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# INTRINSIC NOTIONS OF REGULARITY FOR LOCAL INVERSION, OUTPUT NULLING AND DYNAMIC EXTENSION OF NON-SQUARE SYSTEMS*

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***Abstract.*** We introduce a formalism, based upon the canonical map from initial conditions and inputs, to outputs, for analyzing the regularity conditions used in previous works on system inversion, output nulling and dynamic decoupling. A notion of strong regularity is introduced, and under it, an alternate characterization of the zero-dynamics is given in terms of the canonical map. Relations are established between properties of the zero dynamics and the notions of right- and left-invertibility coming from differential algebraic techniques. The preservation of the proposed regularity condition under dynamic compensation is investigated.

***Key Words***—Dynamic decoupling, inversion, regularity, output nulling, zero dynamics.

## 1. Introduction

It is well-known that, when trying to extend the analyses of finite zeros, right- and left-invertibility, and dynamic input-output decoupling from the class of linear systems to nonlinear systems, singularities may occur. For this reason, whenever systematic procedures have been developed for analyzing any of these properties, certain constant rank conditions have been imposed to ensure smoothness of various functions computed at each step of the procedure and/or finiteness of the number of computations. In this context, our work will build upon the important contributions of Byrnes and Isidori (1984; 1988 a; b), Isidori (1989 a), Isidori and Moog (1988) and van der Schaft (1988) on the zero-dynamics of a nonlinear system. Fliess (1986), Nijmeijer (1986), Respondek (1987), Respondek and Nijmeijer (1988) and Di Benedetto et al. (1989) on right- and left-invertibility, Descusse and Moog (1987), Nijmeijer and Respondek (1986; 1988), Hauser et al. (1988) and Xia (1989) on dynamic input-output decoupling, and Hirschorn (1979), Singh (1981) and Xia and Gao (1988) on system inversion.

In this paper, we introduce a framework, closely related to that of Di

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Benedetto et al. (1989), in which one may unify many of the constant dimensional/rank conditions used in the previously cited works. In Sec. 2, we propose a definition of regularity based upon the canonical map from initial state and inputs to outputs, and independent of any particular algorithm. This regularity condition will turn out to be the union of the conditions previously used in zero-dynamics and dynamic extension, as is detailed in Sec. 5. In Sec. 3, we characterize the proposed notion of regularity in terms of two algorithms for inversion and dynamic extension, thereby showing that their constant rank conditions are identical. Building upon this analysis, in Sec. 4, we take the opportunity to give an alternate characterization of the zero-dynamics in terms of the canonical map introduced in Sec. 2. As a corollary, we relate the notions of left- and right-invertibility, as introduced by Fliess (1986), to properties of the zero-dynamics. Finally, we show, constructively, the existence of dynamic compensators that preserve the regularity property and simultaneously linearize and decouple the input-output map. In Sec. 5, we compare our proposed notion of regularity with those used in Singh (1981), Byrnes and Isidori (1988 b), Isidori (1989 a), Descusse and Moog (1987), Di Benedetto et al. (1989) and Xia (1989).

## 2. A Notion of Regularity

Consider an affine nonlinear control system

$$\Sigma: \left.\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}\right\}, \tag{2.1}$$

where $x(t)\in X$, a simply connected open subset of $\mathbb{R}^n$, $u(t)\in U=\mathbb{R}^m$, $y(t)\in Y=\mathbb{R}^\mu$ and $f(\cdot)$, the columns of $g(\cdot)=[g_1(\cdot),\cdots,g_m(\cdot)]$ and the rows of $h(\cdot)=\mathrm{col}(h_1(\cdot),\cdots,h_\mu(\cdot))$ are analytic functions of $x$. Define $\mathscr{G}(x)=\mathrm{span}\{g_1(x),\cdots,g_m(x)\}$ and suppose that $\dim\mathscr{G}(x)=m$ for every $x\in X$.

In the usual way, one defines, by differentiating along trajectories of (2.1),

$$y^{(1)} = y^{(1)}(x, u) = \frac{\partial h}{\partial x}[f(x)+g(x)u],$$

$$\begin{aligned} y^{(k+1)} &= y^{(k+1)}(x, u, \cdots, u^{(k)}) \\ &= \frac{\partial y^{(k)}}{\partial x}[f(x)+g(x)u] + \sum_{i=0}^{k-1}\frac{\partial y^{(k)}}{\partial u^{(i)}}u^{(i+1)}; \end{aligned}$$

whenever convenient, we let $y^{(0)}(x)=y(x)=h(x)$. At this point, we may view $y^{(k)}$ as an analytic function $y^{(k)}$: $X\times T^{k-1}U\to\mathbb{R}^\mu$ where $T^{k-1}U$ is the $(k-1)$st order tangent bundle of $U$ (Golubitsky and Guillemin, 1973), or as a rational function of the components of $u,\cdots,u^{(k-1)}$ with coefficients analytic in $x$. Adopting this latter point of view for the moment, define, following Di Benedetto et al. (1989), $K_j$ to be the field of rational functions of (the components of) $u,\cdots,u^{(j-1)}$ with meromorphic coefficients in $x$ and set $K:=K_n$. Let $\mathscr{E}$ denote the vector space over $K$ spanned by $\{dx_1,\cdots,dx_n,du_1,\cdots,du_m,\cdots,du_1^{(n-1)},\cdots,du_m^{(n-1)}\}$. From now on, we will abuse notation and write $\{dx\}$ for $\{dx_1,\cdots,dx_n\}$, $\{du\}$ for $\{du_1,\cdots,du_m\}$, $\{dy\}$ for $\{dy_1,\cdots,dy_\mu\}$, etc.; in other words, the differential of a vector valued quantity means the differential of each of its components. Define the nested sequence of subspaces of $\mathscr{E}$ by $\mathscr{E}_0=\text{span}\{dx\}$, $\mathscr{E}_k=\text{span}\{dx,dy^{(1)},\cdots,dy^{(k)}\}$, for $k=1,\cdots,n$; and $\mathscr{F}_k=\text{span}\{dy,\cdots,dy^{(k)}\}$, $k=0,\cdots,n$. In Di Benedetto et al. (1989), it is shown that

$$\dim\mathscr{E}_n - \dim\mathscr{E}_{n-1} = \dim\mathscr{F}_n - \dim\mathscr{F}_{n-1} =: \varrho^*,$$

where $\varrho^*$, the *rank of the system* (2.1) (Fliess, 1985), is a limiting value in the sense that, if one extends $\mathscr{E}$ and $K$ in the obvious way, $\varrho^*=\dim\mathscr{E}_{n+k}-\dim\mathscr{E}_{n+k-1}=\dim\mathscr{F}_{n+k}-\dim\mathscr{F}_{n+k-1}$ for $k\geq 0$.

The above framework was very convenient for proving the conceptual equivalence of several algorithms, which compute system inverses, decoupling compensators and ranks associated with left- and right-invertibility, without specifically addressing the issue of singular points; indeed, they were effectively incorporated into the field $K$. However, when actually constructing a compensator or an inverse, one must work over the field of reals and thus one requires a notion of regularity guaranteeing the existence of an open set in which the necessary operations are well-defined. This motivates the following constructions.

Let $F_0$ denote the map $F_0$: $X\to Y$ by $x\to y=h(x)$ and let $E_0$: $X\to X$ be the identity map. For all $k\geq 1$, let $F_k$ denote the map $F_k$: $X\times T^{k-1}U\to T^kY$ by $(x,u,\cdots,u^{(k-1)})\to(y,y^{(1)},\cdots,y^{(k)})$ and let $E_k$ denote the map $E_k$: $X\times T^{k-1}U\to X\times T^kY$ by $(x,u,\cdots,u^{(k-1)})\to(x,y,\cdots,y^{(k)})$. In the obvious way, for $0\leq k\leq n$, $F_k$ and $E_k$ can be viewed as functions on $X\times T^{n-1}U$. Let $F$ and $E$ denote $F_n$ and $E_n$ respectively, and let $\pi$: $X\times T^{n-1}U\to X$ be the canonical projection.

**Definition 2.1.**

(a) The output $y\equiv 0$ is *strongly regular* for the system (2.1) if $F^{-1}(0)\neq\emptyset$ and for every point $b\in F^{-1}(0)$,

(i) $$\text{rank}_{\mathbb{R}}F_k(b) = \dim_K\mathscr{F}_k, \quad 0\leq k\leq n, \tag{2.2}$$

(ii) $$\text{rank}_{\mathbb{R}}E_k(b) = \dim_K\mathscr{E}_k, \quad 1\leq k\leq n. \tag{2.3}$$

(b) $(x_0,\ y\equiv 0)$ is a *strongly regular pair* if $y\equiv 0$ is a strongly regular output function and $x_0\in\pi(F^{-1}(0))$.

(c) $(x_0,\ y\equiv 0)$ is a *locally strongly regular pair* if $x_0\in\pi(F^{-1}(0))$ and there exists an open neighborhood $\mathscr{O}$ of $x_0$ such that (2.2) and (2.3) hold for every $b\in F^{-1}(0)\cap\pi^{-1}(\mathscr{O})$.

Some comments are in order. Firstly, the fact that working locally in the states, but globally in the inputs, is the correct way to localize the notion of $y\equiv 0$ being strongly regular will be born out in the next section when local strong regularity is characterized in terms of two algorithms associated with left-inverses and dynamic decoupling; this is also clear from the work of Respondek (1987). Secondly, it is perhaps not immediately clear why we focus on $F^{-1}(0)$. This is partly for simplicity, but mostly because we wish to make contact with the zero dynamics algorithm; it is also important when considering output reproducibility about the null output. One could consider an arbitrary point $p=(y,y^{(1)},\cdots,y^{(n)})\in T^nY$ and define regularity with respect to $F^{-1}(p)$. With regard to right-invertibility, this would lead one to studying the reproducibility of output trajectories about a reference output with the first $n+1$ terms of its Taylor expansion given by $p$. Since this does not change the subsequent analysis, but complicates the notation, we focus on $p=(0,\cdots,0)$. Thirdly, for a linear system, the output $y\equiv 0$ is always strongly regular; indeed, the output is strongly regular for any $p\in T^nY$ such that $F^{-1}(p)\neq\phi$. Finally, we have adopted the terminology of *strong* regularity because (2.2) and (2.3) turn out to be precisely the constant rank conditions required to calculate a reduced order left-inverse by Singh (1981); in Sec. 5, these conditions will be shown to be stronger than those associated with output nulling and dynamic input-output decoupling.

We return now to the interpretation of $y^{(k)}$ as an analytic function on $X\times T^{k-1}U$. For $0\leq k\leq n$, $y^{(k)}$ can also be viewed as an analytic function on $X\times T^{n-1}U$. We may then introduce analytic codistributions on $X\times T^{n-1}U$ by, at each point $b$,

$$\Omega_k(b) = \operatorname{span}_{\mathbb{R}}\{dy(b),\ \cdots,\ dy^{(k)}(b)\}, \tag{2.4}$$

$$\Lambda_k(b) = \operatorname{span}_{\mathbb{R}}\{dx,\ dy(b),\ \cdots,\ dy^{(k)}(b)\}. \tag{2.5}$$

Then at every point $b\in X\times T^{n-1}U$,

$$\operatorname{rank}_{\mathbb{R}}F_k(b) = \dim_{\mathbb{R}}\Omega_k(b), \tag{2.6}$$

$$\operatorname{rank}_{\mathbb{R}}E_k(b) = \dim_{\mathbb{R}}\Lambda_k(b), \tag{2.7}$$

because $\Omega_k$ and $\Lambda_k$ represent the row span of the Jacobians of $F_k$ and $E_k$ respectively.

From (2.6) and (2.7), it follows easily that the proposed notion of regularity is invariant under invertible static state variable feedback $u=\alpha(x)+\beta(x)v$, $\beta(x)$ an invertible matrix for every $x\in X$. However, it is in general *not invariant* under the addition of integrators on the input channels. Indeed, consider the system,

$$\left.\begin{aligned} \dot{x}_1 &= x_1u_1 + u_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u_1 \end{aligned}\right\}, \tag{2.8a}$$

$$\left.\begin{aligned} y_1 &= x_1\\ y_2 &= x_2 \end{aligned}\right\}, \tag{2.8b}$$

which is square, globally feedback linearizable and globally statically input-output decouplable; the output $y\equiv 0$ *is* strongly regular (see Proposition 2.2). Nevertheless, if one dynamically extends the system by adding an integrator on the second input channel, viz

$$\dot{u}_2 = v_2, \tag{2.8c}$$

the output $y \equiv 0$ is *no longer* strongly regular with respect to the extended system (2.8a, b, c). If one now adds a second integrator, $\dot{u}_1 = v_1$, then the output $y \equiv 0$ is once again strongly regular. In Sec. 4.3, it will be shown that there do exist useful dynamic compensators that preserve the strong regularity property.

The following preliminary result is useful. Let $r_i$ denote the relative degree (p. 235 of Isidori, 1989 a) of the $i$th output of the system (2.1); that is $r_i$ is the smallest integer such that for some $\bar{x} \in X$, and $1 \leq j \leq m$, $L_{g_j} L_f^{r_i - 1} h_i(\bar{x}) \neq 0$. If for some output $h_i$ no such integer exists, then the relative degree is undefined and the $i$th output is not affected by any of the inputs. Assume then that all of the relative degrees are defined and let $A(x)$ be the *decoupling matrix* (p. 235 of Isidori, 1989 a); that is, the matrix whose *ij-entry* $a_{ij}(x)$ is

$$a_{ij}(x) = L_{g_j} L_f^{r_i - 1} h_i(x). \tag{2.9}$$

**Proposition 2.2.** Consider the system (2.1) and let $x_0 \in X$ be given. Then $(x_0, y \equiv 0)$ is a locally strongly regular pair if,

(a) rank $A(x_0) = \mu$ (the rank of the decoupling matrix equals the number of outputs); and,

(b) for each $1 \leq i \leq \mu$, $0 = h_i(x_0) = \cdots = L_f^{r_i - 1} h_i(x_0)$.

*Proof.* Just use the fact that strong regularity is invariant under invertible static state variable feedback and carry out the relevant computations on the normal form of the decoupled system (p. 261 of Isidori, 1989 a), for which the map from inputs to outputs is linear.

## 3. Characterization of Strong Regularity by the Inversion and Dynamic Extension Algorithms

The goal of this Section is to characterize the notion of regularity, introduced in Sec. 2, in terms of two algorithms, thereby providing computational procedures for checking this property and at the same time showing that the regularity conditions associated with the two algorithms are identical.

To begin, we suppose that $x_0$ is a given point satisfying $x_0 \in \pi(F^{-1}(0))$ and there exists an open neighborhood $\mathcal{O}$ of $x_0$ such that for every $b \in F^{-1}(0) \cap \pi^{-1}(\mathcal{O})$, and for each $1 \leq k \leq n$,

$$\text{rank}_{\mathbf{R}} E_k(b) = \dim_K \mathcal{E}_k =: \varrho_k. \tag{3.1}$$

Note that this is part (ii) of the local definition of strong regularity. We now carry out the inversion algorithm (Singh, 1981) using the version presented in Di Benedetto et al. (1989).

*Step* 1: Calculate $y^{(1)} = a_1(x) + b_1(x)u$ and define $B_1(x) := b_1(x)$. By (3.1), $\text{rank}_K B_1(x) = \text{rank}_{\mathbf{R}} B_1(x_0)$. Hence, there exists a permutation of the outputs, and a partition, such that upon writing $y = \text{col}(\tilde{y}_1, \hat{y}_1)$, where $\tilde{y}_1$ has $\varrho_1 - n$ components and $\hat{y}_1$ has $n + \mu - \varrho_1$ components, then

$$y^{(1)} = \begin{pmatrix} \tilde{y}_1^{(1)} \\ \hat{y}_1^{(1)} \end{pmatrix} = \begin{pmatrix} \tilde{a}_1(x)+\tilde{b}_1(x)u \\ \hat{a}_1(x)+\hat{b}_1(x)u \end{pmatrix}$$

satisfies $\text{rank}_{\mathbb{R}}\tilde{b}_1(x_0)=\varrho_1-n$. Since $\text{rank}_{\mathbb{R}}B_1(x_0)=\varrho_1-n$, there exists a matrix $M_1(x)$, analytic in a neighborhood of $x_0$, such that $\hat{b}_1(x)=M_1(x)\tilde{b}_1(x)$. Therefore, $\hat{y}_1^{(1)}$ can be expressed as

$$\begin{aligned}\hat{y}_1^{(1)} &= \hat{a}_1(x) + M_1(x)(\tilde{y}_1^{(1)}-\tilde{a}_1(x)) \\ &=: \varphi_1(x, \tilde{y}_1^{(1)});\end{aligned} \tag{3.2}$$

that is $\hat{y}_1^{(1)}(x, u)=\varphi_1(x, \tilde{y}_1^{(1)}(x, u))$. It follows that

$$\begin{aligned}\hat{y}_1^{(2)}(x, u, \dot{u}) &= \varphi_1^{(1)}(x, \tilde{y}_1^{(1)}, \tilde{y}_1^{(2)}, u) \\ &= L_f\varphi_1 + \frac{\partial\varphi_1}{\partial\tilde{y}_1^{(1)}}\tilde{y}_1^{(2)} + L_g\varphi_1 u.\end{aligned} \tag{3.3}$$

*Step* 2: Differentiate $\hat{y}_1^{(1)}$. From (3.3), this can be written as

$$\hat{y}_1^{(2)}(x, u, \dot{u}) = a_2(x, \tilde{y}_1^{(1)}, \tilde{y}_1^{(2)}) + b_2(x, \tilde{y}_1^{(1)})u.$$

Define $B_2(x, \tilde{y}_1^{(1)})=\text{col}(\tilde{b}_1(x), b_2(x, \tilde{y}_1^{(1)}))$ and evaluate for $b\in F^{-1}(0)\cap\pi^{-1}(x_0)$,

$$\begin{aligned}\varrho_2 = \text{rank}_{\mathbb{R}}E_2(b) &= \text{rank}_{\mathbb{R}}\left[\begin{array}{c} dx \\ d\dot{y} \\ d\ddot{y} \end{array}\right]\Bigg|_b \\ &= \text{rank}_{\mathbb{R}}\left[\begin{array}{ccc} I & 0 & 0 \\ \hline * & b_1(x_0) & 0 \\ \hline * & * & \tilde{b}_1(x_0) \\ * & b_2(x_0, 0) & 0 \end{array}\right] \\ &= n + \text{rank}_{\mathbb{R}}\tilde{b}_1(x_0) + \text{rank}_{\mathbb{R}}B_2(x_0, 0).\end{aligned}$$

From Di Benedetto et al. (1989), it is known that $\text{rank}_K\mathscr{E}_2=n+\text{rank}_K\tilde{b}_1(x)+\text{rank}_K B_2(x, \tilde{y}_1^{(1)})$. Hence $\text{rank}_{\mathbb{R}}B_2(x_0, 0)=\text{rank}_K B_2(x, \tilde{y}_1^{(1)})$. Therefore there exists a permutation of the components of $\hat{y}_1$, and a subsequent partition, such that $\hat{y}_1=\text{col}(\tilde{y}_2, \hat{y}_2)$, where $\tilde{y}_2$ has $\varrho_2-\varrho_1$ components, and, upon writing

$$a_2 + b_2u = \begin{bmatrix} \tilde{a}_2+\tilde{b}_2u \\ \hat{a}_2+\hat{b}_2u \end{bmatrix},$$

then $\text{rank}_{\mathbb{R}}\text{col}(\tilde{b}_1(x_0), \tilde{b}_2(x_0, 0))=\varrho_2-\varrho_1$. Moreover, since $\text{rank}_{\mathbb{R}}B_2(x_0, 0)=\varrho_2-\varrho_1$, there exists a matrix $M_2(x, \tilde{y}_1^{(1)})$, analytic on an open neighborhood of $(x_0, 0)$ in $\mathcal{O}\times\mathbb{R}^{\varrho_1-n}$ such that $\hat{b}_2(x, \tilde{y}_1^{(1)})=M_2(x, \tilde{y}_1^{(1)})\cdot\text{col}(\tilde{b}_1(x), \tilde{b}_2(x, \tilde{y}_1^{(1)}))$. It follows that $\hat{y}_2^{(2)}$ can be expressed as

$$\hat{y}_2^{(2)} = \hat{a}_2(x) + M_2(x, \tilde{y}_1^{(1)}) \begin{bmatrix} \tilde{y}_1^{(1)} - \tilde{a}_1(x) \\ \tilde{y}_2^{(2)} - \tilde{a}_2(x, \tilde{y}_1^{(1)}, \tilde{y}_1^{(2)} \end{bmatrix}$$
$$=: \varphi_2(x, \tilde{y}_1^{(1)}, \tilde{y}_1^{(2)}, \tilde{y}_2^{(2)}). \tag{3.4}$$

Proceeding in this manner and then reversing the above arguments, one obtains the following result.

**Lemma 3.1.** Suppose that $x_0 \in \pi(F^{-1}(0))$. Then there exist permutations of the outputs for the inversion algorithm such that, for $1 \le k \le n$,

$$\operatorname{rank}_K B_k(x, \tilde{y}_1^{(1)}, \cdots, \tilde{y}_{k-1}^{(k-1)}) = \operatorname{rank}_{\mathbb{R}} B_k(x_0, 0, \cdots, 0),$$

if and only if, there exists an open neighborhood $\mathcal{O}$ of $x_0$ such that for every $b \in F^{-1}(0) \cap \pi^{-1}(\mathcal{O})$, $\operatorname{rank}_{\mathbb{R}} E_k(b) = \operatorname{rank}_K \mathcal{E}_k$.

Using this result, we can now characterize a strongly regular pair by using the functions defined by the inversion algorithm.

**Theorem 3.2.** The pair $(x_0, y \equiv 0)$ is locally strongly regular, if and only if, there exists a permutation of the outputs for the inversion algorithm such that, for each $1 \le k \le n$,

(i) $\operatorname{rank}_K B_k(x, \tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1) = \operatorname{rank}_{\mathbb{R}} B_k(x_0, 0, \cdots, 0)$,

(ii)
$$\operatorname{rank}_K \left[ \frac{\partial h}{\partial x}(x) \right] = \operatorname{rank}_{\mathbb{R}} \left[ \frac{\partial h}{\partial x}(x) \right]$$

and

$$\operatorname{rank}_K \begin{bmatrix} \dfrac{\partial h}{\partial x}(x) \\ \dfrac{\partial \varphi_1}{\partial x}(x, \tilde{y}_1^{(1)}) \\ \vdots \\ \dfrac{\partial \varphi_k}{\partial x}(x, \tilde{y}_1^{(1)}, \cdots, \tilde{y}_k^{(k)}) \end{bmatrix} = \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} \dfrac{\partial h}{\partial x}(x_0) \\ \dfrac{\partial \varphi_1(x_0, 0)}{\partial x} \\ \vdots \\ \dfrac{\partial \varphi_k}{\partial x}(x_0, 0, \cdots, 0) \end{bmatrix},$$

(iii) $h(x_0) = 0, \quad \varphi_1(x_0, 0) = 0, \cdots, \varphi_k(x_0, 0, \cdots, 0) = 0.$

*Proof.* (Necessity) Lemma 3.1 establishes (i). Using (i), it follows that

$$\begin{aligned}
&\operatorname{span}\{dy, dy^{(1)}, \cdots, dy^{(k)}\} \\
&= \operatorname{span}\{dy, d\tilde{y}_i^{(j)}, 1 \le i \le k, i \le j \le k, d\hat{y}_m^{(m)}, 1 \le m \le k\} \\
&= \operatorname{span}\left\{\frac{\partial h}{\partial x}dx, d\tilde{y}_i^{(j)}, 1 \le i \le k, i \le j \le k, \right. \\
&\qquad \left. \frac{\partial \varphi_m}{\partial x}dx + \sum_{i=1}^{m}\sum_{j=i}^{m} \frac{\partial \varphi_m}{\partial \tilde{y}_i^{(j)}} d\tilde{y}_i^{(j)}, 1 \le m \le k\right\} \\
&= \operatorname{span}\{d\tilde{y}_i^{(j)}, 1 \le i \le k, i \le j \le k\} \\
&\oplus \operatorname{span}\left\{\frac{\partial h}{\partial x}dx, \frac{\partial \varphi_m}{\partial x}dx, 1 \le m \le k\right\},
\end{aligned} \tag{3.5}$$

where the spans in (3.5) are all either with respect to $K$, or, with respect to $\mathbb{R}$ and everything is evaluated at $b \in F^{-1}(0) \cap \pi^{-1}(x_0)$. Hence (ii) holds. Part (iii) follows from $x_0 \in \pi(F^{-1}(0))$.

(Sufficiency): If the inversion algorithm can be carried out and (i) holds, then (2.3) of Definition 2.1 follows from Lemma 3.1. Reasoning once again as in (3.5), one establishes (2.2). Finally, (iii) allows one to conclude that $x_0 \in \pi(F^{-1}(0))$.

We now take up characterizing strong regularity in terms of the dynamic extension algorithm of Grizzle et al. (1987) and Di Benedetto et al. (1989). To begin, we suppose once again that (3.1) holds.

*Step* 1: Let $\Sigma_0$ denote the system (2.1). Calculate $y^{(1)}$ and write it as

$$y^{(1)} = a_1(x) + b_1(x)u. \tag{3.6}$$

By the hypothesis (3.1),

$$\varrho_1 - n = \text{rank}_K b_1(x) = \text{rank}_{\mathbb{R}} b_1(x_0). \tag{3.7}$$

Hence, there exists a permutation of the outputs and a partition $y = \text{col}(\bar{y}_1, \hat{y}_1)$ such that $\bar{y}_1$ has $\varrho_1 - n$ components and

$$y^{(1)} = \begin{bmatrix} \bar{y}^{(1)} \\ \hat{y}^{(1)} \end{bmatrix} = \begin{pmatrix} \bar{a}_1(x) + \bar{b}_1(x)u \\ \hat{a}_1(x) + \hat{b}_1(x)u \end{pmatrix} \tag{3.8}$$

satisfies $\varrho_1 - n = \text{rank}_{\mathbb{R}} \bar{b}_1(x_0) = \text{rank}_K \bar{b}_1(x)$. Hence there exists a static state variable feedback $u = \alpha_1(x) + \beta_1(x)v_1$ such that

(i) $\beta_1(x_0)$ is invertible over the reals (and therefore $\beta_1(x)$ is invertible over $K$) (3.9a)

(ii) $\bar{y}_1^{(1)} = \bar{v}_1$ where $\bar{v}_1$ is the first $\varrho_1 - n$ components of $v_1$. (3.9b)

For the resulting closed-loop system, $y^{(1)}(x, \alpha_1(x) + \beta_1(x)v_1)$ only depends on $\bar{v}_1$, for otherwise, by (3.9), the rank of $\partial y^{(1)}/\partial v_1$ would exceed $\varrho_1 - n$, which is impossible by the chain rule. So one can define

$$\psi_1(x, \bar{v}_1) = \hat{a}_1(x) + \hat{b}_1(x)(\alpha_1(x) + \bar{\beta}_1(x)\bar{v}_1), \tag{3.10}$$

where $\beta_1(x) = [\bar{\beta}_1(x), \hat{\beta}_1(x)]$. Now, introduce integrators by

$$\dot{\bar{v}}_1 = \bar{u}_1, \tag{3.11a}$$

and rename the remaining components of $v_1$,

$$\hat{v}_1 = \hat{u}_1. \tag{3.11b}$$

Finally, let $\Sigma_1$ denote the system consisting of $\Sigma_0$, the static state feedback

$u=\alpha(x)+\beta(x)v_1$ and the dynamic extension (3.11). Its state is given by $x_1=\mathrm{col}(x, \bar{v}_1)$, its input is $u_1=\mathrm{col}(\bar{u}, \hat{u}_1)$, and the output remains $y=h(x)$. Denote the extended system as

$$\Sigma_1: \left.\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)u_1 \\ y &= h(x) \end{aligned}\right\}. \tag{3.12}$$

To carry out the $k+1$ step of the algorithm, one differentiates the output $k+1$ times and repeats the procedure (3.6)–(3.12) to construct the dynamically extended system, $\Sigma_{k+1}$. Proceeding in this manner, one obtains the following result.

**Lemma 3.3.** Suppose that $x_0\in\pi(F^{-1}(0))$. There exist permutations of the outputs for the dynamic extension algorithm such that, for each $1\le k\le n$,

$$\mathrm{rank}_K b_k(x, \bar{v}_1, \cdots, \bar{v}_{k-1}) = \mathrm{rank}_{\mathbf{R}} b_k(x_0, 0, \cdots, 0),$$

if and only if, there exists an open set $\mathcal{O}$ about $x_0$ such that $\mathrm{rank}_K \mathcal{E}_k = \mathrm{rank}_{\mathbf{R}} E_k(b)$ for every $b\in F^{-1}(0)\cap\pi^{-1}(\mathcal{O})$.

Using this, one can characterize local strong regularity in terms of the functions defined by the dynamic extension algorithm.

**Theorem 3.4.** The pair $(x_0, y\equiv 0)$ is locally strongly regular if and only if, there exist permutations of the outputs for the dynamic extension algorithm such that, for all $1\le k\le n$,

(i) $$\mathrm{rank}_K b_k(x, \bar{v}_1, \cdots, \bar{v}_{k-1}) = \mathrm{rank}_{\mathbf{R}} b_k(x_0, 0, \cdots, 0),$$

(ii) $$\mathrm{rank}_K\left[\frac{\partial h}{\partial x}(x)\right] = \mathrm{rank}_{\mathbf{R}}\left[\frac{\partial h}{\partial x}(x_0)\right]$$

and

$$\mathrm{rank}_K\begin{bmatrix} \dfrac{\partial h}{\partial x}(x) \\ \dfrac{\partial \psi_1}{\partial x}(x, \bar{v}_1) \\ \vdots \\ \dfrac{\partial \psi_k}{\partial x}(x, \bar{v}_1, \cdots, \bar{v}_{k-1}) \end{bmatrix} = \mathrm{rank}_{\mathbf{R}}\begin{bmatrix} \dfrac{\partial h}{\partial x}(x_0) \\ \dfrac{\partial \psi_1}{\partial x}(x_0, 0) \\ \vdots \\ \dfrac{\partial \psi_k}{\partial x}(x_0, 0, \cdots, 0) \end{bmatrix},$$

(iii) $$h(x_0) = 0, \quad \psi_1(x_0, 0), \cdots, \psi_n(x_0, 0, \cdots, 0) = 0.$$

*Proof.* The proof is almost identical to Theorem 3.2 and is skipped.

By combining the extension algorithm with the rule from Descusse and Moog (1987) for adding integrators, Xia (1989) obtains a minimal order decoupling compensator. We remark, without proof, that the algorithm by Xia (1989) leads to precisely the same regularity conditions.

## 4. Consequences of Strong Regularity

This section shows that the strong regularity condition of Sec. 2 leads to many interesting conclusions for some important synthesis problems, such as output nulling, left- and right-invertibility and dynamic input-output linearization. This development will allow us to establish, in Sec. 5, precise relations between the proposed conditions of regularity and those of previous works.

**4.1 Output nulling** The concept of the zero-dynamics, introduced by Byrnes and Isidori (1984), and analyzed in a sequence of papers (Byrnes and Isidori, 1988 a; b; Isidori, 1989 a; Isidori and Moog, 1988; van der Schaft, 1988) has proven to be important through its applications to local stabilization problems (Byrnes and Isidori, 1988 a; b, 1989; see also Aeyels, 1985; Marino, 1988), model matching (Byrnes et al., 1988; Di Benedetto, 1988), output reproducibility (Byrnes and Isidori, 1988 b; Di Benedetto and Slotine, 1988), exact and approximate linearization via dynamic feedback (Isidori et al., 1986; Hauser et al., 1989) and asymptotic disturbance rejection (Byrnes and Isidori, 1988 c). This section develops an alternate characterization of the zero-dynamics in terms of the maps introduced earlier.

Let the maps $F_k$ be defined as in Sec. 2 and define $N_k := \pi(F_k^{-1}d(0))$.

**Lemma 4.1.** Suppose that $y \equiv 0$ is a strongly regular output function. Then for each $k \geq 1$, $N_k$ is an embedded $C^\omega$ submanifold of $X$. In particular, for each $x_0 \in \pi(F^{-1}(0))$ there exists an open neighborhood $\mathcal{O}$ of $x_0$ such that, for each $1 \leq k \leq n$,

(a) $N_k \cap \mathcal{O} = \{x \in \mathcal{O} \mid h(x) = 0,\ \varphi_1(x, 0) = 0, \cdots, \varphi_k(x, 0, \cdots, 0) = 0\}$, where $\varphi_i$ are the functions computed by the inversion algorithm,

(b) $N_k \cap \mathcal{O} = \{x \in \mathcal{O} \mid h(x) = 0,\ \psi_i(x, 0) = 0, \cdots, \psi_k(x, 0, \cdots, 0) = 0\}$, where $\psi_i$ are the functions computed by the dynamic extension algorithm.

Moreover, the sequence of manifolds $N_k$ converges in a finite number of steps; precisely,

(c) for all $k \geq n+1$, $N_k = N_n$.

*Proof.* See the Appendix.

From now on, we let $N := N_n$. For completeness, we remark that $N$ is not necessarily connected (Isidori, 1989 a); indeed, for the following system:

$$\dot{x}_1 = u,$$

$$\dot{x}_2 = x_1 + x_2,$$

$$y = \sin(x_1 + x_2),$$

one can check that $y \equiv 0$ is a strongly regular output function and that $N$ has a countably infinite number of components given by $\bigcup_{|k|=0}^{\infty} \{(x_1, x_2) \mid x_1 + x_2 = k\pi\}$.

Next, we define a dynamical system on $N$, using the language of affine distributions introduced by Nijmeijer (1981); an affine distribution is the specification, at each point of the tangent space, of an affine set, that is, the translate of a subspace.

**Theorem 4.2.** Suppose that $y\equiv 0$ is a strongly regular output function for (2.1). Let $N=\pi(F^{-1}(0))$, and define, for each $x\in N$,

$$\Delta(x) = \{f(x) + g(x)u \mid u \in U \text{ satisfies } f(x) + g(x)u \in T_xN\}.$$

Then, if the dimension of $N$ is nonzero, $\Delta$ is a non-empty $C^{\omega}$-affine distribution on $N$. Moreover, the distribution $\Delta_0$ defined at each point $x\in N$ by $\Delta_0(x):=\Delta(x)-\Delta(x):=\{Z_1(x)-Z_2(x) \mid Z_i(x)\in\Delta(x)\}$, is constant dimensional.

*Proof.* Fix $x_0\in\pi(F^{-1}(0))$. Then from Lemma 4.1 there exists an open neighborhood $\mathcal{O}$ of $x_0$ such that for $x\in N\cap\mathcal{O}$, $T_xN$ is isomorphic to

$$\left\{v \in T_xX \mid \frac{\partial h(x)}{\partial x}v = 0, \cdots, \frac{\partial \varphi_n}{\partial x}(x, 0, \cdots, 0)v = 0\right\}.$$

Hence,

$$\Delta(x) = \{f(x)+g(x)u \mid L_{f+gu}h(x) = 0, \cdots, L_{f+gu}\varphi_n(x, 0, \cdots, 0) = 0\},$$

which by construction of the inversion algorithm gives, $\Delta(x)=\{f(x)+g(x)u \mid u$ satisfies (4.1)$\}$:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{a}_1(x)+\tilde{b}_1(x)u \\ \vdots \\ \tilde{a}_n(x, 0, \cdots, 0)+\tilde{b}_n(x, 0, \cdots, 0)u \end{bmatrix}. \tag{4.1}$$

Thus, in view of part (i) of Theorem 3.2, and the fact that $\mathcal{G}(x)$ is constant dimensional, $\Delta$ and $\Delta_0$ are both analytic distributions on $N$ and $\Delta_0$ is constant dimensional. Finally, $\Delta(x)$ is nonempty since $f^*(x)=f(x)+g(x)\alpha(x)\in\Delta(x)$ where $\alpha(\cdot)$ is as in the proof of Lemma 4.1 (when $k=n$).

*Remark* 4.3: From the proofs of Lemma 4.1 and Theorem 4.2 one sees that $\Delta$ can also be characterized as, $\forall x\in N$,

$$\Delta(x) = \{f(x) + g(x)u \mid \exists u, \dot{u}, \cdots, u^{(n-1)} \text{ satisfying } F(x, u, \dot{u}, \cdots, u^{(n-1)}) = 0\}.$$

Let $M_k$, $1\le k\le n$, be the sequence of submanifolds defined at each step of the zero dynamics algorithm (Isidori, 1989 a, p. 290) and let $M^*:=M_n$ be the zero-dynamics manifold. Suppose that $TM^*\cap\mathcal{G}=\{0\}$, and define $f^*$ to be the zero-dynamics vector field (see Byrnes and Isidori, 1988 a; b, for the original reference). For the case of square systems, the local characterization of $N$ given in Lemma 4.1 in conjunction with the analysis of Isidori and Moog (1988) shows that the pair $(N, \Delta)$ corresponds to $(M^*, f^*)$. More precisely, if $(x_0, y\equiv 0)$ is a strongly regular pair, there exists an open neighborhood $\mathcal{O}$ of $x_0$ such that

$$N_k \cap \mathcal{O} = M_k \cap \mathcal{O}$$

and

$$\Delta(x) = f^*(x), \quad \forall x \in N \cap \mathcal{O}.$$

In a similar way, one can establish the equivalence between $(N, \Delta)$ and the output nulling dynamics as defined by Van der Schaft (1988) for nonsquare systems. For this reason, $(N, \Delta)$ will be referred to as the *output nulling* dynamics and $N$ as the *output nulling manifold*, thereby using the same terminology introduced by Anderson (1975; 1976) for linear systems.

Finally, the output nulling manifold can also be characterized in terms of smooth input functions yielding a zero output.

**Proposition 4.4.** (see also Byrnes and Isidori, 1988 b, Lemma 2.1; or p. 292 of Isidori, 1989 a, for the case of square invertible systems) Suppose that $y \equiv 0$ is a strongly regular output function. Then

$$N = \{x_0 \in h^{-1}(0) \,|\, \exists \varepsilon > 0 \text{ and a } C^\infty \text{ input function}$$
$$u\colon (0, \varepsilon) \to \mathbb{R}^m \text{ such that } \forall t \in (0, \varepsilon),\ \Phi^u_{x_0}(t) \in h^{-1}(0)\},$$

where $\Phi^u_{x_0}(t)$ denotes the solution of (2.1) corresponding to the input $u$ and $x(0) = x_0$.

*Proof.* If such an input function exists, then $\forall k \geq 1$, $(d^k/dt^k)y(t)|_{t=0} = 0$; therefore $(x_0, u(0), \cdots, u^{(n-1)}(0)) \in F^{-1}(0)$, which shows that $x_0 \in \pi(F^{-1}(0)) = N$. Conversely, suppose $x_0 \in N$. Let $\alpha(x)$ be the feedback function constructed in the proof of Lemma 4.1, for $k = n$. Then for every $x \in N$, $f^*(x) = f(x) + g(x)\alpha(x) \in T_x N$. Let $\Psi(t)$ be the flow of $f^*$ corresponding to $x_0$; i.e., $\Psi(0) = x_0$. Then the input $u(t) := \alpha(\Psi(t))$ is $C^\infty$ and satisfies, for $t > 0$, but sufficiently small, $\Phi^u_{x_0}(t) \in N \subset h^{-1}(0)$.

**4.2 Left- and right-invertibility** Since the work of Hirschorn (1979) and Singh (1981), a great deal of effort has been devoted to understanding left- and right-invertibility of nonlinear systems, and their relation to decoupling (Descusse and Moog, 1987; Nijmeijer and Respondek, 1986; 1988) and output reproducibility (Respondek and Nijmeijer, 1988). Recent work by Respondek (1987) treats this subject in detail, with the emphasis being on conditions that hold on an open dense subset of the state space. Fliess (1985; 1986) achieved a synthesis of left-invertibility, right-invertibility and dynamic decoupling through the use of differential algebra. Following his terminology, we say that the system (2.1) is *left-invertible* if its rank, $\varrho^*$, equals $m$, the number of input components, and it is *right-invertible* if its rank equals $\mu$, the number of output components.

It is known (Di Benedetto et al., 1989) that the integer $\varrho^*$ coincides with the differential output rank, whenever both are defined. Moreover, if $(x_0, y \equiv 0)$ is a strongly regular pair, it follows easily from Di Benedetto et al. (1989) and Sec. 3 that

(i) $$\varrho^* = \text{rank}_{\mathbb{R}} B_n(x_0, 0, \cdots, 0), \tag{4.2}$$

where $B_n(x, \dot{\bar{y}}_1, \cdots, \bar{y}^{(n-1)}_{n-1})$ is calculated from the inversion algorithm,

(ii) $$\varrho^* = \text{rank}_{\mathbb{R}} b_n(x_0, 0, \cdots, 0), \tag{4.3}$$

where $b_n(x, \bar{v}_1, \cdots, \bar{v}_{n-1})$ is calculated from the dynamic extension algorithm. In

Di Benedetto et al. (1989), it was also established that $\varrho^*$ coincides with the ranks of the Jacobian matrices introduced by Nijmeijer (1986),

$$\varrho^* = \operatorname{rank}_K \frac{\partial(\dot{y}, \cdots, y^{(n)})}{\partial(u, \cdots, u^{(n-1)})} - \operatorname{rank}_K \frac{\partial(\dot{y}, \cdots, y^{(n-1)})}{\partial(u, \cdots, u^{(n-2)})}.$$

It then follows from Sec. 3 that

$$\text{(iii)} \quad \varrho^* = \operatorname{rank}_{\mathbb{R}} \frac{\partial(\dot{y}, \cdots, y^{(n)})}{\partial(u, \cdots, u^{(n-1)})}\bigg|_b - \operatorname{rank}_{\mathbb{R}} \frac{\partial(\dot{y}, \cdots, y^{(n-1)})}{\partial(u, \cdots, u^{(n-2)})}\bigg|_b \tag{4.4}$$

for any $b \in F^{-1}(0) \cap \pi^{-1}(\mathcal{O})$, $\mathcal{O}$ a sufficiently small open neighborhood of $x_0$.

The above are relationships between algebraic quantities. The following result is of interest because it establishes the equivalence between an algebraic quantity and a geometric quantity. It is the analogue of $\varrho^* = \dim(\operatorname{Im}B) - \dim(\operatorname{Im}B \cap V^*)$, for a linear system, where $V^*$ is the maximal controlled-invariant subspace contained in the kernel of the output map. Recall that $\mathcal{G}(x) = \operatorname{span}\{g_1(x), \cdots, g_m(x)\}$ and is assumed to be $m$-dimensional throughout $X$.

**Theorem 4.5.** Suppose that $(x_0, y \equiv 0)$ is a strongly regular pair for (2.1) and let $(N, \Delta)$ be the output nulling dynamics. Then $\varrho^* = \dim\mathcal{G}(x_0) - \dim(\mathcal{G}(x_0) \cap T_{x_0}N)$.

*Proof.* Let $\varphi_k(x, \tilde{y}_1^{(1)}, \cdots, \tilde{y}_{k-1}^{(k-1)})$ be the functions produced by the inversion algorithm. Then by construction,

$$\operatorname{rank}_{\mathbb{R}} B_n(x_0, 0, \cdots, 0) = \operatorname{rank}_{\mathbb{R}} B_{n+1}(x_0, \cdots, 0)$$
$$= \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} \frac{\partial h}{\partial x}(x_0) \cdot g(x_0) \\ \vdots \\ \frac{\partial \varphi_n}{\partial x}(x_0, 0, \cdots, 0) g(x_0) \end{bmatrix}.$$

Therefore, $\operatorname{rank}_{\mathbb{R}} B_n(x_0, 0, \cdots, 0) = \dim\mathcal{G}(x_0) - \dim\{v \in \mathcal{G}(x_0) \mid (\partial h/\partial x)(x_0)v = 0, \cdots, (\partial\varphi_n/\partial x)(x_0, 0, \cdots, 0)v = 0\} = \dim\mathcal{G}(x_0) - \dim\mathcal{G}(x_0) \cap T_{x_0}N$. In view of (4.2), this completes the proof.

Hence, under the strong regularity condition, a system is right-invertible if and only if, $\dim\mathcal{G}(x_0) - \dim\mathcal{G}(x_0) \cap T_{x_0}N = \mu$, the number of outputs, and left-invertible if and only if, $\dim\mathcal{G}(x_0) - \dim\mathcal{G}(x_0) \cap T_{x_0}N = m$, the number of inputs. When a system in square $(m = \mu)$, these conditions reduce to $\dim\mathcal{G}(x_0) \cap T_{x_0}N = 0$, which is how invertibility was used in Byrnes and Isidori (1988 a; b).

It is interesting to observe that Byrnes and Isidori (1988 a; b), Descusse and Moog (1987), Di Benedetto et al. (1989), Fliess (1985; 1986), Nijmeijer and Respondek (1986; 1988), Respondek (1987), Respondek and Nijmeijer (1988) and Singh (1981) are thus all consistent on their use of the term "invertible".

**4.3 Dynamic extension** We will show next that a dynamic compensator can be constructed that simultaneously preserves strong regularity, decouples the system, and does not modify the output nulling dynamics. (Recall the example of Sec. 2.)

**Theorem 4.6.** Suppose that (2.1) is right-invertible and that $(x_0, y\equiv 0)$ is a strongly regular pair. Then there exists an open neighborhood $\mathcal{O}_1$ of $x_0$, an integer $q\geq 0$, and a dynamic state variable feedback

$$\left.\begin{aligned} \dot{z} &= \gamma(x, z) + \delta(x, z)v \\ u &= \alpha(x, z) + \beta(x, z)v \end{aligned}\right\}, \qquad (4.5)$$

where $v(t)\in\mathbb{R}^m$, $z(t)\in\mathcal{O}_2$, an open neighborhood of the origin of $\mathbb{R}^q$, such that:

(a) the closed-loop system (2.1)–(4.5) is decoupled on $\mathcal{O}_1\times\mathcal{O}_2$ and the rank of the decoupling matrix (over $\mathbb{R}$) equals the number of outputs (Descusse and Moog, 1987; Nijmeijer and Respondek, 1986; 1988),

(b) $y\equiv 0$ is a strongly regular output function for the closed-loop system (2.1)–(4.5) restricted to $\mathcal{O}_1\times\mathcal{O}_2$,

(c) the output nulling dynamics of the system (2.1) restricted to $\mathcal{O}_1$ is isomorphic to the output nulling dynamics of the closed-loop system (2.1)–(4.5) on $\mathcal{O}_1\times\mathcal{O}_2$,

(d) there exist coordinates $(\bar{\xi}_1,\cdots,\bar{\xi}_{\mu+1})$ on $\mathcal{O}_1\times\mathcal{O}_2$, $\bar{\xi}_i$ possibly vector valued, in which the closed-loop systems (2.1)–(4.5) has the form (see Isidori, 1987; Nijmeijer and Respondek, 1988),

$$\left.\begin{aligned} \dot{\bar{\xi}}_i &= A_i\bar{\xi}_i + b_i v_i, \quad i = 1, \cdots, \mu \\ \dot{\bar{\xi}}_{\mu+1} &= \bar{f}(\bar{\xi}_1 \cdots, \bar{\xi}_\mu, \bar{\xi}_{\mu+1}) + \sum_{i=1}^{m}\bar{g}_i(\bar{\xi}_1, \cdots, \bar{\xi}_\mu, \bar{\xi}_{\mu+1})v_i \\ y_i &= C_i\bar{\xi}_i, \quad i = 1, \cdots, \mu \end{aligned}\right\}, \qquad (4.6)$$

where each pair $(A_i, b_i)$ is in Brunovsky canonical form, $C_i=[1,\cdots,0]$ and

$$\dot{\eta} = \bar{f}(0, \cdots, 0, \eta) + \sum_{i=\mu+1}^{m}\bar{g}_i(0, \cdots, 0, \eta)v_i \qquad (4.7)$$

represents the output nulling dynamics.

*Proof.* See the Appendix.

It is remarked that part (c) of Theorem 4.6 proves that by judiciously appending integrators, the possibility of an integrator introducing a singularity can be avoided. In Isidori (1989 a, p. 389–390), it has been established that no matter how an integrator is appended, the extended system will always possess a zero-dynamics diffeomorphic to that of the original system.

**4.4 Minimum-phase property** In the case of a general nonsquare nonlinear system, the output nulling dynamics $(N, \Delta)$ is an affine control system and not a single vector field; following van der Schaft (1988), to obtain an analogue of the transmission zeros of a linear system, we must calculate the strong accessability distribution $\mathscr{R}^*$ of $(N, \Delta)$ and form the quotient, whenever it is well-defined. Assume $\mathscr{R}^*$ is constant dimensional. Then, the quotient $M^* := N/\mathscr{R}^*$ defines the *zero-dynamics manifold.* Let $\tau$: $N \to N/\mathscr{R}^*$ denote the canonical projection and define the affine distribution, $\hat{\Delta} := \tau_*(\Delta)$, to be the *zero dynamics* (van der Schaft, 1988).

In the case of a linear system, it is easy to verify that $M^*$ is the vector space $V^*/\mathscr{R}^*$, where $V^*$ is the maximal controlled invariant subspace contained in the kernel of the output and $\mathscr{R}^*$ is the maximal controllability subspace contained in $V^*$; $\hat{\Delta}$ is a linear map on $V^*/\mathscr{R}^*$ whose spectrum corresponds to the transmission zeros (whenever the original system is minimal) (Wonham, 1979). One then says that a system is minimum-phase if the transmission zeros are in the open left-half plane. Since the induced dynamics of $V^*$ restricted to $\mathscr{R}^*$ is controllable, and therefore stabilizable, the minimum-phase property is equivalent to the stabilizability of the induced dynamics on $V^*$. Since for nonlinear systems strong accessability does not imply stabilizability, we do not choose to define a minimum-phase property in terms of $\hat{\Delta}$ but instead in terms of $\Delta$, which in addition obviates the need to assume $\mathscr{R}^*$ constant dimensional.

The following definition of a minimum-phase system extends to nonsquare systems, the one introduced by Byrnes and Isidori (1984) for square systems.

**Definition 4.7.** (Based on Byrnes and Isidori, 1984; 1988 a). Suppose that $y \equiv 0$ is a strongly regular output function for (2.1) and that $x_e \in \pi(F^{-1}(0))$ is an equilibrium point. Then the system (2.1) is said to be ***minimum phase*** at $x_e$ if, denoting by $N^c$ the connected component of $N$ passing through $x_e$, $(N^c, \Delta)$ is smoothly asymptotically stabilizable about $x_e$; i.e., there exists an analytic feedback $u = \alpha(x)$ defined locally about $x_e$ on $N^c$ such that $f(x) + g(x)\alpha(x) \in T_x N^c$ (for $x$ in the domain of $\alpha(x)$) and $x_e$ is an asymptotically stable equilibrium point of the vector field $f(x) + g(x)\alpha(x) \,|\, N^c$.

The minimum phase property has been shown to be a sufficient condition for the stabilization of square nonlinear systems via static state variable feedback (Byrnes and Isidori, 1988 a). The following result reduces this problem for nonsquare systems to the well understood case of square systems.

**Theorem 4.8.** Suppose that (2.1) is right invertible, $m > \mu$, $y \equiv 0$ is a strongly regular output function and $x_0 \in \pi(F^{-1}(0))$. Then there exists an open neighborhood $\mathcal{O}$ of $x_0$ upon which is defined an analytic **singular** feedback $u = \alpha(x) + \beta(x)v$, $v \in \mathbb{R}^\mu$, such that the closed-loop system

$$\left.\begin{aligned} \dot{x} &= \bar{f}(x) + \bar{g}(x)v \\ y &= h(x) \end{aligned}\right\}, \tag{4.8}$$

where $\bar{f}(x) = f(x) + g(x)\alpha(x)$ and $\bar{g}(x) = g(x)\beta(x)$, satisfies

(i) rank of (2.1) = rank of (4.8),

(ii) the output nulling dynamics of (4.8) can be taken as the drift term of the output nulling dynamics of (2.1) restricted to $\mathcal{O}$,

(iii) if (2.1) is minimum phase at $x_0$, so in particular $x_0$ is an equilibrium point of (2.1), then (4.8) is also minimum phase at $x_0$.

*Proof.* See the Appendix.

## 5. A Comparison of Regularity Conditions

The goal of this section is to compare the notion of regularity proposed in Sec. 2 with the regularity conditions introduced in previous works on system inversion (Singh, 1981; Isidori and Moog, 1988), output nulling (Byrnes and Isidori, 1988 a; b; Isidori, 1989 a), and input-output dynamic decoupling (Descusse and Moog, 1987; Di Benedetto et al., 1989; Xia, 1989). We will see that the conditions associated with dynamic extension, as well as those associated with output nulling, are weaker than the strong regularity of $y \equiv 0$. However, in the case of square systems, the two conditions taken together precisely correspond to strong regularity of $y \equiv 0$ and the latter coincides with the requirement for constructing a reduced-order left-inverse by Singh (1981).

**5.1 System inversion** Isidori and Moog (1988) show that, in general, the notion of the zeros of a linear system may be extended to nonlinear systems in three distinct ways: (1) the dynamics associated with $y \equiv 0$ (i.e., the zero or output nulling dynamics), (2) the dynamics of a left-inverse system, and (3) the dynamics associated to a maximal loss of observability. Under the hypotheses that the system is square and invertible and the output $y \equiv 0$ is strongly regular, the first two notions always coincide (Isidori and Moog, 1988). To construct a "full-order" inverse, let $u = \gamma(x, \tilde{y}_j^{(i)} \mid 1 \le i \le n, i \le j \le n)$ be the solution of the system of equations

$$\begin{aligned}\tilde{y}_k^{(k)} &= \tilde{a}_k(x, \tilde{y}_j^{(i)} \mid 1 \le i \le k, i \le j \le k-1) \\ &\quad + \tilde{b}_k(x, \tilde{y}_j^{(i)} \mid 1 \le i \le k-1, i \le j \le k-1)u\end{aligned}$$

for $1 \le k \le n$. Compare this to (4.1). The matrix multiplying $u$ will have full rank for $\tilde{y}_j^{(i)}$ sufficiently small if and only if (2.3) of the definition of strong regularity holds. Define the vector field $f_{INV} = f + g\gamma$. Then, by the standard existence and uniqueness theorems,

$$\left.\begin{aligned}\dot{\zeta} &= f_{INV}(\zeta, \tilde{y}_j^{(i)} \mid 1 \le i \le n, i \le j \le n) \\ u &= \gamma(\zeta, \tilde{y}_j^{(i)} \mid 1 \le i \le n, i \le i \le n)\end{aligned}\right\} \tag{5.1}$$

with $\zeta(0) = x(0)$, is a local left-inverse of (2.1). Still following Isidori and Moog (1988), a reduced-order inverse is obtained by solving the set of equations

$$\tilde{y}_k^{(k)} = \varphi_k(x, \tilde{y}_j^{(i)} \mid 1 \le i \le k, i \le j \le k), \quad 1 \le k \le n,$$

for $\delta = n - \dim(N)$ components of $x$ and substituting these into (5.1). A smooth solution will exist whenever (2.2) of the definition of strong regularity holds. The reduced-order inverse then has $\dim(N)$ state variables. Whenever $\zeta \in \pi(F^{-1}(0))$, $f_{INV}(\zeta, 0) \in T_\zeta N$; therefore, whenever $x(0) \in \pi(F^{-1}(0))$, the restriction of (5.1) to $N$ is isomorphic to the zero dynamics of the system. In summary, the regularity conditions of a reduced-order left-inverse by Singh (1981) are the following:

a) $$\operatorname{rank}_R F_k(b) = \dim_K \mathscr{F}_k, \quad 0 \le k \le n,$$

b) $\text{rank}_{\mathcal{R}} E_k(b) = \dim_K \mathcal{E}_k, \quad 1 \le k \le n,$

where $b \in F^{-1}(0) \cap \pi^{-1}(\mathcal{O})$ and $\mathcal{O}$ is an open neighborhood of $x_0$.

**5.2 Zero-dynamics algorithm** Consider now the zero-dynamics algorithm of Byrnes and Isidori (1988 b), Isidori and Moog (1988) and Isidori (1989 a). For the purpose of comparing the regularity conditions used in the above cited works with those given in Definition 2.1, we rewrite the zero-dynamics algorithm using the formalism of Sec. 2.

Let $x_0$ be a given point of $X$ such that $f(x_0)=0$ and $h(x_0)=0$. Set $M_0=\{x \in X \mid h(x)=0\}$. Assume that there exists an open neighborhood $U_0'$ of $x_0$ such that $\partial h/\partial x$ has constant rank on $M_0 \cap U_0'$. Let $M_0^c$ be the connected component of $M_0 \cap U_0'$ which contains the point $x_0$.

*Step* 1: Calculate $y^{(1)}=c_1(x)+d_1(x)u$ and define $D_1(x):=d_1(x)$. Suppose there exists an open neighborhood $U_0$ of $x_0$, contained in $U_0'$, such that $D_1(x)$ has constant rank, denoted $r_1$, on $M_0^c \cap U_0$. Hence, there exists a permutation of the outputs, and a partition, such that upon writing $y=\text{col}(\tilde{y}_1, \hat{y}_1)$, where $\tilde{y}_1$ has $r_1$ components and $\hat{y}_1$ has $\mu - r_1$ components, then

$$y^{(1)} = \begin{bmatrix} \tilde{y}_1^{(1)} \\ \hat{y}_1^{(1)} \end{bmatrix} = \begin{bmatrix} \tilde{c}_1(x)+\tilde{d}_1(x)u \\ \hat{c}_1(x)+\hat{d}_1(x)u \end{bmatrix}$$

satisfies $\text{rank}_{\mathcal{R}} \tilde{d}_1(x_0)=r_1$. Since $\text{rank}_{\mathcal{R}} D_1(x_0)=r_1$, there exists an analytic matrix $R_1(x)$, defined on $M_0^c \cap U_0$, such that $\hat{d}_1(x)=R_1(x)\tilde{d}_1(x)$ on $M_0^c \cap U_0$. Therefore, $\hat{y}_1^{(1)}$ can be expressed as

$$\begin{aligned} \hat{y}_1^{(1)} &= \hat{c}_1(x) + R_1(x)(\tilde{y}_1^{(1)} - \tilde{c}_1(x)) \\ &=: \theta_1(x, \tilde{y}_1^{(1)}), \end{aligned}$$

that is $\hat{y}_1^{(1)}(x, u)=\theta_1(x, \tilde{y}_1^{(1)}(x, u))$. It follows that

$$\begin{aligned} \hat{y}_1^{(2)}(x, u, \dot{u}) &= \theta_1^{(1)}(x, \tilde{y}_1^{(1)}, \tilde{y}_1^{(2)}, u) \\ &= L_f\theta_1 + \frac{\partial \theta_1}{\partial \tilde{y}_1^{(1)}} \tilde{y}_1^{(2)} + L_g\theta_1 u. \end{aligned} \tag{5.2}$$

Note that as $M_0^c \cap U_0$ is an embedded submanifold of $X$, after possibly shrinking $U_0$, $R_1$ can be extended to an analytic function on all of $U_0$.

Define $M_1=\{x \in M_0^c \cap U_0 \mid \theta_1(x, 0)=0\}$. Assume that there exists an open neighborhood $U_1'$ of $x_0$ such that $\begin{bmatrix} (\partial h/\partial x)(x) \\ (\partial \theta_1(x, 0))/\partial x \end{bmatrix}$ has constant rank on $M_1 \cap U_1'$; let $M_1^c$ be the connected component of $M_1 \cap U_1'$ containing $x_0$.

*Step* 2: Differentiate $\hat{y}_1^{(1)}$. From (5.2), this can be written as

$$\hat{y}_1^{(2)}(x, u, \dot{u}) = c_2(x, \tilde{y}_1^{(1)}, \tilde{y}_1^{(2)}) + d_2(x, \tilde{y}_1^{(1)})u.$$

Define $D_2(x, \tilde{y}_1^{(1)})=\text{col}(\tilde{d}_1(x), d_2(x, \tilde{y}_1^{(1)}))$. Suppose there exists an open neighborhood $U_1$ of $x_0$, contained in $U_1'$, such that $D_2(x, 0)$ has constant rank, denoted $r_2$, on $M_1^c \cap U_1$. For $b \in F_1^{-1}(0) \cap \pi^{-1}(M_1^c \cap U_1)$,

$$\operatorname{rank}_{\mathbb{R}} E_2(b) = \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} dx \\ d\dot{y} \\ d\ddot{y} \end{bmatrix}\Bigg|_b$$

$$= \operatorname{rank}_{\mathbb{R}} \left[\begin{array}{ccc} I & 0 & 0 \\ \hline * & d_1(x_0) & 0 \\ \hline * & * & \tilde{d}_1(x_0) \\ * & d_2(x_0, 0) & 0 \end{array}\right]$$

$$= n + \operatorname{rank}_{\mathbb{R}} \tilde{d}_1(x_0) + \operatorname{rank}_{\mathbb{R}} D_2(x_0, 0) = n + r_1 + r_2.$$

Therefore there exists a permutation of the components of $\hat{y}_1$, and a subsequent partition, such that $\hat{y}_1 = \operatorname{col}(\tilde{y}_2, \hat{y}_2)$, where $\tilde{y}_2$ has $r_2 - r_1$ components, and, upon writing

$$c_2 + d_2 u = \begin{bmatrix} \tilde{c}_2 + \tilde{d}_2 u \\ \hat{c}_2 + \hat{d}_2 u \end{bmatrix},$$

then $\operatorname{rank}_{\mathbb{R}} \operatorname{col}(\tilde{d}_1(x_0), \tilde{d}_2(x_0, 0)) = r_2$. Therefore, there exists an analytic matrix $R_2(x, \tilde{y}_1^{(1)})$, defined on $(M_1^c \cap U_1) \times V_1$, where $V_1$ is a sufficiently small open neighborhood of the origin in $\mathbb{R}^{r_1}$, such that $\hat{d}_2(x, \tilde{y}_1^{(1)}) = R_2(x, \tilde{y}_1^{(1)}) \cdot \operatorname{col}(\tilde{d}_1(x), \tilde{d}_2(x, \tilde{y}_1^{(1)}))$. It follows that $\hat{y}_2^{(2)}$ can be expressed as

$$\hat{y}_2^{(2)} = \hat{c}_2(x) + R_2(x, \tilde{y}_1^{(1)}) \begin{bmatrix} \tilde{y}_1^{(1)} - \tilde{c}_1(x) \\ \tilde{y}_2^{(2)} - \tilde{c}_2(x, \tilde{y}_1^{(1)}, \tilde{y}_1^{(2)}) \end{bmatrix}$$
$$=: \theta_2(x, \tilde{y}_1^{(1)}, \tilde{y}_1^{(2)}, \tilde{y}_2^{(2)}),$$

etc.

The regularity conditions of Byrnes and Isidori (1989 b) for the zero-dynamics algorithm can therefore be restated as, for $1 \le k \le n$,

a) $$\operatorname{rank}_{\mathbb{R}} D_k(x, 0, \cdots, 0) \text{ constant on } M_{k-1}^c \cap U_{k-1}, \tag{5.3}$$

b) $$\operatorname{rank}_{\mathbb{R}} \begin{bmatrix} \frac{\partial h}{\partial x}(x) \\ \frac{\partial \theta_1}{\partial x}(x, 0) \\ \vdots \\ \frac{\partial \theta_k}{\partial x}(x, 0, \cdots, 0) \end{bmatrix} \text{ constant on } M_k^c \cap U_k, \tag{5.4}$$

c) $$\operatorname{rank}_{\mathbb{R}} D_n(x_0, 0, \cdots, 0) = m \tag{5.5}$$

for a sequence of nested open neighborhoods $U_0 \supset \cdots \supset U_n$ of $x_0$; superscript $c$ denotes the connected component containing $x_0$.

By construction,

$$M_k^c \cap U_k = \pi(F_k^{-1}(0))^c \cap U_k.$$

Then, proceeding as in the proof of Theorem 3.2, conditions (5.3), (5.4) and (5.5) are equivalent to

a′) $$\text{rank}_{\mathbf{R}} E_k \text{ is constant on } F_{k-1}^{-1}(0) \cap \pi^{-1}(U_{k-1}), \tag{5.6}$$

b′) $$\text{rank}_{\mathbf{R}} F_k \text{ is constant on } F_k^{-1}(0) \cap \pi^{-1}(U_k), \tag{5.7}$$

c′) $$\text{rank}_{\mathbf{R}} E_n(b) - \text{rank}_{\mathbf{R}} E_{n-1}(b) = m, \; b \in F_{n-1}^{-1}(0) \cap \pi^{-1}(U_{n-1}), \tag{5.8}$$

for a sequence of nested open neighborhoods $U_0 \supset \cdots \supset U_n$ of $x_0$. Moreover, in Di Benedetto and Grizzle (1990), it is shown that for square systems, a′) and c′) imply

b″) $$\text{rank}_{\mathbf{R}} F_k = (k+1)\mu, \tag{5.9}$$

on an open neighborhood $\mathcal{O}$ of $X \times T^{n-1} U$, $x_0 \in \pi(\mathcal{O})$.

As a consequence of the above, we deduce that condition (5.3) is weaker than (2.3), and condition (5.9) is equivalent to (2.2). Moreover, condition (5.5) is equivalent to left-invertibility under the regularity conditions (5.3) and (5.4). Indeed, (5.5) is equivalent to (5.8) which, in turn, implies that the linearization of (2.1) at the equilibrium $x_0$ is left-invertible. Since the rank of the linearized system is always less than or equal to the rank of the original nonlinear system, the assertion follows. This latter point settles a question posed to the authors by Isidori (1989 b).

**5.3 Input-output dynamic decoupling algorithm** Let us first consider the dynamic extension algorithm of Sec. 3. At each step of the algorithm, a static state variable feedback is constructed to maximally decouple the outputs. In order that it be smooth, the regularity condition,

$$\text{rank}_{\mathbf{R}} b_k(x_0, 0, \cdots, 0) = \text{rank}_K b_k(x, \bar{v}_1, \cdots, \bar{v}_{k-1}),$$

is imposed. This is equivalent, by Lemma 3.3, to

$$\text{rank}_{\mathbf{R}} E_k(b) = \dim_K \mathscr{E}_k, \quad 1 \le k \le n,$$

for $b \in F^{-1}(0) \cap \pi^{-1}(\mathcal{O})$, $\mathcal{O}$ an open neighborhood of $x_0$.

Until now, all of the regularity conditions presented have involved evaluating, in a sequential manner, conditions on the differentials of the output and its derivatives; moreover, each output component was differentiated up to the same order. This simple pattern no longer holds for the algorithms of Descusse and Moog (1987) and Nijmeijer and Respondek (1986; 1988), where, when evaluating the constant rank conditions, (i) each component of the output may be differentiated a different number of times and (ii) certain lower order derivatives may be excluded. This situation is analyzed in Di Benedetto and Grizzle (1990).


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## Appendix

### Proof of Lemma 4.1.

*Part* (a): Fix $k$. Let $P_k = \{x \in X \mid h(x)=0,\ \varphi_1(x,0)=0,\cdots,\varphi_k(x,0,\cdots,0)=0\}$. The goal is to produce an open neighborhood $\mathcal{O}$ of $x_0$ such that $\pi(F_k^{-1}(0)) \cap \mathcal{O} = P_k \cap \mathcal{O}$.

Since $x_0 \in \pi(F^{-1}(0))$, $(x_0,\ y \equiv 0)$ is a strongly regular pair. Therefore part (i) of Theorem 3.2 and the implicit function theorem establish the existence of an open neighborhood $\mathcal{O}$ of $x_0$ and a $C^\omega$-function $u = \alpha(x)$, defined on $\mathcal{O}$, solving

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{a}_1(x)+\tilde{b}_1(x)u \\ \vdots \\ \tilde{a}_k(x, 0, \cdots, 0)+\tilde{b}_k(x, 0, \cdots, 0)u \end{bmatrix}. \tag{A.1}$$

Define $f^*(x)=f(x)+g(x)\alpha(x)$ on $\mathcal{O}$ and set, for $i\geq 0$,

$$u^{(i)}(x) = L^i_{f^*}\alpha(x). \tag{A.2}$$

From (A.1), $\tilde{y}_i^{(i)}(x,u(x),\cdots,u^{(i-1)}(x))=0$, for $1\leq i\leq k$ and $x\in\mathcal{O}$. This yields $\tilde{y}_i^{(j)}(x,u(x),\cdots,u^{(j-1)}(x))=0$, $1\leq i\leq k$, $i\leq j$ because, for $i\leq j\tilde{y}_i^{(j)}=L_{f^*}^{(j-i)}\tilde{y}_i^{(i)}\equiv 0$ on $\mathcal{O}$. Finally, for all $x\in\mathcal{O}$, $1\leq i\leq k$, $\hat{y}_i^{(i)}(x,u(x),\cdots,u^{(i-1)}(x))=\varphi_i(x,0,\cdots,0)$. Therefore, if $x\in P_k\cap\mathcal{O}$, then $x\in\pi(F_k^{-1}(0))\cap\mathcal{O}$.

Now suppose that $x\in\pi(F_k^{-1}(0))$. Then $h(x)=0$ and there exist $(u,\cdots,u^{(k-1)})$ such that $y^{(i)}(x,u,\cdots,u^{(i-1)})=0$, $1\leq i\leq k$. Therefore, $\tilde{y}_i^{(j)}=0$, $1\leq i\leq k$, $i\leq j\leq k$ and then $0=\hat{y}_i^{(i)}(x,u,\cdots,u^{(i-1)})=\varphi_i(x,0,\cdots,0)$. In other words, $\pi(F_k^{-1}(0))\subset P_k$ and therefore, $\pi(F_k^{-1}(0))\cap\mathcal{O}\subset P_k\cap\mathcal{O}$, completing a double inclusion.

*Part* (b): The proof is almost identical to Part (a) and is therefore omitted.

*Part* (c): We will show that if $x_0\in N_n$, then $x_0\in N_{n+1}$. Fix $x_0\in N_n$. We must produce $u(x_0),\cdots,u^{(n)}(x_0)$ such that $F_{n+1}(x_0,u(x_0),\cdots,u^{(n)}(x_0))=0$; that is, $y^{(i)}(x_0,u(x_0),\cdots,u^{(i-1)}(x_0))=0$, $1\leq i\leq n+1$. Let $k=n$ in the proof of Part (a) and let $u^{(i)}(x)$ be defined on an open neighborhood $\mathcal{O}$ of $x_0$ as in (A.2). Then, $\tilde{y}_i^{(j)}(x,u(x),\cdots,u^{(j-1)}(x))=0$, for $x\in\mathcal{O}$, $1\leq i\leq n$, $1\leq j\leq n+1$ and $\hat{y}_i^{(i)}(x,u(x),\cdots,u^{(i-1)}(x))=0$, for $x\in\mathcal{O}\cap N_n$, $1\leq i\leq n$. It remains only to show that

$$\hat{y}_n^{(n+1)}(x_0,\ u(x_0),\ \cdots,\ u^{(n)}(x_0)) = 0. \tag{A.3}$$

Note that, by definition,

$$\hat{y}_n^{(n+1)}(x,\ u(x),\ \cdots,\ u^{(n)}(x)) = \langle d\hat{y}_n^{(n)}(x,\ u(x),\ \cdots,\ u^{(n-1)}(x)),\ f^*(x)\rangle, \tag{A.4}$$

and for each $1\leq k\leq n$ and $x\in\mathcal{O}$,

$$\hat{y}_k^{(k)}(x,\ u(x),\ \cdots,\ u^{(k-1)}(x)) = \varphi_k(x,\ 0,\ \cdots,\ 0). \tag{A.5}$$

To simplify the notation, let $\bar{\varphi}_k(x)=\varphi_k(x,0,\cdots,0)$ and $\bar{\varphi}_0(x)=h(x)$. By the construction of the inversion algorithm, for $1\leq i\leq n$, $\bar{\varphi}_i(x)=\mathrm{col}[L^i_{f^*}h_{\varrho_i-n+1}(x),\cdots,L^i_{f^*}h_n(x)]$. By part (ii) of Theorem 3.2, the codistributions $W_0\subset\cdots\subset W_n$ defined by

$$W_i(x) = \mathrm{span}_R\{d\bar{\varphi}_0(x),\ \cdots,\ d\bar{\varphi}_i(x)\} \tag{A.6}$$

are constant dimensional on $\mathcal{O}$. Therefore, since $T^*\mathcal{O}$ is $n$ dimensional, there exists $k\leq n-1$ such that $W_i=W_k$ for all $i\geq k$. In particular, each row of $d\bar{\varphi}_n$ is contained in $W_{n-1}$. We will now show that for $x\in N_n\cap\mathcal{O}$, $f^*(x)\in W_{n-1}^{\perp}(x)$, thereby establishing (A.3). If $x\in N_n\cap\mathcal{O}$, then $\bar{\varphi}_k(x)=0$ for $0\leq k\leq n$. But this yields the result because, for $0\leq k\leq n-1$, and $x\in N_n\cap\mathcal{O}$,

$$L_{f^*}\bar{\varphi}_k(x) = \begin{bmatrix} \tilde{y}_{k+1}^{(k+1)}(x,\ u(x),\ \cdots,\ u^{(k)}(x)) \\ \hat{y}_k^{(k+1)}(x,\ u(x),\ \cdots,\ u^{(k)}(x)) \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ \bar{\varphi}_{k+1}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This completes the proof that $x_0 \in N_n$ implies $x_0 \in N_{n+1}$. Since the reverse inclusion is immediate, we conclude that $N_n = N_{n+1}$. Repeating the above steps, one completes the proof of the theorem.

**Proof of Theorem 4.6.** We use the dynamic compensator constructed by the dynamic extension algorithm. The integer $q$ then equals $\dim \bar{v}_1 + \cdots + \dim \bar{v}_{n-1}$. The decoupling matrix of the closed-loop system is equal to $b_n(x, \bar{v}_1, \cdots, \bar{v}_{n-1})$. By (i) of Theorem 3.4, there exists an open neighborhood $\mathcal{O}_1$ of $x_0$ and $\mathcal{O}_2$ of the origin of $\mathbb{R}^q$ such that at each point of $\mathcal{O}_1 \times \mathcal{O}_2$, the rank of $b_n$ equals the number of outputs, $\mu$. (Recall that right-invertibility implies that $\operatorname{rank}_{\mathbb{R}} b_n(x_0, 0, \cdots, 0) = \varrho^* = \mu$). Hence, we can assume without loss of generality that the dynamically extended system is already decoupled. Thus (a) is established, and Proposition 2.2 then yields (b). It follows from Lemma 4.1 (Part (b)) that the output nulling manifold $N$ of (2.1) can be locally represented, in terms of the functions computed in the dynamic extension algorithm, as

$$N \cap \mathcal{O}_1 = \{x \in \mathcal{O}_1 \,|\, h(x) = 0,\ \psi_1(x,\ 0) = 0,\ \cdots,\ \psi_n(x,\ 0,\ \cdots,\ 0) = 0\},$$

after possibly shrinking $\mathcal{O}_1$; moreover, $\psi_n$ is the empty vector because (2.1) is right-invertible. Applying the dynamic extension algorithm to the closed-loop system consisting of (2.1) and (4.5) yields that its output nulling manifold, $N_{cl}$, equals

$$\{(x,\ \bar{v}_1,\ \cdots,\ \bar{v}_{n-1}) \in \mathcal{O}_1 \times \mathcal{O}_2 \,|\, h(x) = 0,\ \bar{y}_1^{(1)} = 0,$$
$$\psi_1(x,\ \bar{v}_1) = 0,\ \cdots,\ \bar{y}_{n-1}^{(n-1)} = 0,\ \psi_{n-1}(x,\ \bar{v}_1,\ \cdots,\ \bar{v}_{n-1}) = 0\},$$

that is,

$$N_{cl} = \{(x,\ 0) \in \mathcal{O}_1 \times \mathcal{O}_2 \,|\, h(x) = 0,\ \psi_1(x,\ 0) = 0,\ \cdots,\ \psi_{n-1}(x,\ 0,\ \cdots,\ 0) = 0\}.$$

Therefore $N \cap \mathcal{O}_1$ and $N_{cl}$ are diffeomorphic. Showing that $\Delta$ is isomorphic to $\Delta_{cl}$ is then straightforward, using Remark 4.3 for example. The "normal form" (4.6) is standard and can be found in many references (see Isidori, 1989 a; Nijmeijer and Respondek, 1988, for example). That (4.7) represents the output nulling dynamics is easily seen by setting the output $y$ identically equal to zero and calculating $\pi(F^{-1}(0))$.

**Proof of Theorem 4.8.** Let $\tilde{b}_1, \cdots, \tilde{b}_n$ be as calculated by the inversion algorithm. Without loss of generality, it can be assumed that there exists an open neighborhood $\mathcal{O}$ of $x_0$ such that

$$\begin{bmatrix} \tilde{b}_1(x) \\ \vdots \\ \tilde{b}_n(x,\ 0,\ \cdots,\ 0) \end{bmatrix} \begin{bmatrix} 0_2 \\ \cdots \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{A.7}$$

where $I_2$ is the $(m-\mu)\times(m-\mu)$ identity matrix and $O_2$ is a $\mu\times(m-\mu)$ matrix of zeros. Indeed, if (A.7) is not satisfied, part (i) of Theorem 3.2 and the Implicit Function Theorem guarantee the existence of an open neighborhood $\mathcal{O}$ of $x_0$ and an $m\times(m-\mu)$ matrix of analytic function, $\beta_2(x)$, such that

$$\begin{bmatrix} \tilde{b}_1(x, 0) \\ \vdots \\ \tilde{b}_n(x, 0, \cdots, 0) \end{bmatrix} \beta_2(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{A.8}$$

and rank $\beta_2(x_0)=m-\mu$. Then choose an $m\times\mu$ matrix of analytic functions $\beta_1(x)$ such that $\beta(x)=[\beta_1(x)\vdots\beta_2(x)]$ has rank $m$; this is always possible, after shrinking $\mathcal{O}$, if necessary. An easy calculation gives, for the closed loop system corresponding to $u=\beta(x)v$, that (A.7) is then satisfied. Hence, (A.7) is assumed to hold. Let $\alpha(x)$ be as in the proof of Lemma 4.1, part (a), for $k=n$. Then the output nulling dynamics of (2.1) restricted to $\mathcal{O}$ is

$$\Delta(x) = \{f^*(x) + \sum_{i=\mu+1}^{m} g_i(x)u_i \mid x \in N \cap \mathcal{O}\}, \tag{A.9}$$

where $f^*(x)=f(x)+g(x)\alpha(x)$. Consider the singular feedback,

$$u = \alpha(x) + \begin{bmatrix} I_1 \\ \cdots \\ 0_1 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_\mu \end{bmatrix}, \tag{A.10}$$

where $I_1$ is the $\mu\times\mu$ identity matrix and $\mathcal{O}_1$ is an $(m-\mu)\times\mu$ matrix of zeros, resulting in the closed loop system,

$$\left.\begin{aligned} \dot{x} &= f^*(x) + \sum_{i=1}^{\mu} g_i(x)v_i \\ y &= h(x) \end{aligned}\right\}, \tag{A.11}$$

which is square, and by (A.7), it is right-invertible and strongly regular; indeed, (A.7) implies that the controls $u_{\mu+1},\cdots,u_m$ do not contribute to the rank calculations in the inversion algorithm. Moreover, its output nulling dynamics is given by

$$\Delta_{cl}(x) = \{f^*(x) \mid x \in N \cap \mathcal{O}\}, \tag{A.12}$$

which is the output nulling dynamics of (2.1) with $u_{\mu+1}=\cdots=u_m=0$. It is now claimed that whenever (2.1) is minimum phase at $x_0$, $f^*$ can be assumed to be asymptotically stable. This is easy, because the minimum phase property assures, after possibly shrinking $\mathcal{O}$, the existence of an analytic feedback $\delta(x)$, defined on $N\cap\mathcal{O}$, such that $x_0$ is an asymptotically stable equilibrium point of $f^*(x)+[g_{\mu+1}(x),\cdots,g_{m(x)}]\delta(x)$. Since $N$ is an embedded submanifold of $X$, after possibly shrinking $\mathcal{O}$ again, we can extend $\delta$ to an analytic function on $\mathcal{O}$. Therefore, by replacing $\alpha(x)$ by $\alpha(x)+\delta(x)$, we arrive at an asymptotically stable $f^*(x)$.

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