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INT. J. CONTROL, 1986, VOL. 43, NO. 5, 1517–1530

Local input–output decoupling of discrete-time non-linear systems

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A local treatment of the block input–output decoupling problem is given for discrete-time non-linear control systems. The major tools employed are the invariant and locally-controlled invariant distributions that have recently been extended to the discrete-time domain. A sufficient condition for the solvability of the problem with local stability about an equilibrium point is also given.

1. Introduction

The problem of modifying a system's behaviour via feedback so that certain of the inputs only interact with specified components of the outputs is classical in control theory. A vast literature exists on this problem for the class of linear systems, where only Wonham (1979) and Morse and Wonham (1971) are cited as examples. More recently, the class of non-linear continuous-time systems has also received a lot of attention, being investigated through a variety of techniques (Claude, Isidori *et al.* 1981, Nijmeijer and Schumacher 1983, Sinha 1977, van der Schaft 1985). However, considerably less is known about this problem for the class of discrete-time non-linear systems.

The goal of this paper is to give a local treatment of the restricted block input–output decoupling problem for the class of analytic non-linear discrete-time systems. The major tools employed will be the invariant and locally-controlled invariant distributions studied by Grizzle (1985 a, b), where they were used to solve the disturbance decoupling problem locally.

A special case of this problem has been solved by Monaco and Normand-Cyrot (1984 a), where, given some non-singularity hypotheses, necessary and sufficient conditions are given for an affine system

$$\mathbf{x}_{k+1} = \mathbf{f}(x_k) + \sum_{i=1}^{m} \mathbf{u}_i \mathbf{g}_i(\mathbf{x}_k)$$

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k)$$

to be feedback equivalent to a parallel cascade of single-input/single-output linear systems plus an unobservable non-linear part. It should be noted that although this class of system is rather restricted, the result obtained is explicit and, modulo certain singularities, also *global*. On the other hand, a much more general class of system will be studied here, but only *local* results will be obtained. A worthwhile goal would be to combine the approach advocated by Monaco and Normand-Cyrot (1984 a) with that given here in the hope of obtaining some 'intermediate' results. Perhaps an indication of how this might be done is given by Monaco and Normand-Cyrot (1984 b), where invariant distributions are treated from an algebraic point of view.

Finally, the problem of finding a feedback law that simultaneously decouples a

Received 23 May 1985.

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system and locally stabilizes it about an equilibrium point is studied. The usual linearization method (i.e. Lyapunov's first method) is combined with locally-controlled invariant distributions to obtain a sufficient condition for decoupling with stability.

2. Definitions and preliminaries

This section fixes the notation and setting employed in this study of discrete-time non-linear systems and summarizes some results owing to Grizzle (1985 a) on controlled invariant distributions.

Definition 2.1

A non-linear discrete-time system is a five-tuple $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}, \mathbf{h}, \mathbf{N})$, where $\pi: \mathbf{B} \to \mathbf{M}$ is an analytic fibre bundle, $\mathbf{N}$ is an analytic manifold and $\mathbf{f}: \mathbf{B} \to \mathbf{M}$ and $\mathbf{h}: \mathbf{M} \to \mathbf{N}$ are analytic mappings. The points of $\mathbf{M}$ are the states of the system, the fibres of $\mathbf{B}$ are the (possibly state-dependent) input spaces, and the outputs are valued in $\mathbf{N}$. The system's dynamics are defined by $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k)$, $\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k)$, for $\mathbf{u}_k \in \pi^{-1}(\mathbf{x}_k)$. Finally, it is supposed that both $\pi: \mathbf{B} \to \mathbf{M}$ and $\mathbf{N}$ are connected.

Definition 2.2

A *feedback* function γ is a bundle isomorphism from $\mathbf{B}$ to $\mathbf{B}$; i.e. γ is a diffeomorphism such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\gamma} & \mathbf{B} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbf{M} & \end{array} \tag{2.1}$$

In local trivializing coordinates $(\mathbf{x}, \mathbf{u})$ for $\mathbf{B}$, one has $\gamma(\mathbf{u}, \mathbf{x}) = (\mathbf{x}, \gamma_{\mathbf{x}}(\mathbf{u}))$.

Since γ is non-singular, feedback can (and will) be viewed simply as a state-dependent change of the input coordinates.

Definition 2.3

Let Δ be an involutive analytic distribution on $\mathbf{M}$. Then Δ is an *invariant distribution* of $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f})$, with respect to a given open cover of local trivializations $(\mathbf{x}, \mathbf{u})_\alpha$ of $\mathbf{B}$, if

$$\mathbf{f}(\cdot, \mathbf{u})_* \Delta \subset \Delta \tag{2.2}$$

for each local coordinate chart (of the given open cover). Δ is *locally controlled invariant* if for each $\mathbf{b}_0 \in \mathbf{B}$ there exists locally a feedback γ (i.e. γ is defined on some open set about $\mathbf{b}_0$) such that Δ is an invariant distribution of the closed-loop system $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f} \circ \gamma)$.

It turns out that locally-controlled invariant distributions can be quite easily characterized. In the following, $\mathbf{V}(\mathbf{B}) = \{\mathbf{X} \in \mathbf{TB} \mid \pi_* \mathbf{X} = 0\}$ denotes the vertical distribution on $\mathbf{B}$.

Theorem 2.1

If Δ is an analytic involutive locally-controlled invariant distribution on $\mathbf{M}$, then for each vector field $\mathbf{X} \in \pi_*^{-1}(\Delta)$ and $\mathbf{b} \in \mathbf{B}$,

$$\mathbf{f}_* \mathbf{X}_b \subset \Delta(\mathbf{f}(\mathbf{b})) + \mathbf{f}_*(\mathbf{b})\mathbf{V}(\mathbf{B}) \tag{2.3}$$

Moreover, if Δ, $\mathbf{f}_*^{-1}(\Delta) \cap \mathbf{V}(\mathbf{B})$ and $\mathbf{f}$ restricted to the fibres of $\mathbf{B}$ all have constant rank, then (2.3) is also sufficient for Δ to be locally-controlled invariant.

Remark 2.1:

Owing to the analyticity assumption, there will always exist open dense subsets $\mathbf{M}' \subset \mathbf{M}$ and $\mathbf{B}' \subset \mathbf{B}$, $\pi(\mathbf{B}') \supset \mathbf{M}'$, on which the aforementioned constant-rank hypotheses are satisfied.

Though it is not true that a maximal locally-controlled invariant distribution contained in a given distribution always exists, something very close to this does in fact hold.

Definition 2.4

An analytic distribution Δ is said to satisfy the local controlled invariance (LCI) condition if there exists an open dense subset $\mathbf{B}' \subset \mathbf{B}$ such that

$$\mathbf{f}_* \mathbf{X}_b \subset \Delta(\mathbf{f}(\mathbf{b})) + \mathbf{f}_*(\mathbf{b})\mathbf{V}(\mathbf{B})$$

for all $\mathbf{X} \in \pi_*^{-1}(\Delta)$ and $\mathbf{b} \in \mathbf{B}'$.

This leads to the following important result.

Theorem 2.2

Let $\mathbf{K}$ be an analytic involutive distribution on $\mathbf{M}$. Then $\mathbf{K}$ contains a maximal distribution satisfying the LCI condition; moreoever, it is necessarily involutive and, on the open dense subsets of $\mathbf{M}$ and $\mathbf{B}$ where the constant-rank hypotheses of Theorem 2.1 are satisfied, it is also locally-controlled invariant.

An algorithm for calculating the maximal LCI distribution contained in $\mathbf{K}$ is given by Grizzle (1985 a).

Doing local-coordinate calculations for discrete-time systems is more delicate than in the case of continuous-time systems, since one must usually work with a pair of coordinate charts: one about the domain of $\mathbf{f}$, and the other about its image. Let $(\mathbf{x}_0, \mathbf{u}_0) \in \mathbf{B}$, and consider $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) \in \mathbf{M}$. Choose a coordinate chart $(\phi_M, \mathbf{V}_M)$ about $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)$ and consider the open set $\mathbf{f}^{-1}(\mathbf{V}_M)$ about $(\mathbf{x}_0, \mathbf{u}_0)$. Choose a trivializing coordinate chart $(\phi_B, \mathbf{V}_B)$ about $(\mathbf{x}_0, \mathbf{u}_0)$ such that $\mathbf{V}_B \subset \mathbf{f}^{-1}(\mathbf{V}_M)$. $(\phi_B, \mathbf{V}_B)$, $(\phi_M, \mathbf{V}_M)$ will be called a *coordinate-chart pair*. Denote coordinates for $(\phi_B, \mathbf{V}_B)$ by $(\mathbf{x}, \mathbf{u})$, and for $(\phi_M, \mathbf{V}_M)$ by $\mathbf{x}$. (This abuse of notation is useful and permits one to perform local calculations as if one were working in a single coordinate chart.) The coordinate-chart pair will be denoted simply by $(\mathbf{x}, \mathbf{u})$. If Δ is an involutive distribution on $\mathbf{M}$ having constant dimension, then by the Frobenius theorem one can assume that

$$\Delta = \text{span}\left\{\frac{\partial}{\partial \mathbf{x}^1}, \ldots, \frac{\partial}{\partial \mathbf{x}^k}\right\}$$

in each chart $(\phi_B, \mathbf{V}_B)$, $(\phi_M, \mathbf{V}_M)$, and hence the notation

$$\Delta = \text{span}\left\{\frac{\partial}{\partial \mathbf{x}^1}, \ldots, \frac{\partial}{\partial \mathbf{x}^k}\right\}$$

is not ambiguous. Note that $\pi(\mathbf{V_B})$ and $\mathbf{V_M}$ may or may not intersect and may or may not coincide. However, if $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = \mathbf{x}_0$, then one can always choose $\mathbf{V_B}$ such that $\pi(\mathbf{V_B}) \subset \mathbf{V_M}$ and $\phi_B | \pi(\mathbf{V_B}) = \phi_M | \pi(\mathbf{V_B})$.

3. Restricted block decoupling problem

Let $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}, \mathbf{h}, \mathbf{N})$ be a discrete-time non-linear control system for which the outputs have been grouped into blocks; i.e. $\mathbf{h} = (\mathbf{h}^1, \ldots, \mathbf{h}^l)$, where $\mathbf{h}^i : \mathbf{M} \to \mathbf{N}^i$ and $\mathbf{N} = \mathbf{N}^1 \mathbf{x} \ldots \mathbf{x} \mathbf{N}^l$. The restricted block decoupling problem† (RBDP) is to find, if it exists, a non-singular feedback $\gamma(\mathbf{x}, \mathbf{u})$ and a partitioning of the inputs into $\mathbf{u} = (\mathbf{u}^1, \ldots, \mathbf{u}^l, \mathbf{u}^{l+1})$, each $\mathbf{u}^i$ possibly being a vector, such that $\mathbf{u}^i$ does not affect $\mathbf{y}^j = \mathbf{h}^j(\mathbf{x})$ for all $j \neq i$, $i = 1, \ldots, l$, and $\mathbf{u}^{l+1}$ does not affect the outputs at all (the possibility of $\mathbf{u}^{l+1}$ being zero-dimensional is not excluded). If one adds the condition that $\mathbf{u}^i$ 'controls' $\mathbf{y}^i$, then one has the restricted block non-interacting control problem. However, since non-singular state variable feedback cannot modify the accessibility nor the output accessibility properties of the system, only the first problem need be addressed.

Starting from the above purely input–output definition of 'decoupledness', a global state-space characterization of this property will be given in terms of invariant equivalence relations. This will lead to a natural localization of the problem in terms of invariant distributions to which the results of Grizzle (1985 a) can be applied to study the local solvability of the RBDP.

Proposition 3.1

Let $\Sigma(\mathbf{M}\mathbf{x}\mathbf{U}^1\mathbf{x} \ldots \mathbf{x}\mathbf{U}^{l+1}, \mathbf{f}, \mathbf{h}, \mathbf{N}^1\mathbf{x} \ldots \mathbf{x}\mathbf{N}^l)$ be a discrete-time non-linear control system. Then Σ is input–output decoupled with respect to the given partition of the inputs and outputs if and only if there exist l equivalence relations $\mathbf{R}^1, \ldots, \mathbf{R}^l$ on $\mathbf{M}$ such that whenever $\mathbf{x}\mathbf{R}^i\bar{\mathbf{x}}$,

$$\mathbf{h}^i(\mathbf{x}) = \mathbf{h}^i(\bar{\mathbf{x}}) \tag{3.1 a}$$

and

$$\mathbf{f}(\mathbf{x}, \mathbf{u}^1, \ldots, \mathbf{u}^{l+1})\mathbf{R}^i\mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}}^1, \ldots, \bar{\mathbf{u}}^{i-1}, \mathbf{u}^i, \bar{\mathbf{u}}^{i+1}, \ldots, \bar{\mathbf{u}}^{l+1}) \tag{3.1 b}$$

for all $\mathbf{u}^j, \bar{\mathbf{u}}^j \in \mathbf{U}^j$, $i, j = 1, \ldots, l$.

Proof

Assume that such equivalence relations exist; consider one of them, say $\mathbf{R}^i$, and let $\mathbf{M}/\mathbf{R}^i$ denote the set of equivalence classes associated with $\mathbf{R}^i$. Then (3.1) gives that Σ projects, in a set-theoretic sense, to a system on $\mathbf{M}/\mathbf{R}^i$:

$$\tilde{\mathbf{x}}_{k+1} = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}_k, \mathbf{u}_k^i)$$
$$\mathbf{y}_k^i = \tilde{\mathbf{h}}^i(\tilde{\mathbf{x}}_k)$$

from which it is clear that $\mathbf{y}^i$ does not depend on $\mathbf{u}^j$ for $j \neq i$. On the other hand, for fixed $\mathbf{x}$ and $\bar{\mathbf{x}}$, define $\mathbf{x}\mathbf{R}^i\bar{\mathbf{x}}$ if, for arbitrary input sequences $(\mathbf{u}_j)_{j=1}^{\infty}$ and $(\hat{\mathbf{u}}_j)_{j=1}^{\infty}$ such that $\mathbf{u}_j = (\mathbf{u}_j^1, \ldots, \mathbf{u}_j^{l+1})$ and $\hat{\mathbf{u}}_j = (\hat{\mathbf{u}}_j^1, \ldots, \hat{\mathbf{u}}_j^{i-1}, \mathbf{u}_j^i, \hat{\mathbf{u}}_j^{i+1}, \ldots, \hat{\mathbf{u}}_j^{l+1})$.

$$\mathbf{y}_k^i(\mathbf{x}; (\mathbf{u}_j)_{j=1}^{\infty}) = \mathbf{y}_k^i(\bar{\mathbf{x}}; (\hat{\mathbf{u}}_j)_{j=1}^{\infty})$$

† The problem is restricted in the sense that only non-singular state variable feedback is allowed; in particular, dynamic compensation is not permitted.

for all $k=0,1,\ldots$ where $\mathbf{y}_k^i(\mathbf{x};(\mathbf{u}_j)_{j=1}^{\infty})$ denotes the output of $\Sigma(\mathbf{M}\mathbf{x}\mathbf{U}^1\mathbf{x}\ldots\mathbf{x}\mathbf{U}^{l+1}, \mathbf{M}, \mathbf{f}, \mathbf{h}^i, \mathbf{N}^i)$ with initial condition $\mathbf{x}$ and input sequence $(\mathbf{u}_j)_{j=1}^{\infty}$. Then it is easily checked that $\mathbf{R}^i$ satisfies (3.1). □

Remark 3.1

For linear systems, one can show that the $\mathbf{R}^i$ constructed above corresponds to a linear subspace of the state space.

Hence, to solve the RBDP one must give conditions for the existence of a feedback function γ and a partitioning of the inputs such that the closed-loop system admits l equivalence relations satisfying (3.1). However, since such conditions would necessarily involve global computations, one is led to localizing the problem. The key to doing this is given by the following result, the proof of which has been relegated to the appendix.

Lemma 3.1

Let $\Sigma(\mathbf{M}\mathbf{x}\mathbf{U}^1\mathbf{x}\ldots\mathbf{x}\mathbf{U}^{l+1}, \mathbf{M}, \mathbf{f}, \mathbf{h}, \mathbf{N}^1\mathbf{x}\ldots\mathbf{x}\mathbf{N}^l)$ be a decoupled non-linear discrete-time control system and let $\mathbf{R}^1,\ldots,\mathbf{R}^l$ be the equivalence relations constructed in the proof of Proposition 3.1. Then for each $i=1,\ldots,l$ there exists an analytic involutive distribution Δ^i and an open dense subset $\mathbf{M}^i\subset\mathbf{M}$ on which the orbits of Δ^i and $\mathbf{R}^i$ locally coincide.

When Δ^i is constant dimensional and $\mathbf{M}^i=\mathbf{M}$, $\mathbf{R}^i$ will be said to be *regular*.

Definition 3.1

The restricted block decoupling problem is *regularly solvable* for $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}, \mathbf{h}, \mathbf{N}^1\mathbf{x}\ldots\mathbf{x}\mathbf{N}^l)$ if there exist l regular equivalence relations and a feedback function γ such that $\Sigma_\gamma := \Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}\circ\gamma, \mathbf{N}^1\mathbf{x}\ldots\mathbf{x}\mathbf{N}^l)$ satisfies (3.1). (For $\mathbf{B}$ non-trivial, (3.1 *b*) should be interpreted with respect to a given open cover of charts $(\mathbf{x},\mathbf{u})_\alpha$.) The problem is *locally regularly solvable* if γ exists at least locally.

Simply diffrentiating along the orbits of the equivalence relations $\mathbf{R}^1,\ldots,\mathbf{R}^l$ gives the following results.

Lemma 3.2

The restricted block decoupling problem is *locally regularly solvable* for $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}, \mathbf{h}, \mathbf{N}^1\mathbf{x}\ldots\mathbf{x}\mathbf{N}^l)$ if and only if there exist l involutive constant-dimensional distributions $\Delta^1,\ldots,\Delta^l$ on $\mathbf{M}$ such that for each $\mathbf{b}_0\in\mathbf{B}$ and sufficiently small local trivializations $(\mathbf{x},\mathbf{u})$ of $\mathbf{B}$ about $\mathbf{b}_0$ there exist a local feedback function γ and a partition $(\mathbf{u}^1,\ldots,\mathbf{u}^{l+1})$ of the inputs satisfying:

(*a*) $\mathbf{h}_*^i(\Delta^i)=0,\quad i=1,\ldots,l;$

(*b*) $\mathbf{f}\circ\gamma(\cdot,\mathbf{u})_*\Delta^i\subset\Delta^i,\quad i=1,\ldots,l;$

(*c*) $\mathbf{f}\circ\gamma_*\left(\operatorname{span}\left\{\frac{\partial}{\partial\mathbf{u}^j}\middle| j\neq i\right\}\right)\subset\Delta^i,\quad i=1,\ldots,l.$

Remark 3.2

In continuous time, the non-interacting control problem is usually formulated in terms of controllability subspaces (Wonham 1979) or controllability distributions

(Nijmeijer and Schumacher 1983). To the author's best knowledge, such distributions have not yet been introduced for discrete-time non-linear systems. Let

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \sum_{i=1}^{l+1} \mathbf{B}^i \mathbf{u}_k^i$$

$$\mathbf{y}_k^i = \mathbf{C}\mathbf{x}_k; \quad i = 1, \dots, l$$

be a linear discrete-time system on $\mathbb{R}^n$. Then Definition 3.1 demands the existence of subspaces $\mathbf{S}^1, \dots, \mathbf{S}^l$ of $\mathbb{R}^n$ and a feedback function $\mathbf{v} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}$, $|\mathbf{G}| \neq 0$, such that

(a') $\mathbf{S}^i \subset \ker \mathbf{C}^i$

(b') $(\mathbf{A} + \mathbf{B}\mathbf{F})\mathbf{S}^i \subset \mathbf{S}^i$

(c') $\mathbf{B}\mathbf{G}$ span $\{\mathbf{u}^1, \dots, \mathbf{u}^{i-1}, \mathbf{u}^{i+1}, \dots, \mathbf{u}^{l+1}\} \subset \mathbf{S}^i$.

Now (b') and (c') are the essential ingredients in the definition of a controllability subspace (one need only add that $\mathbf{S}^i$ is the smallest such subspace). One therefore sees that the formulation of the decoupling problem arrived at in Definition 3.1 is in fact analogous to those posed by Wonham (1979), Morse and Wonham (1971), Nijmeijer and Schumacher (1983) and van der Schaft (1985) for continuous-time systems. More importantly, (b) and (c) may lead to a good notion of a controllability distribution for discrete-time non-linear systems.

The main result characterizing the local solvability of the above problem is the following. (One should note that, owing to analyticity, the constant-rank hypotheses that will be made, hold on open dense subsets of $\mathbf{M}$ and $\mathbf{B}$.)

Theorem 3.1

Let $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}, \mathbf{h}, \mathbf{N}^1 \mathbf{x} \dots \mathbf{x}\mathbf{N}^l)$ be a discrete-time non-linear control system. If the restricted block decoupling problem is locally regularly solvable, then there exist l involutive constant-dimensional distributions $\Delta^1, \dots, \Delta^l$ on $\mathbf{M}$ satisfying

(a) $\mathbf{h}_*^i(\Delta^i) = 0$;

(b) for the family of distributions $\mathbf{E}^i := \mathbf{f}_*^{-1}(\Delta^i) \cap \pi_*^{-1}(\Delta^i)$,

(i) $\pi_* \mathbf{E}^i = \Delta^i$

(ii) $\bigcap_{i \in I} (\mathbf{E}^i \cap \mathbf{V}(\mathbf{B})) + \bigcap_{j \in J} (\mathbf{E}^j \cap \mathbf{V}(\mathbf{B})) = \mathbf{V}(\mathbf{B})$

for all non-empty disjoint subsets $I, J \subset \{1, \dots, l\}$

Moreover, if $\mathbf{f}$ restricted to the fibres of $\mathbf{B}$ and $\mathbf{E}^i \cap \mathbf{V}(\mathbf{B})$ all have constant rank, then these conditions are also sufficient.

Proof

Necessity. Suppose the RBDP is locally regularly solvable. Fix $\mathbf{b}_0 \in \mathbf{B}$ and let $(\mathbf{x}, \mathbf{u})$ be a sufficiently small coordinate-chart pair about $\mathbf{b}_0$ and let $\gamma(\mathbf{x}, \mathbf{u})$, $\mathbf{u} = (\mathbf{u}^1, \dots, \mathbf{u}^{l+1})$ and $\Delta^1, \dots, \Delta^l$ be as in Lemma 3.2. Since γ is always of the form $\gamma(\mathbf{x}, \mathbf{u}) = (\mathbf{x}, \gamma_\mathbf{x}(\mathbf{u}))$, (b) of Lemma 3.2 gives that $\pi_* \mathbf{E}^i = \Delta^i$. To establish (ii) of (b), first note that

$$\mathbf{V}(\mathbf{B}) = \text{span} \left\{ \gamma_* \frac{\partial}{\partial \mathbf{u}^1}, \dots, \gamma_* \frac{\partial}{\partial \mathbf{u}^{l+1}} \right\}$$

Now (c) of Lemma 3.2 gives that

$$\operatorname{span}\left\{\gamma_* \frac{\partial}{\partial \mathbf{u}^j}\middle| j \neq i\right\} \subset \mathbf{E}^i$$

so that

$$\mathbf{E}^i \cap \mathbf{V}(\mathbf{B}) \supset \operatorname{span}\left\{\gamma_* \frac{\partial}{\partial \mathbf{u}^j} | j \neq i\right\}$$

Therefore

$$\bigcap_{i \in I} \mathbf{E}^i \cap \mathbf{V}(\mathbf{B}) \supset \operatorname{span}\left\{\gamma_* \frac{\partial}{\partial \mathbf{u}^j}\middle| j \notin I\right\}$$

which establishes (ii) of (b) once one uses the disjointness of I and J.

Sufficiency. The key point is that by the proof of Theorem 5.1 of Isidori *et al.* (1981), Condition (ii) of (b) implies that the family of distributions $\{\mathbf{E}^i \cap \mathbf{V}(\mathbf{B})\}_{i=1}^l$ is simultaneously integrable (Jakubczyk and Respondek 1980). Hence one can choose coordinates $\mathbf{u} = (\mathbf{u}^1, \ldots, \mathbf{u}^{l+1})$ for the fibres of $\mathbf{B}$, each $\mathbf{u}^i$ possibly being a vector, such that

$$\mathbf{E}^i \cap \mathbf{V}(\mathbf{B}) = \operatorname{span}\left\{\frac{\partial}{\partial \mathbf{u}^1}, \ldots, \frac{\partial}{\partial \mathbf{u}^{i-1}}, \frac{\partial}{\partial \mathbf{u}^{i+1}}, \ldots, \frac{\partial}{\partial \mathbf{u}^{l+1}}\right\}$$

Now condition (i) of (b), $\pi_* \mathbf{E}^i = \Delta^i$, implies that Δ^i is locally-controlled invariant (Grizzle 1985 a, b). Moreover, as

$$\frac{\partial}{\partial \mathbf{u}^j} \in \mathbf{E}^i \cap \mathbf{V}(\mathbf{B})$$

for $j \neq i$, a local feedback ${}^i\gamma$ rendering Δ^i invariant can always be chosen to be of the form ${}^i\gamma(\mathbf{x}, \mathbf{u}) = (\mathbf{x}, \mathbf{u}^1, \ldots, \mathbf{u}^{i-1}, \gamma^i(\mathbf{x}, \mathbf{u}^i), \mathbf{u}^{i+1}, \ldots, \mathbf{u}^{l+1})$. Now define $\gamma(\mathbf{x}, \mathbf{u}) := {}^1\gamma \circ \ldots \circ {}^l\gamma(\mathbf{x}, \mathbf{u}) = (\mathbf{x}, \gamma^1(\mathbf{x}, \mathbf{u}^1), \ldots, \gamma^l(\mathbf{x}, \mathbf{u}^l), \mathbf{u}^{l+1})$. It is claimed that γ is a decoupling feedback. To show (b) of Lemma 3.2, let $\mathbf{X} \in \Delta^i$ and consider

$$\mathbf{f} \circ \gamma(\cdot, \mathbf{u})_* \mathbf{X} = \left(\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\gamma(\mathbf{x}, \mathbf{u})} + \sum_{j=1}^{l} \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}^j}\right|_{\gamma(\mathbf{x}, \mathbf{u})} \frac{\partial \gamma^j(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}\right)\mathbf{X}$$

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}^i} \cdot \frac{\partial \gamma^i}{\partial \mathbf{x}}\right)\mathbf{X} \subset \Delta^i$$

by the definition of γ^i. Furthermore, for $j \neq i$,

$$\frac{\partial \gamma^j}{\partial \mathbf{x}} \mathbf{X} \subset \operatorname{span}\left\{\frac{\partial}{\partial \mathbf{u}^j}\right\}$$

which is mapped into Δ^i under

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}^j}$$

by the construction of $\mathbf{E}^i$. Hence (b) of Lemma 3.2; (c) is shown similarly. □

One of the drawbacks of Theorem 3.1 is that it gives no hint as to how Δ^i is to be determined. However, in conjunction with Theorem 2.2 one has the following result.

Corollary 3.1

Let Σ be as in Theorem 3.1 and let Δ^{i*} be the maximal LCI distribution contained in $\ker \mathbf{h}^i_*$. Suppose that Δ^{i*}, $\mathbf{f}_*^{-1}(\Delta^{i*}) \cap \mathbf{V}(\mathbf{B})$ and $\mathbf{f}$ restricted to the fibres of $\mathbf{B}$ all have constant rank. Then the RBDP is locally regularly solvable if and only if

$$\bigcap_{i\in I}(\mathbf{f}_*^{-1}(\Delta^{i*})\cap \mathbf{V}(\mathbf{B}))+\bigcap_{j\in J}(\mathbf{f}_*^{-1}(\Delta^{j*})\cap \mathbf{V}(\mathbf{B}))=\mathbf{V}(\mathbf{B})$$

for all non-empty disjoint subsets I and J of $\{1, 2, ..., l\}$.

4. Decoupling with local stability

4.1. *General*

In the previous two sections, conditions were given under which a system could be input–output decoupled via state-variable feedback. Of course, if decoupling can only be achieved at the price of instability, then it is of no practical interest. Here, an easily checkable sufficient condition, based on the first method of Lyapunov, will be given for the existence of a feedback that simultaneously decouples and stabilizes a system in the neighbourhood of an equilibrium point.

Definition 4.1

Let $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f})$ be a non-linear discrete-time system. A point $\mathbf{b}_e \in \mathbf{B}$ such that $\mathbf{f}(\mathbf{b}_e)=\pi(\mathbf{b}_e)$ is called an equilibrium point. (In a local-coordinate chart $(\mathbf{x}, \mathbf{u})$ this means that $\mathbf{f}(\mathbf{x}_e, \mathbf{u}_e)=\mathbf{x}_e$, where $(\mathbf{x}_e, \mathbf{u}_e):=\mathbf{b}_e$.)

Definition 4.2

Let $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}, \mathbf{h}, \mathbf{N})$ be a non-linear discrete-time system with equilibrium point $\mathbf{b}_e$ and let $(\mathbf{x}, \mathbf{u})$ be local coordinates about $\mathbf{b}_e$. Then the *linearization* of Σ at $\mathbf{b}_e$ is the system

$$\begin{aligned}\delta\mathbf{x}_{k+1}&=\mathbf{A}\delta\mathbf{x}_k+\mathbf{B}\delta\mathbf{u}_k\\ \delta\mathbf{y}_k&=\mathbf{C}\delta\mathbf{x}_k\end{aligned} \tag{4.1}$$

where

$$\mathbf{A}:=\left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\mathbf{b}_e},\quad \mathbf{B}:=\left.\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\right|_{\mathbf{b}_e},\quad \mathbf{C}:=\left.\frac{\partial\mathbf{h}}{\partial\mathbf{x}}\right|_{\mathbf{x}_e},\quad \delta\mathbf{x}:=\mathbf{x}-\mathbf{x}_e,\quad \delta\mathbf{u}:=\mathbf{u}-\mathbf{u}_e$$

and $\delta\mathbf{y}:=\mathbf{y}-\mathbf{y}_e$ for $\mathbf{y}_e:=\mathbf{h}(\mathbf{x}_e)$, and $\mathbf{y}$ local coordinates for $\mathbf{N}$ about $\mathbf{y}_e$.

Remark 4.1

(i) It is easily shown that different choices of local coordinates $(\mathbf{x}, \mathbf{u})$ result in linearizations that are equivalent under the linear feedback group.

(ii) The state space of the linearized system (4.1) is $\mathbf{T}_{\mathbf{x}_e}\mathbf{M}$.

Definition 4.3

Let Δ be an involutive distribution. The set of local non-singular feedbacks γ that make Δ an invariant distribution of the closed-loop system $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}\circ\gamma)$ is denoted by $\mathscr{F}(\Delta)$ (read 'friends of delta'). For the linearized system (4.1), $\mathscr{F}(\mathbf{V})$ denotes the set of linear feedback $\mathbf{u}=\mathbf{F}\mathbf{x}+\mathbf{v}$ such that $(\mathbf{A}+\mathbf{B}\mathbf{F})\mathbf{V}\subset\mathbf{V}$.

The main result of this section is the following theorem, whose proof will be developed in the sub-sections that follow.

Theorem 4.1

Let $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f}, \mathbf{h}, \mathbf{N}^1\mathbf{x} \ldots \mathbf{x}\mathbf{N}^l)$ be an analytic non-linear discrete-time system with equilibrium point $\mathbf{b}_e$. Suppose that Σ satisfies the sufficient conditions of Theorem 3.1 for the local regular solvability of the RBDP. Then

(i) $\mathbf{V}^i := \Delta^i(\mathbf{x}_e)$ are controlled invariant subspaces of the linearized system (4.1);

(ii) if there exists an $\mathbf{F} \in$

$$\bigcap_{i=1}^{l} \mathscr{F}(\mathbf{V}^i)$$

that stabilizes (4.1), the RBDP is in fact solvable with local stability about $\mathbf{x}_e$. In particular, this is the case if (4.1) is completely controllable and

$$\bigcap_{i=1}^{l} \mathbf{V}^i = \{0\}$$

4.2. *Invariance with local stability*

This sub-section investigates the existence of a local feedback $\gamma \in \mathscr{F}(\Delta)$ that locally stabilizes a system about an equilibrium point. The main idea is to combine the standard linearization method of analyzing local stability with the geometric theory of invariant distributions.

Towards this end, let $\Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f})$ be a system with equilibrium point $\mathbf{b}_e$ and let γ be a feedback function. The closed-loop system $\Sigma_\gamma := \Sigma(\mathbf{B}, \mathbf{M}, \mathbf{f} \circ \gamma)$ will have equilibrium point $\bar{\mathbf{b}}_e = \gamma^{-1}(\mathbf{b}_e)$. (Note, $\pi(\bar{\mathbf{b}}_e) = \pi(\mathbf{b}_e)$.) In local coordinates $(\mathbf{x}, \mathbf{v})$ about the equilibrium point $\bar{\mathbf{b}}_e$, one easily calculates the linearization of Σ_γ to be

$$\delta\mathbf{x}_{k+1} = (\mathbf{A} + \mathbf{BF})\delta\mathbf{x}_k + \mathbf{B}\delta\mathbf{v}_k \tag{4.2}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\delta\mathbf{x}$ are as in (4.1), $\delta\mathbf{v} = \mathbf{v} - \bar{\mathbf{v}}_e$, and $\mathbf{F}$ is given by

$$\left.\frac{\partial\gamma_{\mathbf{x}}}{\partial\mathbf{x}}\right|_{\bar{\mathbf{b}}_e}$$

Hence, a sufficient condition for γ to be locally stabilizing is that $(\mathbf{A} + \mathbf{BF})$ has its eigenvalues in the open unit circle. In addition, if $(\mathbf{A} + \mathbf{BF})$ has no eigenvalue on the unit circle, then this condition is also necessary. (When there *are* eigenvalues on the unit circle, one must appeal to more sophisticated techniques, such as those based on centre manifolds (Carr 1981) in order to deduce stability or instability.)

Now suppose that the involutive distribution Δ is locally-controlled invariant and that $\gamma \in \mathscr{F}(\Delta)$. Then from (2.3), it follows that $\mathbf{V} := \Delta(\mathbf{x}_e)$ is a controlled invariant subspace of the linearized system (4.1) and that

$$\left.\frac{\partial\gamma_{\mathbf{x}}}{\partial\mathbf{x}}\right|_{\bar{\mathbf{b}}_e} =: \mathbf{F} \in \mathscr{F}(\mathbf{V})$$

That *every* $\mathbf{F} \in \mathscr{F}(\mathbf{V})$ arises in just this way, given certain constant-rank assumptions, is the statement of the following result, whose proof is given in the appendix.

Lemma 4.1

Let Σ be a non-linear system with equilibrium point $\mathbf{b}_e$, let Δ be an involutive distribution satisfying the sufficient conditions of Theorem 2.1 for local controlled invariance, and let $\mathbf{V} := \Delta(\mathbf{x}_e)$. Then for every $\mathbf{F} \in \mathscr{F}(\mathbf{V})$ there exists a $\gamma \in \mathscr{F}(\Delta)$ such that

$$\mathbf{F} = \left.\frac{\partial \gamma_x}{\partial \mathbf{x}}\right|_{\bar{\mathbf{b}}_e}$$

Combining Lemma 4.1 with the previous discussion on stability analysis via linearization, one obtains a sufficient condition for the existence of a feedback that simultaneously makes a given distribution invariant and stabilizes the system about an equilibrium point.

Theorem 4.2

Let Σ, Δ, and $\mathbf{V}$ be as in Lemma 5.1. Then, if there exists an $\mathbf{F} \in \mathscr{F}(\mathbf{V})$ such that $(\mathbf{A} + \mathbf{BF})$ is asymptotically stable, there exists a $\gamma \in \mathscr{F}(\Delta)$ that locally stabilizes Σ about $\mathbf{x}_e$.

Remark 4.2

There exists an $\mathbf{F} \in \mathscr{F}(\mathbf{V})$ such that $(\mathbf{A} + \mathbf{BF})$ is asymptotically stable if and only if $\mathbf{V} = \mathbf{V}_g^*$, where $\mathbf{V}_g^*$ is the maximal (asymptotically) stabilizable $(\mathbf{A}, \mathbf{B})$ invariant subspace contained in $\mathbf{V}$ (Wonham 1979).

4.3. *Proof of Theorem* 4.1

The following lemma provides the key result and can also be used to give an alternative proof of Lemma 4.1.

Lemma 4.2

Let Σ be a non-linear system with equilibrium point $\mathbf{b}_e$, let Δ be an involutive distribution satisfying the sufficient conditions of Theorem 2.1 for local controlled invariance, and let $\mathbf{V} := \Delta(\mathbf{x}_e)$. Then for every $\gamma \in \mathscr{F}(\Delta)$ and $\bar{\mathbf{F}} \in \mathscr{F}(\mathbf{V})$ there exists a local feedback $\hat{\gamma}$ such that $\bar{\gamma} := \hat{\gamma} \circ \gamma \in \mathscr{F}(\Delta)$ and

$$\left.\frac{\partial \bar{\gamma}_x}{\partial \mathbf{x}}\right|_{\bar{\mathbf{b}}_e} = \bar{\mathbf{F}}$$

where $\bar{\mathbf{b}}_e = \bar{\gamma}^{-1}(\mathbf{b}_e)$.

Proof

Let $\gamma \in \mathscr{F}(\Delta)$ and $\bar{\mathbf{F}} \in \mathscr{F}(\mathbf{V})$ be arbitrary. Choose local coordinates $(\mathbf{x}, \mathbf{u}) = (\mathbf{x}^1, \ldots, \mathbf{x}^n, \mathbf{u}^1, \ldots, \mathbf{u}^n)$ about $\bar{\mathbf{b}}_e = \gamma^{-1}(\mathbf{b}_e)$ such that

$$\Delta = \operatorname{span}\left\{\frac{\partial}{\partial \mathbf{x}^1}, \ldots, \frac{\partial}{\partial \mathbf{x}^k}\right\} \quad \text{and} \quad \mathbf{V}(\mathbf{B}) \cap \mathbf{f}_*^{-1}(\Delta) = \left\{\operatorname{span}\frac{\partial}{\partial \mathbf{u}^1}, \ldots, \frac{\partial}{\partial \mathbf{u}^r}\right\}$$

Note that in such coordinates, $\mathbf{V} = \Delta$ and $\mathbf{B}^{-1}(\mathbf{V}) = \mathbb{R}^m \cap [\mathbf{A}|\mathbf{B}]^{-1}\mathbf{V} = \mathbf{V}(\mathbf{B}) \cap \mathbf{f}_*^{-1}(\Delta)$. Define

$$\mathbf{F} := \left.\frac{\partial \gamma_x}{\partial \mathbf{x}}\right|_{\bar{\mathbf{b}}_e}$$

and define $\hat{\gamma}$ by $\hat{\gamma}_x(\mathbf{u}) := (\bar{\mathbf{F}} - \mathbf{F})\mathbf{x} + \mathbf{u}$. Since $\bar{\mathbf{F}}, \mathbf{F} \in \mathscr{F}(\mathbf{V})$, it follows that $(\bar{\mathbf{F}} - \mathbf{F})\mathbf{V} \subset$

$\mathbf{B}^{-1}(\mathbf{V})$ (Wonham 1979). Therefore, $(\mathbf{f}\circ\bar{\gamma})_*\Delta = \mathbf{f}_*\bar{\gamma}_*\Delta = (\mathbf{f}_*\circ\gamma_*)\Delta + (\mathbf{f}_*)(\bar{\mathbf{F}} - \mathbf{F})\Delta \subset (\mathbf{f}\circ\gamma)_*\Delta + \mathbf{f}_*(\mathbf{V}(\mathbf{B})\cap\mathbf{f}_*^{-1}(\Delta)) \subset \Delta + \Delta = \Delta$, which establishes that $\bar{\gamma}\in\mathscr{F}(\Delta)$. That

$$\left.\frac{\partial\bar{\gamma}_x}{\partial \mathbf{x}}\right|_{\mathbf{b}_e} = \bar{\mathbf{F}}$$

is clear. □

Theorem 4.1 will now be established. First note that $\mathbf{V}^i$ is controlled invariant $\Leftrightarrow \mathbf{A}\mathbf{V}^i \subset \mathbf{V}^i + \text{Im }\mathbf{B} \Leftrightarrow [\mathbf{A}|\mathbf{B}]\mathbf{V}^i\text{x}\mathbb{R}^m \subset \mathbf{V}^i + \text{Im }\mathbf{B} \Leftrightarrow \mathbf{f}_*(\mathbf{b}_0)\pi_*^{-1}(\Delta^i) \subset \Delta^i(\mathbf{f}(\mathbf{b}_0)) + \mathbf{f}_*(\mathbf{b}_0)\mathbf{V}(\mathbf{B})$. Hence Δ^i being locally-controlled invariant, and therefore satisfying (2.3), implies (i). Next note that the hypotheses of the theorem give that

$$\bigcap_{i=1}^{l} \mathscr{F}(\Delta^i) \neq \phi$$

(i.e. the RBDP is solvable). To show (ii), one must prove the existence of a

$$\gamma \in \bigcap_{i=1}^{l} \mathscr{F}(\Delta^i)$$

that is locally stabilizing. Let $\bar{\gamma}$ be any element of

$$\bigcap_{i=1}^{l} \mathscr{F}(\Delta^i)$$

and let $\hat{\gamma}$ be the feedback constructed in Lemma 4.2 for a stabilizing

$$\mathbf{F} \in \bigcap_{i=1}^{l} \mathscr{F}(\mathbf{V}^i)$$

Then

$$\gamma := \hat{\gamma}\cdot\bar{\gamma} \in \bigcap_{i=1}^{l} \mathscr{F}(\Delta^i) \quad\text{and}\quad \left.\frac{\partial\gamma_x}{\partial \mathbf{x}}\right|_{\mathbf{b}_e} = \mathbf{F}$$

Hence γ is locally stabilizing for the non-linear system Σ. □

5. Conclusions and comments

This paper has considered the input–output decoupling problem for non-linear discrete-time systems from a local viewpoint. Starting from a global state-space characterization of what it means for a system to be input–output decoupled, it was shown that a natural (regular) local version of the problem could be formulated in terms of invariant distributions. This local problem was then resolved using some recent results on controlled invariant distributions for discrete-time systems. In a similar manner, one can also treat the triangular decoupling problem.

The input–output decoupling problem with stability was also considered, and sufficient conditions for its resolution were established. The approach centred on 'invariance with local stability' and linked up the classical linearization method with the differential geometric-decoupling theory.

In a related paper, Grizzle and Nijmeijer (1985) have studied the infinite-zero structure of a non-linear discrete-time system and have characterized the solvability of the RBDP in terms of it.

Appendix

A.1. *Proof of Lemma* 3.1

The following result will be useful.

Lemma A.1

Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be analytic manifolds and let $\mathbf{g}:\mathbf{X}\mathrm{x}\mathbf{Y}\rightarrow\mathbf{Z}$ be an analytic map. Define an equivalence relation $\mathbf{R}$ by $\mathbf{R}(\bar{\mathbf{x}})=\{\mathbf{x}\in\mathbf{X}|\mathbf{g}(\mathbf{x},\mathbf{y})=\mathbf{g}(\bar{\mathbf{x}},\mathbf{y})$ for all $\mathbf{y}\in\mathbf{Y}\}$. Then there exists an open dense subset of $\mathbf{X}$ where the orbits of $\mathbf{R}$ locally coincide with the leaves of $\Delta:=\mathbf{TX}\cap\ker\mathbf{g}_*$.

Proof

Δ is clearly an involutive analytic distribution and hence has constant dimension on an open dense subset $\mathbf{X}'\subset\mathbf{X}$. Furthermore, if necessary one can shrink $\mathbf{X}'$ so that

$$\frac{\partial\mathbf{g}}{\partial\mathbf{x}}(\cdot,\mathbf{y})|\mathbf{X}'$$

has constant rank for an open dense subset of $\mathbf{y}$'s in $\mathbf{Y}$. Now fix a point $\bar{\mathbf{x}}\in\mathbf{X}'$ and choose coordinates $(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ for $\mathbf{X}$ such that

$$\Delta=\mathrm{span}\left\{\frac{\partial}{\partial\mathbf{x}_1},\ldots,\frac{\partial}{\partial\mathbf{x}_r}\right\},\quad r:=\dim\Delta$$

Consider the associated leaves of Δ given by

$$\mathbf{F}(\bar{\mathbf{x}})=\{(\mathbf{x}_1,\ldots,\mathbf{x}_n)|\mathbf{x}_{r+1}=\bar{\mathbf{x}}_{r+1},\ldots,\mathbf{x}_n=\mathbf{x}_n\}$$

Claim

$\mathbf{R}(\bar{\mathbf{x}})=\mathbf{F}(\bar{\mathbf{x}})$ in a neighbourhood of $\bar{\mathbf{x}}$.

Proof

In the above coordinates, one can write $\mathbf{g}(\mathbf{x},\mathbf{y})=\mathbf{g}(\mathbf{x}_{r+1},\ldots,\mathbf{x}_n,\mathbf{y})$. Therefore, $\mathbf{R}(\bar{\mathbf{x}})=\{(\mathbf{x}_1,\ldots,\mathbf{x}_n)|\mathbf{g}(\mathbf{x}_{r+1},\ldots,\mathbf{x}_n,\mathbf{y})=\mathbf{g}(\bar{\mathbf{x}}_{r+1},\ldots,\bar{\mathbf{x}},\mathbf{y})$ for all $\mathbf{y}\in\mathbf{Y}\}\supset\{(\mathbf{x}_1,\ldots,\mathbf{x}_n)|$ $\mathbf{x}_{r+1}=\bar{\mathbf{x}}_{r+1},\ldots,\mathbf{x}_n=\bar{\mathbf{x}}_n\}=\mathbf{F}(\bar{\mathbf{x}})$. Going in the other direction, for each $i=r+1,\ldots,n$,

$$\frac{\partial\mathbf{g}}{\partial\mathbf{x}_i}(\bar{\mathbf{x}},\mathbf{y})\neq 0$$

for some $\mathbf{y}\in\mathbf{Y}$ (for otherwise

$$\frac{\partial}{\partial\mathbf{x}_i}$$

would be an element of Δ) and hence $\mathbf{R}(\bar{\mathbf{x}})\subset\mathbf{F}(\bar{\mathbf{x}})$. □

Returning to the proof of Lemma 3.1, define a sequence of equivalence relations

$\mathbf{R}_k^i$, $i = 1, \ldots, l$, $k = 0, 1, \ldots$ by $\mathbf{x}\mathbf{R}_k^i\bar{\mathbf{x}}$ if

$$\mathbf{y}_t^i(\mathbf{x}; (\mathbf{u}_j)_{j=0}^{\infty}) = \mathbf{y}_t^i(\bar{\mathbf{x}}; (\hat{\mathbf{u}}_j)_{j=0}^{\infty})$$

for all $t = 0, \ldots, k$, $(\mathbf{u}_j)_{j=0}^{\infty}$ and $(\hat{\mathbf{u}}_j)_{j=0}^{\infty}$, where the notation of Proposition 3.1 has been employed. Fix i. By Lemma A.1, for each $k = 0, 1, 2, \ldots$ the orbits of $\mathbf{R}_k^i$ locally coincide with the leaves of some analytic involutive distribution Δ_k^i.

One shows quite easily that

$$\mathbf{R}_{k+1}^i(\bar{\mathbf{x}}) = \{\mathbf{x} \in \mathbf{R}_k^i(\bar{\mathbf{x}}) \mid \mathbf{f}(\mathbf{x}, \mathbf{u}^1, \ldots, \mathbf{u}^n)\mathbf{R}_k^i\mathbf{f}(\bar{\mathbf{x}}, \hat{\mathbf{u}}^1, \ldots, \hat{\mathbf{u}}^{i-1}, \mathbf{u}^i, \hat{\mathbf{u}}^{i+1}, \ldots, \hat{\mathbf{u}}^n)$$

for all $\mathbf{u}^1, \ldots, \mathbf{u}^n, \hat{\mathbf{u}}^1, \ldots, \hat{\mathbf{u}}^{i-1}, \hat{\mathbf{u}}^{i+1}, \ldots, \hat{\mathbf{u}}^n$).

Hence $\Delta_{k+1}^i \subset \Delta_k^i$; but since each Δ_k^i has constant dimension on an open dense subset of $\mathbf{M}$, for $k \geqslant \dim \mathbf{M} =: n$, one must have $\Delta_k^i = \Delta_n^i$ on an open dense subset of $\mathbf{M}$. This establishes the result for $\Delta^i := \Delta_n^i$. □

A.2. *Proof of Lemma* 4.1

It suffices to establish the lemma for the case $\mathbf{V}(\mathbf{B}) \cap \mathbf{f}_*^{-1}(\Delta) = \{\mathbf{0}\}$ since if $\gamma \in \mathscr{F}(\Delta)$, $\gamma_* | \mathbf{V}(\mathbf{B}) \cap \mathbf{f}_*^{-1}(\Delta)$ is completely arbitrary. For the linearized system, this condition translates into $\mathbf{V} \cap \operatorname{Im} \mathbf{B} = \{\mathbf{0}\}$. Now recall the construction of $\mathscr{F}(\mathbf{V})$ (Wonham 1979): let $\mathbf{v}^1, \ldots, \mathbf{v}^k$ be a basis for $\mathbf{V}$. Then $\mathbf{A}\mathbf{V} \subset \mathbf{V} + \operatorname{Im} \mathbf{B}$ and $\mathbf{V} \cap \operatorname{Im} \mathbf{B} = \{\mathbf{0}\}$ imply the existence of unique $\mathbf{u}^1, \ldots, \mathbf{u}^k$ such that $\mathbf{A}\mathbf{v}^i + \mathbf{B}\mathbf{u}^i \subset \mathbf{V}$. Define $\mathbf{F}_0 : \mathbf{V} \to \mathbb{R}^m$ by $\mathbf{F}_0\mathbf{v}^i = \mathbf{u}^i$; then $\mathscr{F}(\mathbf{V}) = \{\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n \mid \mathbf{F}|\mathbf{V} = \mathbf{F}_0|\mathbf{V}\}$.

From the proof of Theorem 2.1 of Grizzle (1985 a) and the Frobenius theorem (Spivak 1979), $\mathscr{F}(\Delta)$ is constructed in the following manner:

Let $\mathbf{X}^1, \ldots, \mathbf{X}^k$ be a basis for Δ. Then Δ is constant dimensional, $\mathbf{V}(\mathbf{B}) \cap \mathbf{f}_*^{-1}(\Delta) = \{\mathbf{0}\}$, and $\mathbf{f}_*(\pi_*^{-1}(\Delta)) \subset \Delta + \mathbf{f}_*\mathbf{V}(\mathbf{B})$ imply the existence of unique $\mathbf{Y}^1, \ldots, \mathbf{Y}^k \in \mathbf{V}(\mathbf{B})$ such that $\mathbf{f}_*(\mathbf{X}^i + \mathbf{Y}^i) \subset \Delta$. Then $\gamma \in \mathscr{F}(\Delta)$ is any locally-invertible solution of the set of partial differential equations

$$\frac{\partial \gamma_{\mathbf{x}}(\mathbf{u})}{\partial \mathbf{x}} \mathbf{X}^i(\mathbf{x}) = \mathbf{Y}^i(\gamma(\mathbf{x}, \mathbf{u})), \quad \gamma_{\mathbf{x}_0}(\mathbf{u}) = \mathbf{u}$$

Hence

$$\left. \frac{\partial \gamma_{\mathbf{x}}}{\partial \mathbf{x}} \right|_{\mathbf{b}_e} \mathbf{X}^i(\mathbf{x}_e) = \mathbf{Y}^i(\mathbf{b}_e)$$

is completely specified for $i = 1, \ldots, k$ but

$$\left. \frac{\partial \gamma_{\mathbf{x}}}{\partial \mathbf{x}} \right|_{\mathbf{b}_e} |\operatorname{span} \{\mathbf{X}^{k+1}, \ldots, \mathbf{X}^n\}$$

is arbitrary for $\mathbf{X}^1, \ldots, \mathbf{X}^n$ a local basis for $\mathbf{TM}$. Therefore, as $\mathbf{X}^1(\mathbf{x}_e), \ldots, \mathbf{X}^k(\mathbf{x}_e)$ is a basis for $\mathbf{V}$ and as $\mathbf{A}\mathbf{X}^i(\mathbf{x}_e) + \mathbf{B}\mathbf{Y}^i(\mathbf{b}_e) = \mathbf{f}_*(\mathbf{b}_e)[\mathbf{X}^i + \mathbf{Y}^i] \subset \Delta(\mathbf{f}(\mathbf{b}_e)) = \Delta(\mathbf{x}_e) = \mathbf{V}$, the uniqueness of $\mathbf{F}_0|\mathbf{V}$ implies that

$$\mathbf{F}_0|\mathbf{V} = \left. \frac{\partial \gamma_{\mathbf{x}}}{\partial \mathbf{x}} \right|_{\mathbf{b}_e} |\mathbf{V}$$

Since

$$\left. \frac{\partial \gamma_{\mathbf{x}}}{\partial \mathbf{x}} \right|_{\mathbf{b}_e}$$

is arbitrary 'off of V', the proof is complete. □

ACKNOWLEDGMENTS

This material is based upon work supported in part by the North Atlantic Treaty Organization under a grant awarded in 1984, and in part by NSF Contract No. ECS-8505318.

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