We propose a rigorous approach to find the periodic signal that best approximates the observed breathing trajectory. Approximation error in L_2 norm is minimized by projecting the observed discrete trajectory onto a set of subspaces each of which consists of all signals describing a particular candidate period. Projection Theorem provides a closed form solution for each subspace, and then a line search over different candidate period is conducted to yield the best overall approximation.

In doing so, not only do we get the most descriptive period, but we also obtain an irregularity index with respect to that period. Moreover, description of fundamental period pattern is a free by-product of this process and can facilitate individualized treatment planning.

1. Optimization setup and a sketchy derivation

Let X_T be the subspace of all continuous functions with period T that is band-limited in $[-\gamma, \gamma]$ [1] Let f_t denote a function with period T. We formulate this problem in a multi-layer optimization setup.

$$T^* = \text{argmin}_T \min_{f_T \in X_T} ||f - f_T||_2 \tag{1}$$

where T^* is the period of the closest periodic signal to the original underlying continuous trajectory and accordingly, $\text{argmin}_{f_T \in X_T} ||f - f_T||_2|_{T=T^*}$ is the "best" periodic approximation to the original sequence f in L_2 sense.

Let any signal $f(t) \in X_t$ be uniquely represented by the Fourier Series

$$f(t) = \sum_k c_k e^{-j\frac{2\pi}{T}kt}$$

We then define the sampling operator $A : X_T \rightarrow \Re^N$ as:

$$\begin{aligned}
&\mathbf{a} = \mathbf{Af} \Longleftrightarrow \\
&\quad a(n) = (A\mathbf{f})(\mathbf{n}) = \mathbf{f}(\mathbf{t_n}) \\
&\quad = \sum_{k=-K}^{K} [\frac{1}{T}\int_0^T f(t) e^{-j\frac{2\pi}{T}kt} dt] e^{-j\frac{2\pi}{T}kt_n} \\
&\quad n = 1, 2, \ldots, N
\end{aligned} \tag{2}$$

The corresponding adjoint operator $A^* : \Re^N \rightarrow X_T$ can be shown to be:

$$\begin{aligned}
&\mathbf{f} = \mathbf{A}^*\mathbf{a} \Longleftrightarrow \\
&\quad f(t) = (A^*\mathbf{a})(\mathbf{t}) = \sum_{\mathbf{k=-K}}^{\mathbf{K}} [\frac{\mathbf{1}}{\mathbf{N}} \sum_{\mathbf{n=1}}^{\mathbf{N}} \mathbf{a_n} \mathbf{e}^{-\mathbf{j}\frac{\mathbf{2\pi}}{\mathbf{T}}\mathbf{kt_n}}] \mathbf{e}^{-\mathbf{j}\frac{\mathbf{2\pi}}{\mathbf{T}}\mathbf{kt}}
\end{aligned} \tag{3}$$

By projection theorem [1], the optimal f_* satisfies

$$(A^*A)\mathbf{f}_* = \mathbf{A}^*\mathbf{a}$$

2. Verification

We conducted numerical experiments with clinical data describing amplitude-time relationship (RPM, Varian). Breathing period has been assumed to be between 1 second and 10 seconds, a well accepted physical range. With a resolution of 0.1 seconds, we project the observed discrete RPM data onto different candidate periods, and calculate the corresponding Root Mean Square (RMS) Error in relative amplitude. In the final stage, a line search is conducted to get the local minimum among the RMS Error curve and the corresponding period is claimed to be optimal.[2]

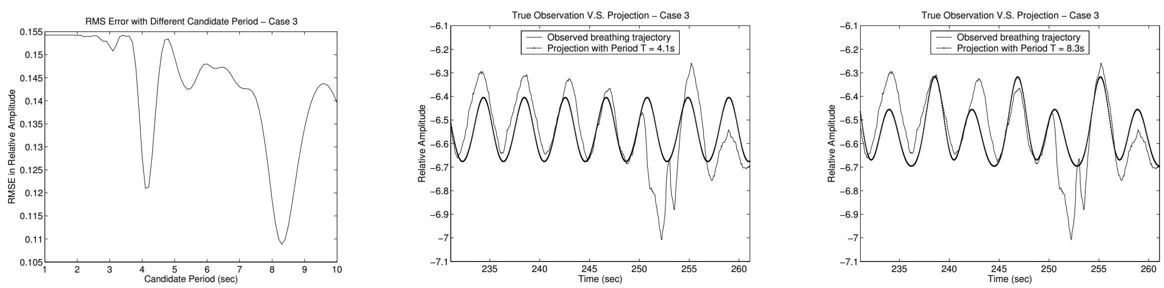
[1] Although no real signal is band-limited, this is not a stringent assumption since physical motions are believed to have bounded frequency. This is also a basic assumption for Fourier transform based analysis.

[2] We have devised our searching scheme to bias toward period less than 6 seconds, where most natural breathing periods fall in, and we refer to it as the more physically sound range.

Case ID	T1	RMSE1	T2	RMSE2	Case ID	T1	RMSE1	T2	RMSE2
1	4.7	0.1769	9.5	0.1653	7	4.4	0.0358	8.8	0.0346
2	4.5	0.0112	9.0	0.0109	8	5.4	0.0151	5.4	0.0151
3	4.1	0.1210	8.3	0.1088	9	4.6	0.0319	9.2	0.0294
4	4.7	0.0204	9.4	0.0185	10	4.0	0.1289	7.2	0.0367
5	3.4	0.0756	9.7	0.0400	11	5.6	0.0811	5.6	0.0811
6	3.1	0.1276	9.6	0.1237	12	5.2	0.1078	10.0	0.0985

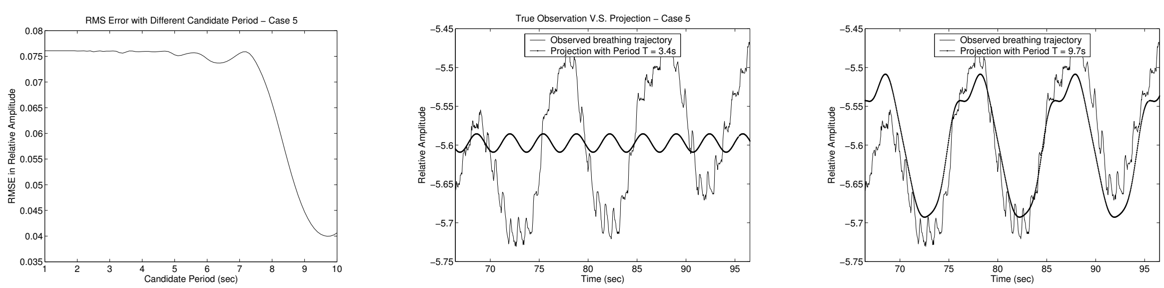
Table 1: Population Test Results: T1, T2 (in sec) are best period found within preassigned physical range (< 6sec) and the extended range (< 10sec) respectively; RMSE's are the corresponding error in relative amplitude.

There are two typical cases:



(a) (b) (c)

Figure 1: Test result for a dataset where incorporating physical constraint yields more reasonable shape though a little higher RMSE (a)RMS error for candidate period in the range 1sec-10sec (b)the best periodic signal with preferred physical range (c)the best periodic signal with extended range



(a) (b) (c)

Figure 2: Test result for a dataset where the original breathing pattern is well outside the imposed physical range: (a)RMS error for candidate period in the range 1sec-10sec (b)the best periodic signal with preferred physical range (c)the best periodic signal with extended range

The above typical instances indicate a compromise between the confidence in physical prior and the fidelity to observation data. In both cases, our approach shows consistency with the underlying physical phenomena and a reasonable fundamental pattern. The first one shows incorporating physical prior increases robustness to outliers while the second case illustrate the necessity of flexibility in priors.

References

[1] D. G. Luenberger, *Optimization by vector space methods.* New York: Wiley, 1969.