Proof of properties

Property 1: Consider hypothesis RH5 in Appendix A (that is, the coordinates of the robot consist of $N-1$ relative angles, $q_1, \cdots, q_{N-1}$, and one absolute angle q_N,) and assume furthermore that q_N is measured in the counterclockwise direction. Then the angular momentum of the robot about the stance leg end during the single support phase is $\sigma = d_N(q)\dot{q}$, where $d_N(q)$ is the last row of D in (1).

Proof: Let (p_c^h, p_c^v) be the Cartesian coordinates of the center of mass of the robot. Let $\vec{r}_1$ be a vector from the stance foot to the center of mass and let $\vec{r}_3$ be the vector from the stance foot to the swing foot. Let Ψ and ψ be the angle between $\vec{r}_1$ and the ground, and the angle between $\vec{r}_1$ and $\vec{r}_3$, respectively. Let q_N be the absolute angle in the counterclockwise direction, see Fig. 1.

Then $|\vec{r}_1|$ and ψ are independent of q_N and $\Psi = \psi + q_N - q_{N,0}$, where $q_{N,0}$ is a scalar such that the swing leg end touches the ground at $q_N = q_{N,0}$ while the relative angles $q_1, \cdots, q_{N-1}$ are unchanged. Therefore, the coordinates of the center of mass are

$$p_c^h = |\vec{r}_1| \cos(\psi + q_N - q_{N,0}) \tag{1}$$

$$p_c^v = |\vec{r}_1| \sin(\psi + q_N - q_{N,0}). \tag{2}$$

The partial derivatives of (1) and (2) with respect to q_N are given as

$$\frac{\partial p_c^h}{\partial q_N} = -|\vec{r}_1| \sin(\psi + q_N - q_{N,0}) = -p_c^v \tag{3}$$

$$\frac{\partial p_c^v}{\partial q_N} = |\vec{r}_1| \cos(\psi + q_N - q_{N,0}) = p_c^h. \tag{4}$$

The Lagrangian is defined as $L = K - V$, where K is the kinetic energy and V is the potential energy of the robot. Since K does not depend on q_N and $V = Mgp_c^v$ with g being gravitational acceleration and M being total mass,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_N} = \frac{\partial L}{\partial q_N} = -\frac{\partial V}{\partial q_N} = -Mgp_c^h. \tag{5}$$

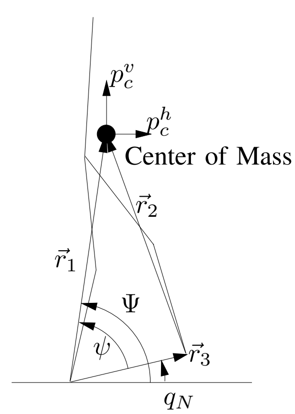

Fig. 1. Coordinates of the center of mass

Let σ be the angular momentum about the stance foot. Then,

$$\frac{d\sigma}{dt} = (\vec{r}_1 \times \vec{F}_c) \cdot \vec{e}_3, \tag{6}$$

where $\vec{F}_c$ is the force acting on the center of mass. Since $\vec{r}_1 = p_c^h \vec{e}_1 + p_c^v \vec{e}_2$ and $\vec{F}_c = -Mg\vec{e}_2$,

$$\frac{d\sigma}{dt} = -Mgp_c^h. \tag{7}$$

From (5) and (7), we can deduce that $\partial L/\partial \dot{q}_N = \sigma + C_0$, with some constant C_0. Since $\partial L/\partial \dot{q}_N$ and σ are zero if $(\dot{q}_1, \cdots, \dot{q}_N) = 0$, $C_0 = 0$. Therefore

$$\frac{\partial L}{\partial \dot{q}_N} = \sigma. \tag{8}$$

Since $d_N(q)\dot{q} = \partial L/\partial \dot{q}_N$, $d_N(q)\dot{q}$ is the angular momentum of the robot. ■

Property 2: Under hypotheses RH5 and GH2 in Appendix A, $d_N(q^-)\dot{q}^- = d_{e,N}(q_e^-)\dot{q}_e^-$, where $d_{e,N}(q_e)$ is the Nth row of D_e in (2).

Proof: Let L and L_e denote the Lagrangians for the single support phase and double support phase, respectively. Define $L = K - V$ and $L_e = K_e - V_e$, where K and K_e are the kinetic energies and V and V_e are the potential energies. By the hypothesis GH2, $L = L_e|_{(p_1^h, p_1^v, \dot{p}_1^h, \dot{p}_1^v)=(c,0,0,0)}$, where c is constant. Since $d_N(q)\dot{q} = \partial L/\partial \dot{q}_N$, $d_N(q^-)\dot{q}^- = \partial L/\partial \dot{q}_N = \partial L_e/\partial \dot{q}_N|_{(p_1^h, p_1^v, \dot{p}_1^h, \dot{p}_1^v)=(c,0,0,0)} = d_{e,N}(q_e^-)\dot{q}_e^-$ ■

Property 3: Assume the hypothesis RH5 in Appendix A, then in the coordinates $q_e^1 = (q_1, \cdots, q_N, p_1^h, p_1^v)$, $d_{e,N}(q_e^{1-})\dot{q}_e^{1-} = d_{e,N}(q_e^{1+})\dot{q}_e^{1+}$, where q_e^{1-}, $\dot{q}_e^{1-}$ are the states before the foot actuation and q_e^{1+}, $\dot{q}_e^{1+}$ are the states after the foot actuation.

Proof: In the given coordinate, the Cartesian coordinate of the stance foot is given by $\Upsilon_1(q_e) = (p_1^h, p_1^v)$. Therefore, $E_N = (\partial \Upsilon_1/\partial q_N)^T = 0$. Thus, $d_{e,N}(q_e^{1+})\dot{q}_e^{1+} - d_{e,N}(q_e^{1-})\dot{q}_e^{1-} = 0$. ■

Property 4: Let $\vec{P}_c^+, \vec{P}_c^-$ be the linear momentum of the robot after and before foot actuation. Then $\vec{P}^+ - \vec{P}_c^- = \mathcal{T} F_f$

Proof: Let $q_c = (q_1, \cdots, q_N, p_c^h, p_c^v)$ be the coordinates of the robot, where p_c^h, p_c^v denote the Cartesian coordinates of the center of mass, see Fig. 1. Then the kinetic energy K_e does not depend on p_c^h, p_c^v and the potential energy is given by $V_e = Mgp_c^v$. Since the force acting on the center of mass is $F_h = 0, F_v = -Mg$,

$$\frac{d}{dt}\frac{\partial L_e}{\partial \dot{p}_c^h} = 0 = F_h \tag{9}$$

$$\frac{d}{dt}\frac{\partial L_e}{\partial \dot{p}_c^v} = -Mg = F_v. \tag{10}$$

Therefore, $\frac{\partial L_e}{\partial \dot{p}_c^h}, \frac{\partial L_e}{\partial \dot{p}_c^v}$ are the linear momentum. In the coordinates q_c, let $\Upsilon_1(q_c)$ be the Cartesian coordinates of the stance foot. Then, $\vec{P}_c^+ - \vec{P}_c^- = \mathcal{T} F_f$. ■

Property 5: Under hypotheses RH5 and GH2 in Appendix A, $d_N(q^{2+})\dot{q}^{2+} = d_{e,N}(q_e^{2-})\dot{q}_e^{2-}$.

Proof: The proof is analogous to the proof of property 2. ■

Property 6: Assume the hypothesis RH5 in Appendix A. Then in the coordinate $q_e^2 = (q_1, \cdots, q_N, p_2^h, p_2^v)$, $d_{e,N}(q_e^{2-})\dot{q}_e^{2-} = d_{e,N}(q_e^{2+})\dot{q}_e^{2+}$, where q_e^{2-}, $\dot{q}_e^{2-}$ are the states before impact and q_e^{2+}, $\dot{q}_e^{2+}$ are the states after impact.

Proof: The proof is analogous to the proof of property 3. ■