theta)^2}{2r} < k\frac{a^2}{2r} << \pi \quad \text{or equivalently,} \quad r >> \frac{a^2}{\lambda}$$

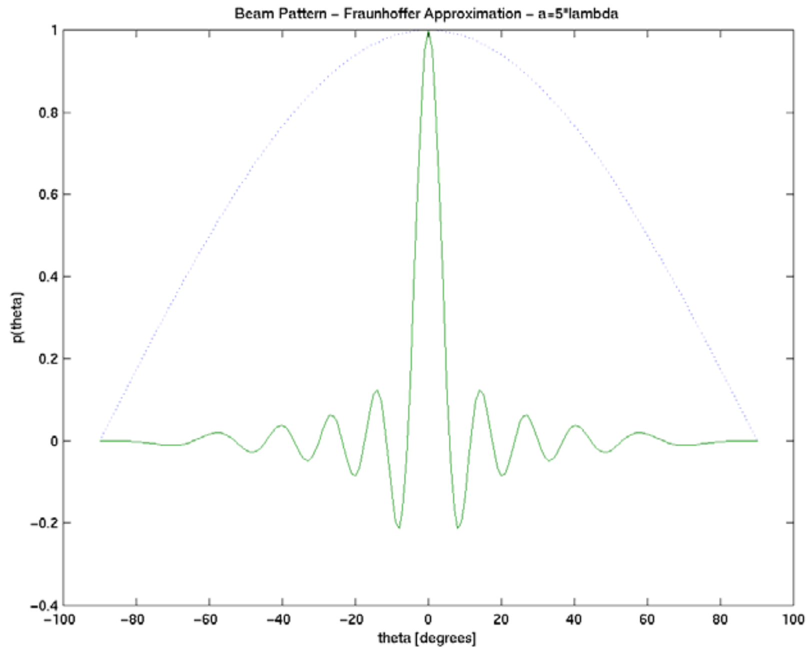
and thus:

$$p(r,\theta) = \frac{\cos\theta \cdot e^{ikr}}{r} \int s(x_0) e^{-ikx_0 \sin\theta} dx_0$$

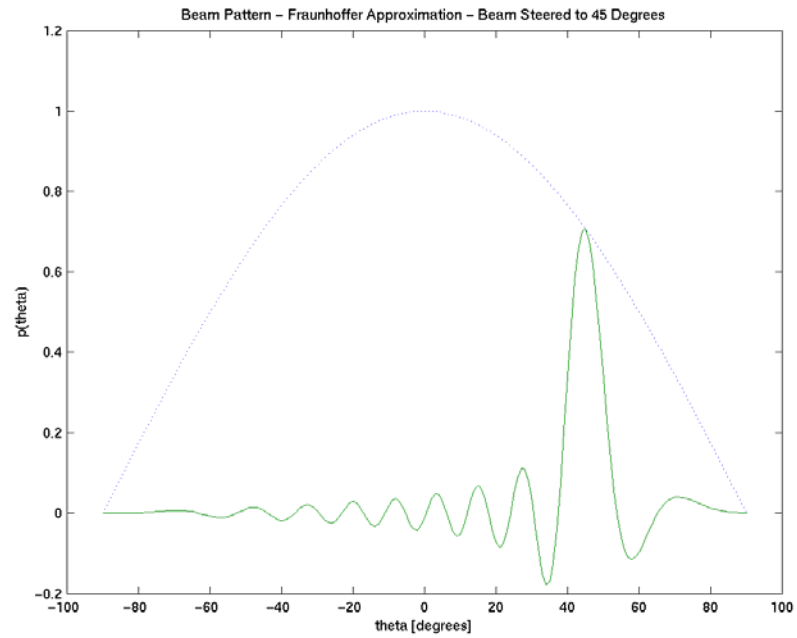
$$= \frac{\cos\theta \cdot e^{ikr}}{r} \Im\{s(x)\}\Big|_{u=\frac{\sin\theta}{\lambda}}$$

In the far field, the beam pattern is the Fourier transform of the aperture function evaluated in angular coordinates. Clearly for small angles, $\cos\theta \approx 1, r \approx z$, and $\sin\theta \approx x_z/z$, which makes this representation identical to the Cartesian version of the Fraunhoffer approximation.

Plot of $\cos\theta \cdot \Im\{s(x)\}\Big|_{u=\frac{\sin\theta}{\lambda}}$ for $s(x) = rect(x/2a)$ and for $a = 5\lambda$.

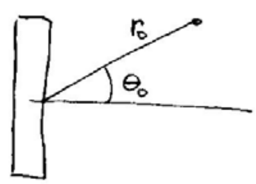

Beam Pattern – Fraunhoffer Approximation – a=5*lambda
p(theta)
theta [degrees]
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−100
−80
−60
−40
−20
0
20
40
60
80
100

Plot of $\cos\theta \cdot \Im\{s(x)\}\big|_{u=\frac{\sin\theta}{\lambda}}$ for $s(x) = rect(x/2a)e^{ikx\sin\theta_0}$, for $a = 5\lambda, \theta_0 = 45^\circ$.



Focusing and Beam Steering in Polar Coordinates in the Fresnel Zone

To focus at depth r_0 and steer to angle θ_0:



a complex aperture function must be used. For example, let

$$s(x_0) = |s(x_0)|e^{ikx_0\sin\theta_0}e^{-ik\frac{(x_0\cos\theta_0)^2}{2r_0}}$$

$$= |s(x_0)|e^{i2\pi x_0\frac{\sin\theta_0}{\lambda}}e^{-i2\pi\frac{(x_0\cos\theta_0)^2}{2r_0\lambda}}$$

The amount of phase applied across the face of the transducer for focussing is related to $(x_0\cos\theta_0)^2$ rather than $x_0{}^2$ for an off-axis focal point. This is because the transducer face appears smaller by a factor of $\cos\theta_0$ due to obliquity.

The expression for the pressure for the Fresnel zone at depth r_0 is then:

$$p(r_0,\theta) = \frac{\cos\theta \cdot e^{ikr_0}}{r_0} \int s(x_0) e^{-i2\pi x_0 \frac{\sin\theta}{\lambda}} e^{i2\pi \frac{(x_0 \cos\theta)^2}{2r_0\lambda}} dx_0$$

$$= \frac{\cos\theta \cdot e^{ikr_0}}{r_0} \int |s(x_0)| e^{i2\pi x_0 \frac{\sin\theta_0}{\lambda}} e^{-i2\pi x_0 \frac{\sin\theta}{\lambda}} dx_0$$

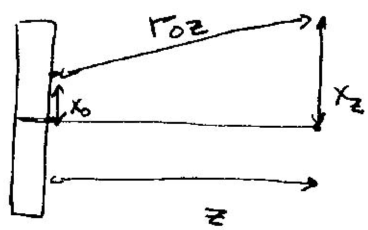
$$= \frac{\cos\theta \cdot e^{ikr_0}}{r_0} F\{|s(x)|\}\Big|_{u=\frac{\sin\theta}{\lambda}-\frac{\sin\theta_0}{\lambda}}$$

Again, this is just like the expression for the Fraunhoffer zone. Not that the origin of the beam is now at located where $(\sin\theta - \sin\theta_0) = 0$, which occurs at $(\theta = \theta_0)$ for $-\pi/2 < \theta < -\pi/2$. For example, if $|s(x)| = \mathrm{rect}(x/2a)$, then

$$p(r_0,\theta) = \frac{\cos\theta \cdot e^{ikr_0}}{r_0} 2a \,\mathrm{sinc}\left(\frac{2a}{\lambda}(\sin\theta - \sin\theta_0)\right)$$

Diffraction Viewed as Propagation Delays

Thus far, we have studied diffraction in the steady state, that is, we have examined the effect of wavefronts adding constructively or destructively. We will again look at diffraction from a slightly different viewpoint – that of diffraction being caused by differences in propagation delay from the source positions to reflector:



- This formulation is a preview to see how one might focus in an array system.

We define the propagation delay as:

$$\tau_{0z} = \tau(x_0, z, x_z) = \frac{r_{0z}}{c} \approx \frac{z}{c} + \frac{(x_0 - x_z)^2}{2zc} \quad \text{(Fresnel approximation)}$$

After lumping a bunch of terms into a constant *K'*, we get the following description of the arrival of a pulse at some point in the object:

$$P(x_z, z, t) = K' \int_{-a}^{a} a(t - \tau_{0z}) e^{-i\omega_0 (t - \tau_{0z})} dx_0$$

Suppose we add some delay (τ') to the pulse for each position along the transducer to focus at a point (x_z=0, z_0):

$$\tau'(x_0) = \frac{x_0{}^2}{2z_0 c}$$

Now the pressure function is:

$$P(x_z, z, t) = K' \int_{-a}^{a} a(t + \tau'(x_0) - \tau_{0z}) e^{-i\omega_0 (t + \tau'(x_0) - \tau_{0z})} dx_0$$

$$= K' \int_{-a}^{a} a\left(t + \frac{x_0{}^2}{2z_0 c} - \frac{z}{c} - \frac{(x_0 - x_z)^2}{2zc} \right) e^{-i\omega_0 \left(t + \frac{x_0{}^2}{2z_0 c} - \frac{z}{c} - \frac{(x_0 - x_z)^2}{2zc} \right)} dx_0$$

If we looking direction that the focal point, (x_z=0, z_0):

$$P(0, z_0, t) = K' \int_{-a}^{a} a\left(t - \frac{z_0}{c} \right) e^{i\omega_0 \left(t - \frac{z_0}{c} \right)} dx_0 = K' \cdot 2a \cdot a\left(t - \frac{z_0}{c} \right) e^{i\omega_0 \left(t - \frac{z_0}{c} \right)}$$

all wavefronts come together to add constructively.

Looking at and (z_0, x_z):

$$P(x_z, z_0, t) = K' \int_{-a}^{a} a\left(t - \frac{z_0}{c} + \frac{x_0 x_z}{z_0 c} - \frac{x_z{}^2}{2z_0 c} \right) e^{-i\omega_0 \left(t - \frac{z_0}{c} + \frac{x_0 x_z}{z_0 c} - \frac{x_z{}^2}{2z_0 c} \right)} dx_0$$

$$= K' e^{-i\omega_0 \left(t - \frac{z_0}{c} - \frac{x_z{}^2}{2z_0 c} \right)} \int_{-a}^{a} a\left(t - \frac{z_0}{c} + \frac{x_0 x_z}{z_0 c} - \frac{x_z{}^2}{2z_0 c} \right) e^{-i\omega_0 \frac{x_0 x_z}{z_0 c}} dx_0$$

Observe that similarities between this and the formulation of the Fraunhoffer zone expression from the first diffraction handout:

$$p(z,x_z) = \frac{e^{ikz}e^{ik\frac{x_z^2}{2z}}}{z}\int s(x_0)e^{-ik\frac{x_0 x_z}{z}}dx_0$$

$$= \frac{e^{ikz}e^{ik\frac{x_z^2}{2z}}}{z}\int_{-a}^{a} e^{-i\omega_0\frac{x_0 x_z}{zc}}dx_0$$

In essence, we have the same formulation result as focussing in the Fresnel zone (same as Fraunhoffer zone), but with the inner part of the diffraction integral weighted by shifted versions of the envelop function, $a(t)$.

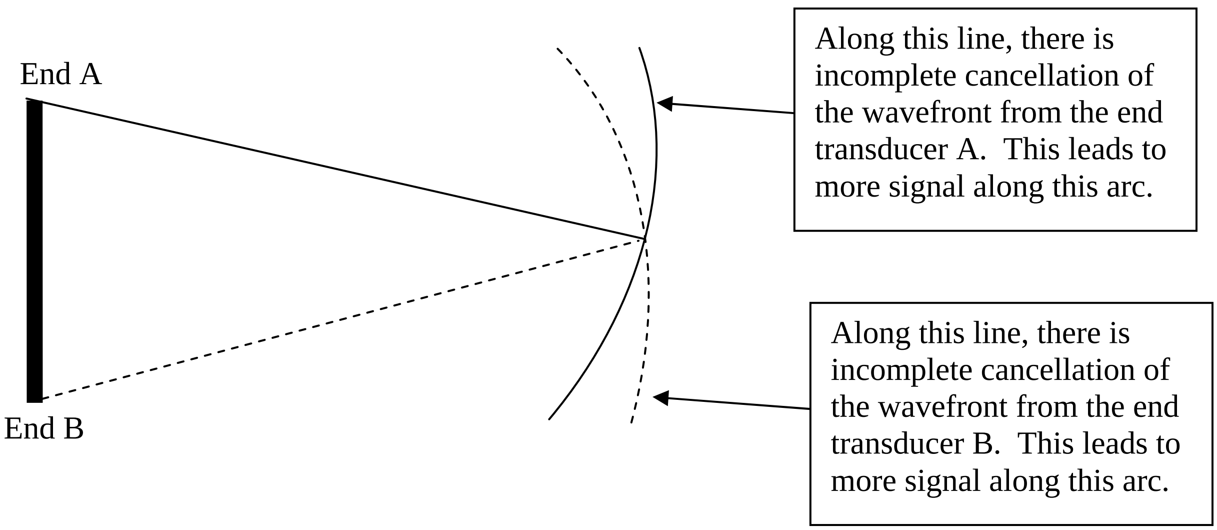
Thus, focussing/complex driving functions can be induced by delays at the transducer (for both transmit and receive).

Broadband or Wideband Diffraction

Just a few comments the above diffraction formulation:

- The FT diffraction relationship depends on the summation (either constructive or destructive) to produce the field pattern. Thus, the beam cross-section is no longer exactly the FT of the aperture function (unless the pulses, $a(t)$, are much longer than the integral function (in which case it can be pulled outside the integral.
- For very short pulses, non-alignment of the envelope ($a(t)$) functions will lead to incomplete interaction of the wavefronts and the beam pattern is thus changed. The pulses, $a(t)$, will tend to reduce the integral which might lead to less perfect suppression of out of beam signals.
- Thus, as we make the pulse shorter, we increase degradation in the lateral diffraction pattern, but get much improved depth localization.
- While the lateral resolution is somewhat degraded by short pulses, a bigger issue is that as we move off-axis, the pulses no longer overlap and this leads to a change in the point spread function to something more like an “X” pattern.
- Here is a description of the effect:
    - Close to axis, the envelopes overlap and we get the expected pattern of the steady state approximation.

- Off-axis, at the focal depth, we get fewer overlapping envelopes and weaker suppression. When we move far enough off-axis so that the number of envelopes gets smaller and the response is quite weak. For some pulse lengths, this response can be a bit cleaner than the steady-state approximation.
- Off-axis and in the arms of the “X” pattern in particular, we have even fewer overlapping envelopes for the transducers at the edges of the array and we get even weaker suppression (interference) of the pulses. This can be seen in this figure:



- Macovski, also does an analysis of this effect (see Wideband Diffraction, pp. 191-195). The results are summarized in this figure (same as above, but linearized):

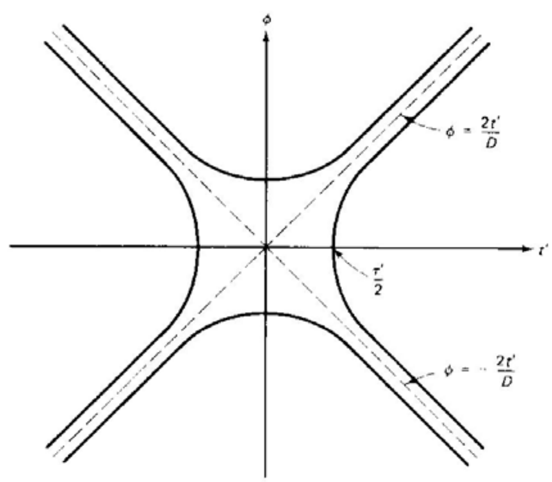


FIG. 9.10 Typical transmitted pattern, using wideband diffraction considerations.

where *t'* is the depth axis and ϕ is the angle away from the focal point. This has the appearance of a "X"-like pattern.

- We will see examples this artifact for point objects in the US project.

Broadband Diffraction

In the following figures, we can see how the point spread function degrades as the pulse length gets shorter. For the shortest pulses, there is no wavefront interaction since there is fewer than one cycle in the pulse function.