

A Min-Max Approach to the Nonuniform N-Dimensional FFT for Rapid Iterative Reconstruction of MR Images

Bradley SUTTON¹, Jeffrey A. FESSLER², Douglas NOLL¹

¹University of Michigan, Dept. of Biomedical Engineering, Ann Arbor, MI USA; ²University of Michigan, Electrical Engineering and Computer Science, Ann Arbor, Michigan USA;

Introduction

The FFT has been widely used in many areas of signal processing for the efficient computation of the DFT in $O(N \log N)$ computations instead of $O(N^2)$. But in many applications, such as non-rectilinear k-space scanning in MRI [5], the data are not uniformly spaced. Many papers have been written on approximating the nonuniform DFT by interpolating an upsampled version of the FFT, beginning with [3] and including [1,2,4,6,7,8,10,11]. This paper presents an algorithm that provides the best approximation in the min-max sense to the nonuniform DFT (NUFFT), i.e. we minimize the worst case approximation error. We apply the method to a spiral SENSE imaging experiment [9].

Theory

To use the FFT to approximate the nonequispaced DFT, we take a min-max approach. The problem is as follows: Given equally spaced object samples, x_n , for $n=0,1,\dots,N-1$ with DSFT, $X(\omega)$, we need to compute the DFT at the locations of the nonuniformly spaced sample points in the frequency domain, y_m for $m=0,1,\dots,M-1$, $y_m = X(\omega_m)$.

The algorithm is as follows:

1. $O(K \log K)$ Choose some $K \geq N$ and compute the K -point FFT, $X_k = X(2\pi k/K)$

2. $O(JM)$ Then interpolate the X_k to approximate the y_m 's using the J_m nearest neighbors of the closest point on the regular grid to ω_m . The approximation, y'_m , is given by,

$$y'_m = \sum_{j=1}^{J_m} P_{mj} X_{k(m,j)}$$

where $P = \{P_{mj}\}$ is the $M \times J_m$ interpolation matrix and the $k(m,j)$ are the closest points on the regular grid to ω_m .

We used the min-max approach in choosing the interpolation matrix, P . The interpolation matrix is chosen such that we minimize the maximum error of our estimator over all objects, \underline{x} . Looking at the m^{th} row of P , $\underline{p}^{(m)}$, we want to minimize the worst case error,

$$\min_{\underline{p}^{(m)}} \max_{\|\underline{x}\|_{\infty}=1} |y'_m - y_m|$$

Looking at the estimator error, we get

$$|y'_m - y_m| = |(\underline{x}, \underline{q}^{(m)})|$$

where \underline{q} is defined by

$$q_n^{(m)} = \sum_{j=1}^{J_m} P_{mj} e^{i2\pi k(m,j)n/K} - e^{i2\pi \omega_m n} \\ = W^{(m)} \underline{p}^{(m)*} - b_n^{(m)}$$

where $W_n^{(m)} = e^{i2\pi k(m,j)n/K}$ and $b_n^{(m)} = e^{i2\pi \omega_m n}$.

By Cauchy-Schwarz,

$$\max_{\|\underline{x}\|_{\infty}=1} |(\underline{x}, \underline{q}^{(m)})| = \|\underline{q}^{(m)}\|$$

So, the maximum error of our estimator is given by $\|\underline{q}^{(m)}\|$.

Minimizing $\underline{q}^{(m)}$ over $\underline{p}^{(m)}$ gives

$$\underline{p}^{(m)*} = (W^{(m)} W^{(m)})^{-1} W^{(m)} \underline{b}^{(m)}$$

In [8], it was shown that if $J_m = J$, $\forall m$, $W^{(m)} W^{(m)}$ does not depend on the sample locations and hence can be computed once for all M samples, this matrix was termed the regular Fourier Matrix by [8]. Similar to [8], the entries are given by,

$$(W^{(m)} W^{(m)})_{j,k} = (1 - e^{-i2\pi N(j-k)/K}) / (1 - e^{-i2\pi(j-k)/K})$$

Looking at $W^{(m)}$, it can be divided into a diagonal matrix and a matrix that does not depend on the sample locations,

$$W_{n,j}^{(m)} = e^{i2\pi k(m,j)n/K} e^{i2\pi jn/K}$$

where $k(m)$ is the first neighboring grid point used. Letting

$$D_{n,j}^{(m)} = e^{i2\pi k(m,j)n/K}, \text{ and } W_{n,j} = e^{i2\pi jn/K},$$

$W^{(m)}$ can be rewritten as,

$$W^{(m)} = D^{(m)} W$$

We have derived the following closed form solution for $D^{(m)} \underline{b}^{(m)}$ that only depends on the residual, $r^{(m)}$, between the sample point, ω_m , and the closest point on the regular grid, $2\pi k(m)/K$,

$$(W^{(m)} \underline{b}^{(m)})_j = (1 - e^{i(r^{(m)} - 2\pi j/K)N}) / (1 - e^{i(r^{(m)} - 2\pi j/K)}),$$

for $j=1,\dots,J$. Hence, we can find the rows of the weighting matrix P in $O(MJ)$, since $(W^{(m)} W^{(m)})^{-1}$ can be precomputed. An accuracy-

computation time tradeoff is available through the choice of values for the sampling factor, K , and the neighborhood size, J .

Results

Unlike the method presented in [8], the algorithm presented here is easily extended to N -dimensional problems encountered in medical imaging. We applied this algorithm to a conjugate gradient method used in reconstructing SENSE images. One spiral of a two interleaved spiral sequence was used in combination with two 5 in. coils placed on either side of a resolution phantom. The reconstruction was performed using the full DFT routine and the NUFFT routine presented here. The full DFT routine took approximately 8min on a Sun Ultra10 to perform 10 iterations to reconstruct a 64×64 image. Figure 1 shows the timing and normalized mean-squared error (NMSE) when the sampling factor, K is 192. Note that the condition number of $(W^{(m)} W^{(m)})$ can become a problem when larger neighborhood sizes are used, but at these sizes, the gain in NMSE may be outweighed by increases in computation time.

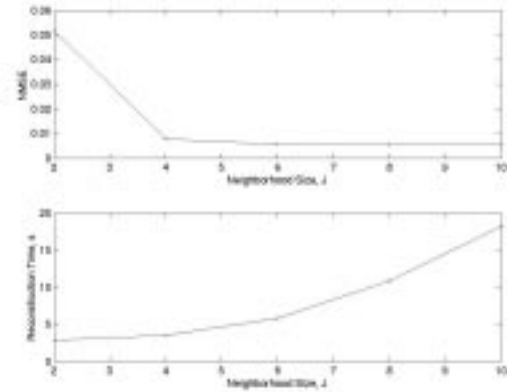


Figure 1: Error and timing results for case of $K=192$ in both dimensions.

Discussion

We have presented a method that efficiently performs the non-uniform fast Fourier Transform for image reconstruction from data that is not sampled on a Cartesian grid. This method can be used in iterative reconstruction techniques to speed MR image reconstruction, including parallel imaging experiments.

References

- [1] C Anderson, MD Dahleh *SIAM J. Sci. Comput.*, 17(4):913-919, 1996.
- [2] G Beylkin, *Appl. Comput. Harmon. Anal.*, 2:363-381, 1995.
- [3] A Dutt, V Rokhlin *SIAM J. Sci. Comput.*, 14(6):1368-1393, 1993.
- [4] A Dutt, V Rokhlin *Appl. Comput. Harmon. Anal.*, 2:85-100, 1995.
- [5] SAR Kannengießer, et al. *Proc. of ISMRM, 8th Sci. Meeting*, 155, 2000.
- [6] QH Liu, N Nguyen *IEEE Microwave and Guid. Wave Letters*, 8(1):18-20, 1998.
- [7] QH Liu, XY Tang *Electronics Letters* 34(20):1913-1914, 1998.
- [8] N Nguyen, QH Liu. *SIAM J. Sci. Comput.*, 21(1):283-293, 1999.
- [9] KP Pruessmann, et al. *Mag. Res. Med.*, 42:952-962, 1999.
- [10] G Steidl. *Advances in Comput. Math.*, 9:337-352, 1998.
- [11] AF Ware. *SIAM Rev.*, 40(4):838-856, 1998.

This work was supported in part by a Whitaker Foundation Graduate Fellowship and CBER, Univ. of Mich.