

Block Noninteracting Control with Stability via Static State Feedback*

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Abstract. Necessary and sufficient conditions for the existence of a static state feedback that achieves noninteraction and internal stability are obtained. This is accomplished by first characterizing the set of all controllability subspaces (distributions) that can arise as solutions to the noninteracting control problem. This characterization is then used to identify a fixed internal dynamics that is common to every noninteractive closed loop. The stability properties of this dynamics is shown to be the key factor in the problem of achieving noninteracting control with internal stability.

Key words. State feedback, Noninteracting control, Controllability distributions, Geometric Methods.

1. Introduction

This paper addresses the problem of achieving noninteracting control with internal stability via invertible static state feedback, for a system whose outputs have been grouped into blocks. Both linear and nonlinear systems are investigated.

If one restricts attention to left-invertible square systems (equal numbers of scalar input and output components) where each output "block" is scalar valued, then the problem of achieving noninteracting control with internal stability by means of static state feedback is well understood. The solution for the class of linear systems was given by Gilbert [G1] in 1969; the solution for nonlinear systems has only recently been obtained [H1], [H2], [HG], [IG]. The papers [G1] and [IG] show that imposing a noninteractive structure, by means of a static state feedback, entails the assignment of a certain fixed internal dynamics. This dynamics, which is independent of the particular static feedback used to achieve noninteraction, must necessarily be asymptotically stable if an asymptotically stable noninteractive closed loop is sought. Moreover, under mild assumptions (asymptotic stabilizability of the original system, noncriticalness), the asymptotic stability of the fixed dynamics is also sufficient for the achievement of noninteraction with stability. Hence, the

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induced fixed dynamics constitutes the only obstruction to the solvability of the noninteracting control problem with internal stability by means of static state feedback.

For the class of *linear* systems, Wonham and Morse [WM] effectively quenched any interest in pursuing the corresponding problem for systems with block outputs when they proved that if one allows dynamic state feedback, then, provided the original system is stabilizable, the conditions required for achieving noninteracting control, and those for achieving noninteracting control *with stability*, are the same; that is, if noninteraction is at all possible, then noninteraction with stability is always possible [W, Theorem 9.3]. Their technique was to devise a dynamically extended system which could be made noninteractive with static state feedback, but which possessed, in light of the results of [G1], and [IG], no fixed dynamics. Actually, Wonham and Morse presented this latter property in a different but equivalent manner, proving that the eigenvalues of the resulting noninteractive system were freely assignable.

Investigations on achieving noninteracting control via dynamic state feedback have also been carried out for nonlinear systems; necessary and sufficient conditions for the existence of such a feedback, as well as procedures for its construction, are now known [DM], [F], [NR]. However, these results were specifically designed to enlarge the class of systems which could be made noninteractive, and not necessarily to gain asymptotic stability. Indeed, even for linear systems the techniques in question, unlike those developed by Wonham and Morse, do not guarantee an internally stable noninteractive closed-loop system.

A natural question, therefore, is whether it is possible to develop general dynamic compensation techniques for nonlinear systems which simultaneously accomplish noninteraction and internal stability. This has been resolved in the negative [IG]. It has been shown in particular that no system of the form $\{\dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = a(x_1, x_2, x_3); y_1 = x_1, y_2 = x_2(\partial a/\partial x_1)(0) \neq 0, (\partial a/\partial x_2)(0) \neq 0, (\partial a/\partial x_3)(0) > 0, (\partial^2 a/\partial x_1 \partial x_2)(0) \neq 0\}$ can be made noninteractive and internally stable by any dynamic (or static) state feedback, despite the fact that (a) each such system can be separately stabilized or made noninteractive, and (b) the linearization of such a system can be made stably noninteractive with a simple one-dimensional dynamic state feedback. The key point of the analysis was that noninteractive control via static state feedback induced an unstable fixed dynamics, and, contrary to what is possible for linear systems, this dynamics could not be removed by any dynamic state feedback resulting in a noninteractive closed-loop. The central role of the fixed dynamics in understanding nonlinear noninteracting control with stability was thus made clear.

The goal of this paper is to extend the analysis of [IG], which is valid only for square systems, to the block noninteracting control problem. The main difficulty to be overcome is the nonunicity of solutions, where, by solution, is meant the specification of a set of controllability subspaces (distributions) from which an appropriate feedback leading to a noninteractive closed loop can be constructed. Results will be established to identify a particular (sub)dynamics that is common to every noninteractive closed loop. This dynamics will be called the *fixed dynamics*, and its central role in the problem of achieving noninteracting control with internal stability will be established.

Section 2 develops the results in the context of linear systems. Even here the consideration of all possible solutions in order to uncover a suitable notion of the fixed dynamics is nontrivial, and most of the detailed arguments are relegated to the appendices. The main ideas, however, are presented in such a way that the nonlinear analysis of Section 3 follows the same sequence of steps. Unfortunately, the possible presence of singularities presents real difficulties, and weaker results are obtained for the class of nonlinear systems.

2. Linear Block Noninteracting Control with Stability

Consider a linear system whose outputs have been grouped into blocks:

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu, \\ y_i = C_i x, \quad 1 \leq i \leq \mu, \end{cases} \quad (2.1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^{p_i}$. The system is said to be *noninteractive* with respect to a given partition $u^T = (u_1, \dots, u_\mu, u_{\mu+1})$ of the inputs, u_i possibly vector valued, if y_i is unaffected by u_j for all $i \neq j$. The *noninteracting control problem* addressed here is to find, if possible, a *regular static state variable feedback*, $u = Fx + Gv$, $|G| \neq 0$ (regular refers to the invertibility of G) and a partitioning of the new command inputs v so that the resulting closed-loop system is noninteractive. If in addition the eigenvalues of $A + BF$ are in the open left half-plane, the problem is said to be solvable with *stability*.

The goal is to give necessary and sufficient conditions for noninteracting control with stability, using static state variable feedback. The geometric tools of [MW] and [W] will be used freely. If M is an $n \times n$ matrix and $\mathcal{S} \subset \mathbb{R}^n$ is a subspace, then $\langle M|\mathcal{S} \rangle := \mathcal{S} + M\mathcal{S} + \dots + M^{n-1}\mathcal{S}$.

Necessary and sufficient conditions for achieving noninteraction without the constraint of internal stability are well known [D], [DLM], [MW]. None of the available criteria is a convenient starting point for the additional analysis needed in order to characterize stabilizing solutions. The approach taken in [G2] appears to be well suited for this purpose. This is first briefly developed.

The following result is a geometric formulation of the notion of noninteraction; see p. 87 of [W].

Lemma 2.1. *The system*

$$\begin{aligned} \dot{x} &= Ax + \sum_{j=1}^{\mu+1} B_j u_j, \\ y_i &= C_i x, \quad 1 \leq i \leq \mu, \end{aligned}$$

is noninteractive with respect to the indicated partition of the inputs if, and only if, there exist subspaces $V^1, \dots, V^\mu \subset \mathbb{R}^n$ such that, for $1 \leq i \leq \mu$,

- (a) $V^i \subset \ker C_i$,
- (b) $AV^i \subset V^i$,
- (c) $\text{Im } B_j \subset V^i$ for all $j \neq i$.

Using this, one obtains the following necessary and sufficient condition for achieving noninteracting control (without the constraint of internal stability). The proof is given in Appendix A.

Theorem 2.2. *The system Σ , (2.1), can be rendered noninteractive with a regular static state variable feedback if and only if there exist controlled-invariant subspaces $V^i \subset \ker C_i$, $1 \leq i \leq \mu$, such that*

$$\text{Im } B \cap V^i + \bigcap_{j \neq i} (\text{Im } B \cap V^j) = \text{Im } B \quad (2.2)$$

for each $1 \leq i \leq \mu$.

Equation (2.2) turns out to be a necessary and sufficient condition for the existence of a single feedback rendering the set of subspaces $\{V^1, \dots, V^\mu\}$ simultaneously invariant, and hence could be called the *compatibility condition*. Whenever a set of subspaces can be made simultaneously invariant with a single feedback, it will be called a *compatible set* (of subspaces).

If V is a controlled-invariant subspace, and if \mathcal{R} is the maximal controllability subspace contained in V , then $V \cap \text{Im } B = \mathcal{R} \cap \text{Im } B$. Using this fact one deduces the following corollary of Theorem 2.2, which will form the basis of all further analysis.

Corollary 2.3. *The system Σ , (2.1), can be rendered noninteractive with a regular static state variable feedback if and only if there exist controllability subspaces $\mathcal{R}_i \subset \ker C_i$, $1 \leq i \leq \mu$, such that*

$$(\mathcal{R}_i \cap \text{Im } B) + \bigcap_{j \neq i} (\mathcal{R}_j \cap \text{Im } B) = \text{Im } B \quad (2.3)$$

for all $1 \leq i \leq \mu$.

A set of subspaces $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ satisfying the hypotheses of Corollary 2.3 will be called a *solution to the noninteracting control problem*. Note that if $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ is any such solution, then

$$\mathcal{R} := \bigcap_{j=1}^{\mu} \mathcal{R}_j \quad (2.4)$$

is a controlled-invariant subspace since it is the intersection of compatible subspaces. The maximal controllability subspace contained in \mathcal{R} will be denoted by \mathcal{D} . Let $\pi: \mathcal{R} \rightarrow \mathcal{R}/\mathcal{D}$ be the canonical projection, and let F be any feedback matrix such that $(A + BF)\mathcal{R} \subset \mathcal{R}$. Then $(A + BF)|(\mathcal{R}/\mathcal{D})$ will denote the induced operator on \mathcal{R}/\mathcal{D} ; i.e., the unique linear operator \hat{A} such that $\pi \circ (A + BF)|_{\mathcal{R}} = \hat{A} \circ \pi|_{\mathcal{R}}$.

Lemma 2.4 (see also Theorem 7.2 of [WM] and Theorem 3 of [D]). *Let $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ be any solution to the noninteracting control problem, and let F_1 and F_2 be any two feedback matrices such that $(A + BF_i)\mathcal{R}_j \subset \mathcal{R}_j$ for $1 \leq j \leq \mu$, $1 \leq i \leq 2$. Then $(A + BF_1)|(\mathcal{R}/\mathcal{D}) = (A + BF_2)|(\mathcal{R}/\mathcal{D})$. That is, the modes $(A + BF)|(\mathcal{R}/\mathcal{D})$ are independent of the particular choice of feedback F rendering $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ simultaneously invariant.*

Proof. This follows easily from Theorem 5.7 of [W]. ■

The dynamics $(A + BF)|(\mathcal{R}/\mathcal{Q})$ will be called the *fixed modes associated with the solution* $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$. Clearly these modes must be stable if there is to exist a feedback associated with the solution $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ which simultaneously achieves noninteraction and stability. If the system is stabilizable, then it can be shown that stability of the fixed modes is also sufficient. Whenever the system in question is left-invertible, then the solution to the noninteracting control problem shall be seen to be unique, and, therefore, Lemma 2.4 easily leads to a necessary and sufficient condition for the existence of a feedback which simultaneously achieves noninteraction and stability. For the general block noninteracting control problem, there may exist many solutions $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$, and hence Lemma 2.4 can at best lead to necessary and sufficient conditions for the existence of a stabilizing feedback associated with a particular choice of $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$, and thus only to weak sufficient conditions for the existence of a stabilizing noninteractive control law. What is needed, in some sense, is a "parametrization" of the set of all solutions. This is developed next.

Let \mathcal{P}_i^* denote the maximal controllability subspace contained in $\ker C_i$, $1 \leq i \leq \mu$, and define $\mathcal{P}^* := \bigcap_{j=1}^{\mu} \mathcal{P}_j^*$. Let \mathcal{Q}^* denote the maximal controllability subspace contained in \mathcal{P}^* .

Lemma 2.5. *Suppose that the noninteractive control problem is solvable and let $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ be any particular solution. Then the set of controllability subspaces $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is always compatible. Moreover, there exists a regular static state feedback and a partitioning of the new command inputs, $u = Fx + \sum_{i=1}^{\mu+1} G_i v_i$, such that*

- (a) $(A + BF)\mathcal{P}_i^* \subset \mathcal{P}_i^*$, $1 \leq i \leq \mu$,
- (b) $(A + BF)\mathcal{R}_i \subset \mathcal{R}_i$, $1 \leq i \leq \mu$,
- (c) $\text{Im } BG_j \subset \mathcal{R}_i$ for all $j \neq i$.

Proof. See Appendix A. ■

This lemma shows in particular that if the noninteractive control problem is solvable, then $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is a solution; in fact, it is the *maximal solution*. The next result shows how any particular solution to the noninteracting control problem relates to the maximal solution. Recall that if \mathcal{S} is a controllability subspace, and if F is any feedback matrix rendering \mathcal{S} invariant, then

$$\mathcal{S} = \langle A + BF | \mathcal{S} \cap \text{Im } B \rangle.$$

Theorem 2.6. *Suppose that the noninteractive control problem is solvable and let $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ be any solution. Then, for each $1 \leq i \leq \mu$,*

$$\mathcal{R}_i + \mathcal{Q}^* = \mathcal{P}_i^*.$$

Proof. Since $\mathcal{Q}^* \subset \mathcal{P}_i^*$ (by definition of \mathcal{Q}^*) and $\mathcal{R}_i \subset \mathcal{P}_i^*$ (by maximality of \mathcal{P}_i^*), $\mathcal{R}_i + \mathcal{Q}^* \subset \mathcal{P}_i^*$. By Lemma 2.5, $\mathcal{R}_i + \mathcal{Q}^*$ is a controlled-invariant subspace compatible with \mathcal{P}_i^* . Let F be any feedback matrix rendering $\mathcal{R}_i + \mathcal{Q}^*$ and \mathcal{P}_i^* simultane-

ously invariant. Then, $\langle A + BF | (\mathcal{R}_i + \mathcal{Q}^*) \cap \text{Im } B \rangle \subset \mathcal{R}_i + \mathcal{Q}^*$. Hence, to establish the theorem, it suffices to show that $(\mathcal{R}_i + \mathcal{Q}^*) \cap \text{Im } B = \mathcal{P}_i^* \cap \text{Im } B$, for it would then follow that $\mathcal{P}_i^* = \langle A + BF | \mathcal{P}_i^* \cap \text{Im } B \rangle = \langle A + BF | (\mathcal{R}_i + \mathcal{Q}^*) \cap \text{Im } B \rangle \subset \mathcal{R}_i + \mathcal{Q}^*$, completing the double inclusion. Toward this end, define $q^* := \dim[\mathcal{Q}^* \cap \text{Im } B]$, $r_i := \dim[(\mathcal{R}_i + \mathcal{Q}^*) \cap \text{Im } B]$, $\bar{r}_i := \dim[\bigcap_{j \neq i} (\mathcal{R}_j + \mathcal{Q}^*) \cap \text{Im } B]$, $d_i := \dim[\mathcal{P}_i^* \cap \text{Im } B]$, and $m := \dim[\text{Im } B]$. Since $\{\mathcal{R}_i\}_{i=1}^\mu$ is a solution to the noninteracting control problem, it follows that, for each $1 \leq i \leq \mu$,

$$(\mathcal{R}_i + \mathcal{Q}^*) \cap \text{Im } B + \bigcap_{j \neq i} (\mathcal{R}_j + \mathcal{Q}^*) \cap \text{Im } B = \text{Im } B. \tag{2.5}$$

Now, note that $\mathcal{Q}^* \cap \text{Im } B \subset \bigcap_{i=1}^\mu [(\mathcal{R}_i + \mathcal{Q}^*) \cap \text{Im } B] \subset \bigcap_{i=1}^\mu [\mathcal{P}_i^* \cap \text{Im } B] = \mathcal{P}^* \cap \text{Im } B = \mathcal{Q}^* \cap \text{Im } B$, and hence

$$\bigcap_{i=1}^\mu [(\mathcal{R}_i + \mathcal{Q}^*) \cap \text{Im } B] = \mathcal{Q}^* \cap \text{Im } B. \tag{2.6}$$

Combining (2.5) and (2.6), one obtains $r_i + \bar{r}_i = m + q^*$. Using similar reasoning, one has

$$\mathcal{P}_i^* \cap \text{Im } B + \bigcap_{j \neq i} (\mathcal{R}_j + \mathcal{Q}^*) \cap \text{Im } B = \text{Im } B \tag{2.7}$$

and

$$[\mathcal{P}_i^* \cap \text{Im } B] \cap \left[\bigcap_{j \neq i} (\mathcal{R}_j + \mathcal{Q}^*) \cap \text{Im } B \right] = \mathcal{Q}^* \cap \text{Im } B. \tag{2.8}$$

This yields $d_i + \bar{r}_i = m + q^*$. Therefore, $r_i = d_i$ which proves $(\mathcal{R}_i + \mathcal{Q}^*) \cap \text{Im } B = \mathcal{P}_i^* \cap \text{Im } B$. ■

The above result establishes the connection between an arbitrary solution $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ and the maximal solution $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$. As a general remark, there is not as much freedom in the set of possible solutions as one may have expected *a priori*. In particular, if the system Σ is left-invertible, the V^* , the maximal controlled-invariant subspace contained in $\ker C = \ker [C_1^T, \dots, C_\mu^T]^T$, satisfies $V^* \cap \text{Im } B = \{0\}$, which in turn yields $\mathcal{Q}^* = \{0\}$ and the unicity of the solution $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$. In this special case, Lemma 2.4 identifies the fixed modes of the system and its conclusion is actually just a reformulation of results of [WM]. Using the results of [NS2], one can show that \mathcal{P}^* corresponds to the “radical” employed in [WM], and that this result can be extended to the class of nonlinear systems.

The goal now is to relate the fixed modes (see Lemma 2.4) of the maximal solution $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ to the fixed modes of any other solution $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$. Recall that $\mathcal{P}^* := \bigcap_{j=1}^\mu \mathcal{P}_j^*$, $\mathcal{Q}^* =$ maximal controllability subspace contained in \mathcal{P}^* , and $\mathcal{R} := \bigcap_{j=1}^\mu \mathcal{R}_j$.

Theorem 2.7. *Let \mathcal{P}^* , \mathcal{R} , and \mathcal{Q}^* be as above. Then*

$$\mathcal{P}^* = \mathcal{R} + \mathcal{Q}^*.$$

Proof. See Appendix A. ■

As an immediate corollary one has

Corollary 2.8. *Let F be any feedback matrix rendering $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ invariant. Then*

$$(A + BF) \left| \frac{\mathcal{P}^*}{\mathcal{Q}^*} \right. = (A + BF) \left| \frac{\mathcal{R}}{\mathcal{R} \cap \mathcal{Q}^*} \right.$$

To summarize what has been established up to this point, by Lemma 2.5 there exists a feedback matrix F rendering $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ simultaneously invariant; clearly the same matrix also renders \mathcal{Q}^* , \mathcal{P}^* , \mathcal{Q} , and \mathcal{R} invariant. By Lemma 2.4, the fixed modes $(A + BF)|_{(\mathcal{R}/\mathcal{Q})}$ and $(A + BF)|_{(\mathcal{P}^*/\mathcal{Q}^*)}$ are independent of the particular choice of F , and hence the eigenvalues of $(A + BF)|_{(\mathcal{R}/\mathcal{Q})}$ and $(A + BF)|_{(\mathcal{P}^*/\mathcal{Q}^*)}$ will be eigenvalues of any noninteractive closed-loop system obtained from $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ or $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$, respectively. By construction, all of the subspaces are invariant under $(A + BF)$, and thus $(A + BF)|_{(\mathcal{R}/(\mathcal{R} \cap \mathcal{Q}^*))}$ is well defined. Now, since $\mathcal{Q} \subset \mathcal{R}$ and $\mathcal{Q} \subset \mathcal{Q}^*$, one has $\mathcal{Q} \subset \mathcal{Q}^* \cap \mathcal{R}$, so the set of eigenvalues of $(A + BF)|_{(\mathcal{R}/(\mathcal{R} \cap \mathcal{Q}^*))}$ is a subset of the eigenvalues of $(A + BF)|_{(\mathcal{R}/\mathcal{Q})}$. But by Corollary 2.8, $(A + BF)|_{(\mathcal{R}/(\mathcal{R} \cap \mathcal{Q}^*))} = (A + BF)|_{(\mathcal{P}^*/\mathcal{Q}^*)}$, so the eigenvalues of $(A + BF)|_{(\mathcal{P}^*/\mathcal{Q}^*)}$ are also eigenvalues of $(A + BF)|_{(\mathcal{R}/\mathcal{Q})}$. Stating this result formally, one has the following necessary condition for the achievement of noninteracting control with stability.

Lemma 2.9. *Suppose that the system Σ , (2.1), can be rendered noninteractive with regular static state feedback. Then, if the fixed modes of the maximal solution are unstable, it is impossible to achieve noninteraction and stability simultaneously.*

The converse to this result is essentially contained in Theorem 7.2 of [WM] and Theorem 3 of [D], but expressed in different terminology and with a different set of controllability subspaces. For completeness, it will also be developed here. This is most easily done via some general decomposition and structural results, which are new.

Lemma 2.10. *Suppose Σ can be rendered noninteractive with a regular static state feedback. Let $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ be any solution and let $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ be the maximal solution. Denote by C^* the controllable subspace of Σ . Then the set of subspaces $\{C^*, \mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{Q}, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*, \mathcal{R} \cap \mathcal{Q}^*\}$ is simultaneously decomposable. That is, there exists a decomposition of the state space as $\mathbb{R}^n = V_1 \oplus V_2 \oplus \dots \oplus V_{\mu+3}$, and a further splitting of V_i for $1 \leq i \leq \mu + 1$ as $V_i = V_i^a \oplus V_i^b$, so that*

$$C^* = \sum_{j=1}^{\mu+2} V_j, \quad \mathcal{R}_i = \sum_{j \notin \{i, \mu+3\}} V_j,$$

$$\mathcal{P}_i^* = \mathcal{R}_i \oplus V_i^b, \quad \mathcal{Q}^* \cap \mathcal{R} = V_{\mu+1}, \quad \mathcal{Q} = V_{\mu+1}^a.$$

Proof. See Appendix A. ■

In terms of the above subspaces, one easily calculates that $\mathcal{R} := \bigcap_{j=1}^{\mu} \mathcal{R}_j = V_{\mu+1} \oplus V_{\mu+2}$, $\mathcal{P}^* := \bigcap_{j=1}^{\mu} \mathcal{P}_j^* = \sum_{j=1}^{\mu} V_j^b \oplus V_{\mu+1} \oplus V_{\mu+2}$, $\mathcal{R}/\mathcal{Q} = V_{\mu+1}^b \oplus V_{\mu+2}$, and

$\mathcal{R}/(\mathcal{R} \cap \mathcal{Q}^*) = V_{\mu+2}$. Hence, one can use this result to produce an alternate proof of Lemma 2.9, since, by Corollary 2.8, $(A + BF)|(\mathcal{P}^*/\mathcal{Q}^*) = (A + BF)|(\mathcal{R}/\mathcal{R} \cap \mathcal{Q}^*)$.

Using this lemma, it is possible to exhibit the internal structure of a noninteractive system.

Corollary 2.11. *Suppose that Σ has been rendered interactive with a feedback compatible with $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$. Let $\mathcal{R}_i = \mathcal{P}_i^*$ in Lemma 2.10 and let $(x_1, \dots, x_{\mu+3})$ be a coordinate system adapted to $\{V_1, \dots, V_{\mu+3}\}$; that is, $x_i \in V_i$ for $1 \leq i \leq \mu + 3$. Then in such coordinates Σ decomposes as*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_\mu \\ \dot{x}_{\mu+1} \\ \dot{x}_{\mu+2} \\ \dot{x}_{\mu+3} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 & 0 & 0 & * \\ 0 & A_{22} & & \vdots & 0 & 0 & * \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{\mu\mu} & 0 & 0 & * \\ * & * & \cdots & * & A_{(\mu+1)(\mu+1)} & * & * \\ * & * & \cdots & * & 0 & A_{(\mu+2)(\mu+2)} & * \\ 0 & 0 & \cdots & 0 & 0 & 0 & A_{(\mu+3)(\mu+3)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\mu \\ x_{\mu+1} \\ x_{\mu+2} \\ x_{\mu+3} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \\ * \\ * \\ 0 \end{bmatrix} u_1 + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ * \\ * \\ 0 \end{bmatrix} u_\mu + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ B_{\mu+1} \\ 0 \\ 0 \end{bmatrix} u_{\mu+1},$$

$$y_i = C_{ii}x_i + C_{i(\mu+3)}x_{\mu+3}, \quad 1 \leq i \leq \mu.$$

Moreover, the fixed modes are given by $A_{(\mu+2)(\mu+2)}$.

Since the proof follows readily from Lemma 2.10, it will be omitted. It is simply noted that the structure of the matrix multiplying $u_{\mu+1}$ comes from $V_{\mu+1} = \mathcal{Q}^*$.

It is now easy to state and prove the final result.

Theorem 2.12. *Consider the system Σ , (2.1). Let \mathcal{P}_i^* denote the maximal controllability subspace contained in $\ker C_i$. Define $\mathcal{P}^* := \bigcap_{j=1}^\mu \mathcal{P}_j^*$ and let \mathcal{Q}^* denote the maximal controllability subspace contained in \mathcal{P}^* . Then Σ can be rendered noninteractive with regular state variable feedback if and only if*

(a) $\mathcal{P}_i^* \cap \text{Im } B + \bigcap_{j \neq i} (\mathcal{P}_j^* \cap \text{Im } B) = \text{Im } B, 1 \leq i \leq \mu.$

Moreover, if noninteractive control is possible, then a stable noninteractive closed loop can be achieved if and only if the following two additional conditions hold:

- (b) $(A + BF)|(\mathcal{P}^*/\mathcal{Q}^*)$ is stable, where F is any fixed feedback matrix rendering $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ invariant.
- (c) Σ is stabilizable (i.e., $A|(\mathbb{R}^n/C^*)$ is stable where C^* is the controllable subspace of Σ).

Proof. The first statement follows from Corollary 2.3. Concerning noninteracting control with stability, the necessity of (c) is clear; the necessity of (b) is the statement of Lemma 2.9. The sufficiency of (b) and (c) follow from Corollary 2.11 because $(A + BF)(\mathcal{P}^*/\mathcal{Q}^*) = A_{(\mu+2)(\mu+2)}$, $A|(\mathbb{R}^n/C^*) = A_{(\mu+3)(\mu+3)}$, and all other pairs (A_{ii}, B_i) , $1 \leq i \leq \mu + 1$, are easily shown to be controllable. ■

3. Nonlinear Block Noninteracting Control with Stability

Consider now a nonlinear system whose outputs have been grouped into blocks:

$$\Sigma: \begin{cases} \dot{x} = f(x) + \sum_{j=1}^m g_j(x)u_j, \\ y_i = h_i(x), \quad 1 \leq i \leq \mu, \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u_j \in \mathbb{R}$, $y_i \in \mathbb{R}^{p_i}$, and f, g_j, h_i are analytic. The system is said to be *noninteractive* with respect to a given partition $u^T = (u_1, \dots, u_\mu, u_{\mu+1})$ of the inputs, u_i possibly vector valued, if y_i is unaffected by u_j for $i \neq j$. The *noninteractive control problem* is to find, if possible, a *regular static state variable feedback*, $u = \alpha(x) + \beta(x)v$, $|\beta(x)| \neq 0$ for all x (regular refers to the invertibility of β) and a partitioning of the new command inputs v so that the resulting closed-loop system is noninteractive. If an addition, the zero-input dynamics (or drift term) $\dot{x} = f(x) + \sum_{i=1}^m g_i(x)\alpha_i(x)$, where $\alpha(x) = (\alpha_1(x), \dots, \alpha_m(x))^T$, is asymptotically stable about a given equilibrium point x_0 , the problem is said to be *solvable with stability*.

In the following, necessary and sufficient conditions for solving the above problem *locally* will be sought. That is, a solution will be sought on some open neighborhood of a given equilibrium point. The nonlinear geometric tools pioneered in [H3] and [IKGM] and recently synthesized in [I] will be used freely. The development will closely follow that of Section 2 and the recent paper [IG]; consequently many of the proofs will be abridged or omitted all together.

It will be seen that there are two main obstacles to achieving a complete generalization of the previous results to the nonlinear setting. The first is the occurrence of singularities. The second is related to a deficiency in the notion of a controllability distribution of a nonlinear system: there is no “dynamics assignability,” and hence no guarantee of “invariance with stability,” as one has in the linear setting (see, for example, Chapter 5 of [W]).

Criteria for the achievement of noninteraction without the constraint of internal stability are now well known [NS1], [NS2], but, once again, they are not convenient starting points for the additional analysis needed to characterize stabilizing solutions. The approach of Section 2, however, works equally well in the nonlinear setting.

Let $I_1, \dots, I_{\mu+1}$ be a partition of the set $\{1, \dots, m\}$; that is $I_j \subset \{1, \dots, m\}$, $I_j \cap I_k = \emptyset$ for all $j \neq k$, and $\bigcup_{k=1}^{\mu+1} I_k = \{1, \dots, m\}$. These index sets will be used to indicate a partitioning of the inputs, with respect to which a system is to be noninteractive. The notion of noninteraction is now formulated in geometric terms. Whenever convenient, f will also be denoted as g_0 .

Lemma 3.1 [I, Theorems 3.2 and 3.12]. *Let $I_1, \dots, I_{\mu+1}$ be a partition of $\{1, \dots, m\}$ and consider the system*

$$\Sigma: \begin{cases} \dot{x} = g_0(x) + \sum_{k=1}^{\mu+1} \left(\sum_{j \in I_k} g_j(x) u_j \right), \\ y_i = h_i(x), \quad 1 \leq i \leq \mu. \end{cases}$$

Then the following statements are equivalent:

- (a) Σ is noninteractive with respect to the indicated partition of the inputs.
- (b) For $k = 1, 2, \dots, Lg_{i_1} \cdots Lg_{i_k} h_i(x) = 0$ for all x whenever at least one $i_j \notin I_i \cup \{0\}$.
- (c) *There exist analytic involutive distributions $\Delta_1, \dots, \Delta_\mu$ such that, for $1 \leq i \leq \mu$,*
 - (i) $\Delta_i \subset \ker dh_i$,
 - (ii) $[g_k, \Delta_i] \subset \Delta_i, 0 \leq k \leq m$,
 - (iii) $g_j \in \Delta_i$ for all $j \notin I_i \cup \{0\}$.

It is emphasized that at this point the distribution $\Delta_1, \dots, \Delta_\mu$ do not need to be constant dimensional; (c) is easily shown to imply (b) on the open and dense subset where the Δ_i are constant dimensional. But, as the functions in (b) are smooth, vanishing on an open and dense subset entails vanishing everywhere.

Using this lemma, one obtains the following necessary conditions and sufficient conditions for achieving noninteracting control (without the constraint of internal stability). The proof is given in Appendix B. As a point of notation, \mathcal{G} will denote the distribution

$$\mathcal{G}(x) := \text{span}\{g_1(x), \dots, g_m(x)\}; \quad (3.2)$$

throughout the paper we assume that $\mathcal{G}(x)$ has dimension m .

Theorem 3.2. *Suppose that the system (3.1) can be rendered noninteractive with a regular static state feedback. Then there exist controlled invariant distributions $\Delta_1, \dots, \Delta_\mu$ satisfying*

$$\Delta_i \subset \ker dh_i, \quad (3.3)$$

$$\mathcal{G} \cap \Delta_i + \bigcap_{j \neq i} (\mathcal{G} \cap \Delta_j) = \mathcal{G} \quad (3.4)$$

for each $1 \leq i \leq \mu$. Conversely, if in a neighborhood of some point x_0 there exist controlled-invariant distributions $\Delta_1, \dots, \Delta_\mu$ satisfying (3.3) and (3.4), and if in addition, the distributions $\mathcal{G}, \Delta_i, \Delta_i \cap \mathcal{G}$, and $\bigcap_{j \neq i} (\mathcal{G} \cap \Delta_j)$, $1 \leq i \leq \mu$, are constant dimensional, then, on a possibly smaller neighborhood of x_0 there exists a regular static state feedback rendering (3.1) noninteractive.

Equation (3.4) turns out to be the necessary and sufficient condition (modulo singularities) for the local existence of a single feedback rendering the set of distributions $\{\Delta_1, \dots, \Delta_\mu\}$ simultaneously invariant, and hence could be called the *compatibility condition*. Whenever a set of distributions can be made simultaneously invariant with a single feedback, it will be called a *compatible set* (of distributions).

If Δ is an involutive controlled-invariant distribution, and if \mathcal{R} is the maximal controllability distribution contained in Δ , then $\Delta \cap \mathcal{G} = \mathcal{R} \cap \mathcal{G}$. Using this fact, one deduces the following corollary of Theorem 3.2, which will form the basis of all subsequent analysis.

Corollary 3.3. *Suppose that the system (3.1) can be rendered noninteractive with a regular static state feedback. Then there exist controllability distributions $\mathcal{R}_1, \dots, \mathcal{R}_m$ satisfying*

$$\mathcal{R}_i \subset \ker dh_i, \tag{3.5}$$

$$\mathcal{G} \cap \mathcal{R}_i + \bigcap_{j \neq i} (\mathcal{G} \cap \mathcal{R}_j) = \mathcal{G} \tag{3.6}$$

for each $1 \leq i \leq \mu$. Conversely, if in a neighborhood of some point x_0 there exist controllability distributions $\mathcal{R}_1, \dots, \mathcal{R}_\mu$ satisfying (3.5) and (3.6), and if in addition the distributions $\mathcal{G}, \mathcal{R}_i, \mathcal{R}_i \cap \mathcal{G}$, and $\bigcap_{j \neq i} (\mathcal{G} \cap \mathcal{R}_j)$, $1 \leq i \leq \mu$, are constant dimensional, then, on a possibly smaller neighborhood of x_0 , there exists a regular static state feedback rendering (3.1) noninteractive.

A set of controllability distributions $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ satisfying the constant dimensionality assumptions of Corollary 3.3 will be called a *regular set*; at times the qualifier, “about the point x_0 ,” will be added for clarity. If in addition $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ satisfies (3.5) and (3.6), it will be called a *regular solution to the noninteracting control problem*. Note that if $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ is a regular solution, then

$$\mathcal{R} := \bigcap_{j=1}^{\mu} \mathcal{R}_j \tag{3.7}$$

is a controlled-invariant distribution (at least on a neighborhood of x_0) since it is the intersection of compatible distributions. The maximal controllability distribution contained in \mathcal{R} will be denoted by \mathcal{L} .

The goal now, as in the linear analysis, is to identify the *fixed modes* [IG] associated with a given regular solution. For linear systems, the analysis was implicitly carried out on an open set (all of \mathbb{R}^n) about an equilibrium point, the origin, and the feedback in question always preserved the equilibrium point. These details will have to be assumed when dealing with nonlinear systems.

Temporarily, let \mathcal{R} be any involutive controlled-invariant distribution, \mathcal{L} the maximal controllability distribution contained in \mathcal{R} (so that $\mathcal{R} \cap \mathcal{G} = \mathcal{L} \cap \mathcal{G}$), let x_0 be an equilibrium point of (3.1) (i.e., $f(x_0) = 0$), and assume that $\mathcal{R}, \mathcal{L}, \mathcal{L} \cap \mathcal{G}$, and \mathcal{G} have constant dimension around x_0 . Since \mathcal{R} is controlled-invariant, there exists a feedback function $\alpha(x)$ such that $[f + g\alpha, \mathcal{R}] \subset \mathcal{R}$ (this entails automatically that $[f + g\alpha, \mathcal{L}] \subset \mathcal{L}$). Moreover, α can always be chosen such that $\alpha(x_0) = 0$, i.e., it preserves the equilibrium point [IKGM]. Therefore $f + g\alpha$ is tangent to M , the leaf of \mathcal{R} passing through x_0 , and $f + g\alpha|M$ is a well-defined vector field of M . Consider the quotient manifold M/\mathcal{L} which, because of constant dimensionality assumptions, locally around x_0 is a smooth manifold of dimension $\dim \mathcal{R} - \dim \mathcal{L}$. Let π denote the canonical projection $\pi: M \rightarrow M/\mathcal{L}$. Since \mathcal{L} is invariant under $f + g\alpha|M$, locally around $\pi(x_0)$ there is a well-defined induced vector field on M/\mathcal{L} ; i.e., a vector field

\hat{f} such that $\pi_*(f + g\alpha|M) = \hat{f}\pi$. This vector field will be denoted by $f + g\alpha|(M/\mathcal{Q})$. Note that if (x_1, x_2) is a choice of local coordinates on M such that $\mathcal{Q} = \text{span}\{\partial/\partial x_1\}$, the submanifold $L = \{(x_1, x_2): x_1 = 0\}$ is a local representation of M/\mathcal{Q} . In these coordinates the vector field $f + g\alpha|M$ is represented as

$$f + g\alpha|M = \begin{bmatrix} \tilde{f}_1(x_1, x_2) \\ \tilde{f}_2(x_2) \end{bmatrix}$$

and $\tilde{f}_2(x_2)$ is a local representation of $f + g\alpha|(M/\mathcal{Q})$.

It is now claimed that the vector field $f + g\alpha|(M/\mathcal{Q})$ is independent of the particular choice of α , as long as α preserves the equilibrium point and renders \mathcal{R} invariant. To see this, let α^1 and α^2 be two such feedbacks. Since \mathcal{R} and \mathcal{Q} are both controlled-invariant and $\mathcal{Q} \subset \mathcal{R}$, one can always find an $m \times m$ matrix of functions β , $\beta(x_0)$ invertible, such that $[(g\beta)_j, \mathcal{R}] \subset \mathcal{R}$, $[(g\beta)_j, \mathcal{Q}] \subset \mathcal{Q}$, $1 \leq j \leq m$, and $\mathcal{Q} \cap \mathcal{S} = \text{span}\{(g\beta)_1, \dots, (g\beta)_k\}$. Since $[f + g\alpha^i, \mathcal{R}] \subset \mathcal{R}$, one deduces that

$$[f + (g\beta)\beta^{-1}\alpha^1 - f - (g\beta)\beta^{-1}\alpha^2, \mathcal{R}] \subset \mathcal{R};$$

that is, for all vector fields $Y \in \mathcal{R}$,

$$[(g\beta)\beta^{-1}(\alpha^1 - \alpha^2), Y] \in \mathcal{R}.$$

Hence,

$$\sum_{j=1}^m \{[(g\beta)_j, Y](\beta^{-1}\alpha^1 - \beta^{-1}\alpha^2)_j - (g\beta)_j L_Y(\beta^{-1}\alpha^1 - \beta^{-1}\alpha^2)_j\} \in \mathcal{R}$$

for all $Y \in \mathcal{R}$. Now, using $[(g\beta)_j, Y] \in \mathcal{R}$ and $\mathcal{R} \cap \text{span}\{(g\beta)_{k+1}, \dots, (g\beta)_m\} = \{0\}$, one obtains that, for $k + 1 \leq j \leq m$,

$$L_Y(\beta^{-1}\alpha^1 - \beta^{-1}\alpha^2)_j = 0 \quad \text{for all } Y \in \mathcal{R}.$$

This implies that $(\beta^{-1}\alpha^1 - \beta^{-1}\alpha^2)_j$ is constant on leaves of \mathcal{R} , and therefore, since $\alpha^1(x_0) = \alpha^2(x_0)$, that, for $k + 1 \leq j \leq m$, $(\beta^{-1}\alpha^1)_j$ and $(\beta^{-1}\alpha^2)_j$ agree on M . To finish the proof of the claim, note that, by construction of β , $0 = \pi_*(g\beta)_i$ for $1 \leq i \leq k$. Hence, on L one has the following string of equalities:

$$\begin{aligned} \pi_* \left[f + \sum_{i=1}^m g_i \alpha_i^1 \right] &= \pi_* \left[f + \sum_{i=1}^m (g\beta)_i (\beta^{-1}\alpha^1)_i \right] \\ &= \pi_* \left[f + \sum_{i=k+1}^m (g\beta)_i (\beta^{-1}\alpha^1)_i \right] \\ &= \pi_* \left[f + \sum_{i=k+1}^m (g\beta)_i (\beta^{-1}\alpha^2)_i \right] \\ &= \pi_* \left[f + \sum_{i=1}^m (g\beta)_i (\beta^{-1}\alpha^2)_i \right] \\ &= \pi_* \left[f + \sum_{i=1}^m g_i \alpha_i^2 \right]. \end{aligned}$$

This establishes the claim. Hence all of the above can be summarized by stating that

$f + g\alpha(M/\mathcal{Q})$ is independent of the particular choice of α , as long as α renders \mathcal{R} invariant and preserves the equilibrium point.

Applying the above analysis to the noninteracting control problem gives

Theorem 3.4. *Let x_0 be an equilibrium point of (3.1). Suppose that on a neighborhood of x_0 , $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ is a regular solution to the noninteracting control problem, and, moreover, that $\mathcal{R} := \bigcap_{j=1}^\mu \mathcal{R}_j$ and \mathcal{Q} , the maximal controllability distribution contained in \mathcal{R} , are constant dimensional. Let $u = \alpha(x) + \beta(x)v$ be any regular static state feedback constructed from $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ which renders (3.1) noninteractive and preserves the equilibrium point. Then the leaf of \mathcal{R} passing through x_0 , denoted M , is an invariant submanifold of $\dot{x} = (f + g\alpha)(x)$ and, moreover, $f + g\alpha(M/\mathcal{Q})$ does not depend upon the particular choice of α .*

The vector field $f + g\alpha(M/\mathcal{Q})$ will be called the *fixed dynamics* (or fixed modes) of the solution $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$. Clearly this dynamics must be stable if an asymptotically stable noninteractive closed-loop is to be constructed from $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$. For the class of linear systems, it was shown in Section 2 that the *minimal* fixed dynamics was obtained by choosing the *maximal* solution to the noninteracting control problem. This is developed next.

Lemma 3.5. *Suppose that the noninteractive control problem admits a regular solution $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ and that the set $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is regular. Then the set of controllability distributions $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is locally compatible and so, in particular, $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is a regular solution. Moreover, there locally exists a regular static state feedback and a partition of the new command inputs, $u = \alpha(x) + \sum_{i=1}^{\mu+1} (\sum_{j \in I_i} \beta_j(x)v_j)$, such that*

- (a) $[f + g\alpha, \mathcal{P}_i^*] \subset \mathcal{P}_i^*$, $[(g\beta)_k, \mathcal{P}_i^*] \subset \mathcal{P}_i^*$, $1 \leq k \leq m$,
- (b) $[f + g\alpha, \mathcal{R}_i] \subset \mathcal{R}_i$, $[(g\beta)_k, \mathcal{R}_i] \subset \mathcal{R}_i$, $1 \leq k \leq m$,
- (c) $(g\beta)_j \in \mathcal{R}_i$ for all $j \notin I_i$.

Proof. See Appendix B. ■

Lemma 3.5 yields the following result, relating arbitrary regular solutions to the maximal solution, when the latter is also regular.

Theorem 3.6. *Suppose that $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ is any regular solution to the noninteracting control problem and suppose that the maximal solution $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is also regular. Then, for each $1 \leq i \leq \mu$,*

$$\mathcal{R}_i + \mathcal{Q}^* = \mathcal{P}_i^*.$$

Proof. Simply apply the linear argument point-wise to obtain the result; in doing so, one should note that $\bigcap_{i=1}^\mu (\mathcal{P}_i^* \cap \mathcal{S})$ is constant dimensional as a consequence of the regularity of the maximal solution. ■

From this theorem, one sees that whenever $\mathcal{Q}^* = \{0\}$, there is only one solution to the noninteracting control problem, and hence, in this case, Theorem 3.4 constitutes a general necessary condition for the existence of a noninteractive stabilizing solution. This, already, is a generalization of Theorem 5.1 of [IG]. When $\mathcal{Q}^* \neq \{0\}$, one must dig a little further.

Theorem 3.7. *Suppose that $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ is any regular solution to the noninteracting control problem and suppose that the maximal solution $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is also regular. Let $\mathcal{R} := \bigcap_{j=1}^\mu \mathcal{R}_j$, $\mathcal{P}^* := \bigcap_{j=1}^\mu \mathcal{P}_j^*$, and $\mathcal{Q}^* =$ maximal controllability distribution contained in \mathcal{P}^* . Then, if $\mathcal{R} \cap \mathcal{Q}^*$ is constant dimensional,*

$$\mathcal{P}^* = \mathcal{R} + \mathcal{Q}^*.$$

Proof. See Appendix B. ■

As an immediate corollary one has

Corollary 3.8. *Let \mathcal{P}^* , \mathcal{Q}^* , and \mathcal{R} be as above. Suppose that \mathcal{P}^* , \mathcal{Q}^* , and $\mathcal{Q}^* \cap \mathcal{R}$ all have constant dimension about some equilibrium point x_0 , and that α preserves the equilibrium point and renders $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ invariant. Let M^* denote the leaf of \mathcal{P}^* passing through x_0 and M denote the leaf of \mathcal{R} passing through x_0 . Then*

$$f + g\alpha \left| \frac{M^*}{\mathcal{Q}^*} \right. = f + g\alpha \left| \frac{M}{\mathcal{R} \cap \mathcal{Q}^*} \right.$$

To summarize what has been established up to this point, by Lemma 3.5 there exists a regular static feedback $u = \alpha(x) + \beta(x)v$ rendering $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ simultaneously invariant; hence the same feedback also renders \mathcal{Q}^* , \mathcal{P}^* , \mathcal{Q} , and \mathcal{R} invariant. By Theorem 3.4, the fixed dynamics $f + g\alpha|(M/\mathcal{Q})$ and $f + g\alpha|(M^*/\mathcal{Q}^*)$ are independent of the particular choice of α , as long as x_0 is an equilibrium point preserved by α . Therefore, they will each be part of the drift (i.e., zero input) dynamics of any noninteractive loop obtained from $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ and $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$, respectively. By construction, everything is invariant under $f + g\alpha$, and thus $f + g\alpha|(M/(\mathcal{R} \cap \mathcal{Q}^*))$ is well defined. Now, since $\mathcal{Q} \subset \mathcal{R}$ and $\mathcal{Q} \subset \mathcal{Q}^*$, it follows that $\mathcal{Q} \subset \mathcal{Q}^* \cap \mathcal{R}$, so the dynamics $f + g\alpha|(M/(\mathcal{R} \cap \mathcal{Q}^*))$ is contained within the dynamics $f + g\alpha|(M/\mathcal{Q})$; more precisely, $f + g\alpha|(M/(\mathcal{R} \cap \mathcal{Q}^*))$ is a projection of $f + g\alpha|(M/\mathcal{Q})$. Hence, applying Corollary 3.8, one obtains that the dynamics $f + g\alpha|(M^*/\mathcal{Q}^*)$ is a projection of the dynamics $f + g\alpha|(M/\mathcal{Q})$. If one chooses local coordinates (x_1, x_2, x_3, x_4) on M^* such that $\mathcal{Q} = \text{span}\{\partial/\partial x_1\}$, $\mathcal{Q}^* \cap \mathcal{R} = \text{span}\{\partial/\partial x_1, \partial/\partial x_2\}$, $\mathcal{R} = \text{span}\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_4\}$, and $\mathcal{Q}^* = \text{span}\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3\}$, then the vector field $f + g\alpha|M^*$ is represented as $f_1(x_1, x_2, x_3, x_4)(\partial/\partial x_1) + f_2(x_2, x_3, x_4)(\partial/\partial x_2) + f_3(x_3)(\partial/\partial x_3) + f_4(x_4)(\partial/\partial x_4)$, and $f + g\alpha|M = f_1(x_1, x_2, 0, x_4)(\partial/\partial x_1) + f_2(x_2, 0, x_4)(\partial/\partial x_2) + f_4(x_4)(\partial/\partial x_4)$, $f + g\alpha|(M/\mathcal{Q}) = f_2(x_2, 0, x_4)(\partial/\partial x_2) + f_4(x_4)(\partial/\partial x_4)$, $f + g\alpha|(M^*/\mathcal{Q}^*) = f + g\alpha|(M/(\mathcal{R} \cap \mathcal{Q}^*)) = f_4(x_4)(\partial/\partial x_4)$.

This analysis yields the following “necessary” condition for the achievement of noninteracting control with stability.

Theorem 3.9. *Suppose that in a neighborhood of an equilibrium point x_0 , $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is a regular solution to the noninteracting control problem. Suppose in addition that \mathcal{P}^* and \mathcal{Q}^* have constant dimension near x_0 and that $u = \alpha(x) + \beta(x)v$ is any regular static state feedback rendering $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ invariant. Then, if the fixed dynamics $f + g\alpha|(M^*/\mathcal{Q}^*)$ is not asymptotically stable, there does not exist any regular solution $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$, with \mathcal{R} , \mathcal{Q} , and $\mathcal{R} \cap \mathcal{Q}^*$ constant dimensional, that admits an asymptotically stable noninteractive closed loop.*

The above result is rather weak because there is no effective means of testing the hypotheses, due to the assumption on $\mathcal{R} \cap \mathcal{Q}^*$. However, it does show that the only other way for a stabilizing noninteractive control law to exist is through the occurrence of singularities.

The next step is to investigate whether the stability of the fixed dynamics constitutes a sufficient condition for achieving noninteracting control with asymptotic stability. It will be shown that if one excludes critical asymptotic stability, then this is indeed the case. This will be accomplished via a general decomposition result, which is of independent interest.

Lemma 3.10. *Fix $x_0 \in \mathbb{R}^n$. Suppose that $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ is a regular solution about x_0 to the noninteractive control problem and suppose that the maximal solution $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is also regular near x_0 ; moreover, assume that, \mathcal{R} , \mathcal{Q} , \mathcal{P}^* , \mathcal{Q}^* , and $\mathcal{R} \cap \mathcal{Q}^*$ have constant dimension near x_0 . Finally, suppose that C^* , the strong accessibility distribution, also has constant dimension near x_0 . Then there exists an open set about x_0 , and a coordinate system $x = (x_1, \dots, x_{\mu+3})$, each x_i possibly vector valued, such that*

$$\begin{aligned} C^* &= \text{span} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{\mu+2}} \right\}, \\ \mathcal{R}_i &= \text{span} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{i-1}}, \frac{\partial}{\partial x_{i+1}}, \dots, \frac{\partial}{\partial x_{\mu+2}} \right\}, \\ \mathcal{P}_i^* &= \mathcal{R}_i \oplus \text{span} \left\{ \frac{\partial}{\partial x_i^b} \right\}, \\ \mathcal{Q}^* \cap \mathcal{R} &= \text{span} \left\{ \frac{\partial}{\partial x_{\mu+1}} \right\}, \\ \mathcal{Q} &= \text{span} \left\{ \frac{\partial}{\partial x_{\mu+1}^a} \right\}, \end{aligned}$$

where, for $1 \leq i \leq \mu + 1$, $x_i = (x_i^a, x_i^b)$ is a further splitting of x_i .

Proof. See Appendix B. ■

In terms of the above coordinates, $\mathcal{R} = \text{span}\{\partial/\partial x_{\mu+1}, \partial/\partial x_{\mu+2}\}$, $\mathcal{P}^* = \mathcal{R} \oplus \text{span}\{\partial/\partial x_1^b, \dots, \partial/\partial x_\mu^b\}$, $\mathcal{R}/\mathcal{Q} = \text{span}\{\partial/\partial x_{\mu+1}^b, \partial/\partial x_{\mu+2}\}$ and $\mathcal{R}/(\mathcal{R} \cap \mathcal{Q}^*) = \text{span}\{\partial/\partial x_{\mu+2}\}$. Hence, one could use this result in conjunction with Corollary 3.8

to show how the fixed dynamics of $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ contains the fixed dynamics of $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$.

Using this lemma, the internal structure of a general block noninteractive system is exhibited, along with a local coordinate representation of the fixed-dynamics; see also Theorem 4.2 of [IG].

Corollary 3.11. *Fix $x_0 \in \mathbb{R}^n$. Suppose that in a neighborhood of x_0 , $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ is a regular solution to the decoupling problem, that \mathcal{P}^* and \mathcal{Q}^* have constant dimension, and that Σ , (3.1), has already been rendered noninteractive with a feedback which is compatible with $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$. Then in the coordinates $(x_1, \dots, x_{\mu+3})$ of Lemma 3.10, where \mathcal{R}_i is taken to be equal to \mathcal{P}_i^* , Σ decomposes as*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + \sum_{k \in I_1} g_{k1}(x_1)u_k, \\ &\vdots \\ \dot{x}_\mu &= f_\mu(x_\mu) + \sum_{k \in I_\mu} g_{k\mu}(x_\mu)u_k, \\ \dot{x}_{\mu+1} &= f_{\mu+1}(x_1, \dots, x_{\mu+3}) + \sum_{j=1}^m g_{j(\mu+1)}(x_1, \dots, x_{\mu+3})u_j, \\ \dot{x}_{\mu+2} &= f_{\mu+2}(x_1, \dots, x_\mu, x_{\mu+2}, x_{\mu+3}) + \sum_{j=1}^{\mu} \left(\sum_{k \in I_j} g_{k(\mu+2)}(x_1, \dots, x_\mu, x_{\mu+2}, x_{\mu+3})u_k \right), \\ \dot{x}_{\mu+3} &= f_{\mu+3}(x_{\mu+3}), \\ y_i &= h_i(x_i, x_{\mu+3}), \quad 1 \leq i \leq \mu, \end{aligned}$$

where $\{I_j\}_{j=1}^{\mu+1}$ is a suitable partition of the inputs. Moreover, if x_0 is an equilibrium point, the coordinates can always be chosen in such a way that $x_0 = 0$, and then the fixed dynamics is precisely

$$\dot{x}_{\mu+2} = f_{\mu+2}(0, \dots, 0, x_{\mu+2}, 0). \quad (3.8)$$

Since the proof, as in the case of linear system, follows readily from Lemma 3.10, it will be omitted. It is now possible to state and prove a sufficient condition for decoupling with stability.

Theorem 3.12. *Consider the system Σ , (3.1). Let \mathcal{P}_i^* denote the maximal controllability distribution contained in $\ker dh_i$. Define $\mathcal{P}^* := \bigcap_{j=1}^{\mu} \mathcal{P}_j^*$, \mathcal{Q}^* the maximal controllability distribution contained in \mathcal{P}^* , and $\mathcal{G} = \text{span}\{g_1, \dots, g_m\}$. Assume that \mathcal{G} , \mathcal{Q}^* , \mathcal{P}^* , \mathcal{P}_i^* , $\mathcal{P}_i^* \cap \mathcal{G}$, and $\bigcap_{j \neq i} (\mathcal{P}_j^* \cap \mathcal{G})$ are all constant dimensional on a neighborhood of a given point $x_0 \in \mathbb{R}^n$. Then there exists a neighborhood of x_0 on which is defined a regular static state feedback rendering Σ noninteractive if and only if*

$$\mathcal{P}_i^* \cap \mathcal{G} + \bigcap_{j \neq i} (\mathcal{P}_j^* \cap \mathcal{G}) = \mathcal{G} \quad (3.9)$$

for $1 \leq i \leq \mu$. Moreover, if in addition x_0 is an equilibrium point, the linearization of Σ about x_0 is (asymptotically) stabilizable and the linearization of the fixed-dynamics (3.8) about x_0 is asymptotically stable, then there exists a regular static state feedback

which simultaneously achieves noninteraction and asymptotically stabilizes the drift dynamics.

Proof. The first statement follows from Corollary 3.3. The second statement is an easy consequence of Corollary 3.11. ■

4. Conclusions

This paper has extended the analyses of [G1] and [IG] to the block noninteracting control problem, showing that imposing a noninteractive structure by means of static state feedback entails the assignment of a certain fixed internal dynamics. This dynamics, which is independent of the particular static feedback used to achieve noninteraction, must necessarily be asymptotically stable if an asymptotically stable noninteractive system is sought. Conversely, if the fixed dynamics is asymptotically, but not critically, stable, and if the original system is asymptotically, but not critically, stabilizable, then noninteractive control with internal stability can be achieved with a static feedback. This shows that whenever a system is (noncritically) stabilizable, the only obstruction to achieving noninteracting control with stability is purely geometric and is characterized by the fixed dynamics. When the solution to the noninteracting control problem is unique (i.e., $\mathcal{Q}^* = \{0\}$), then even in cases of critical stability, an asymptotically stable noninteractive closed loop can be obtained if one allows dynamic compensation [IG].

Appendix A

A.1. Proof of Lemma 2.5

A preliminary lemma is first established.

Lemma A.1. Let $\{V_1, \dots, V_p\}$ be a collection of subspaces of some vector space X . Then $\{V_1, \dots, V_p\}$ is simultaneously decomposable in the sense that there exists a collection of subspaces $\{X_1, \dots, X_{p+1}\}$ contained in X satisfying

$$(a) \quad X = X_1 \oplus X_2 \oplus \dots \oplus X_{p+1} \text{ and}$$

$$(b) \quad V_i = \sum_{j \neq i} X_j$$

if and only if, for all $1 \leq i \leq p$,

$$V_i + \bigcap_{j \neq i} V_j = X. \quad (A.1)$$

Proof. Necessity being obvious, only sufficiency will be proved; the proof will be constructive. Define subspaces $W_i \subset X^*$, the dual of X , by $W_i := V_i^\perp$. Dualizing (A.1) yields

$$W_i \cap \sum_{j \neq i} W_j = \{0\}, \quad (A.2)$$

and hence the subspaces $\{W_1, \dots, W_p\}$ are independent. Therefore, there exists a

subspace W_{p+1} such that

$$W_1 \oplus W_2 \oplus \cdots \oplus W_{p+1} = X^*. \quad (\text{A.3})$$

Define X_i by

$$X_i := \left(\sum_{j \neq i} W_j \right)^\perp. \quad (\text{A.4})$$

It is straightforward to verify that the set $\{X_1, \dots, X_{p+1}\}$ so defined is independent; that is, for all $1 \leq i \leq p+1$,

$$X_i \cap \sum_{j \neq i} X_j = \{0\}. \quad (\text{A.5})$$

It is now claimed that

$$V_i = \sum_{j \neq i} X_j. \quad (\text{A.6})$$

By duality, (A.6) holds if and only if

$$W_i = \bigcap_{j \neq i} X_j^\perp. \quad (\text{A.7})$$

But $X_j^\perp = \sum_{k \neq j} W_k$, so that

$$\bigcap_{j \neq i} X_j^\perp = \bigcap_{j \neq i} \left(\sum_{k \neq j} W_k \right) = W_i. \quad \blacksquare$$

Continuing the proof of Lemma 2.5, since, for each $1 \leq i \leq \mu$, $\text{Im } B \cap \mathcal{R}_i + \bigcap_{j \neq i} (\text{Im } B \cap \mathcal{R}_j) = \text{Im } B$, Lemma A.1 guarantees that one can decompose $\text{Im } B$ in such a way that $\text{Im } B = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_{\mu+1}$ and $\text{Im } B \cap \mathcal{R}_i = \sum_{j \neq i} \mathcal{B}_j$. Let \bar{B}_i be matrices such that $\text{Im } \bar{B}_i = \mathcal{B}_i$, columns of \bar{B}_i linearly independent, and define $\bar{B} = [\bar{B}_1 | \bar{B}_2 | \cdots | \bar{B}_{\mu+1}]$.

Since $A\mathcal{R}_i \subset \mathcal{R}_i + \text{Im } B$, and $\mathcal{R}_i \cap \text{Im } B = \sum_{j \neq i} \mathcal{B}_j$, it follows that

$$A\mathcal{R}_i \subset \mathcal{R}_i + \text{Im } \bar{B}_i. \quad (\text{A.8})$$

Similarly, since \mathcal{P}_i^* is controlled-invariant and $\mathcal{R}_i \subset \mathcal{P}_i^*$, one has

$$A\mathcal{P}_i^* \subset \mathcal{P}_i^* + \text{Im } \bar{B}_i. \quad (\text{A.9})$$

Therefore, for each $1 \leq i \leq \mu$, there exists a matrix \bar{F}_i such that

$$(A + \bar{B}_i \bar{F}_i) \mathcal{P}_i^* \subset \mathcal{P}_i^* \quad (\text{A.10})$$

and

$$(A + \bar{B}_i \bar{F}_i) \mathcal{R}_i \subset \mathcal{R}_i. \quad (\text{A.11})$$

Now define $\bar{F} = (\bar{F}_1^T, \bar{F}_2^T, \dots, \bar{F}_{\mu+1}^T)^T$, where $\bar{F}_{\mu+1}$ is chosen arbitrarily so that \bar{F} is an $m \times n$ matrix. Choose a matrix G , $|G| \neq 0$, such that $\bar{B} = BG$ and partition $G = [G_1 | G_2 | \cdots | G_{\mu+1}]$ so that $\bar{B}_i = BG_i$. Finally, define $F = G\bar{F}$.

It is now claimed that $u = Fx + \sum_{i=1}^{\mu+1} G_i v_i$ is the desired feedback and partitioning of the new command inputs. Part (c) of the lemma is clear, since, by construction,

$$\text{Im } BG_i = \text{Im } \bar{B}_i = \mathcal{B}_i, \quad (\text{A.12})$$

and $\mathcal{B}_j \subset \mathcal{R}_i$ for all $j \neq i$. To show part (b), note that $A + BF = A + \sum_{j=1}^{\mu+1} \bar{B}_j \bar{F}_j$, and consider

$$(A + BF)\mathcal{R}_i = (A + \bar{B}_i \bar{F}_i)\mathcal{R}_i + \left(\sum_{j \neq i} \bar{B}_j \bar{F}_j \right) \mathcal{R}_i. \quad (\text{A.13})$$

Then $(A + \bar{B}_i \bar{F}_i)\mathcal{R}_i \subset \mathcal{R}_i$ by construction of \bar{F}_i , and, for $j \neq i$, $\bar{B}_j \bar{F}_j \mathcal{R}_j \subset \text{Im } \bar{B}_j = \mathcal{B}_j \subset \mathcal{R}_i$. Part (a) follows in a similar manner. ■

A.2. Proof of Theorem 2.2

The necessity of the condition (2.2) follows easily from Lemma 2.1. The proof of its sufficiency follows exactly the same steps as the proof of Lemma 2.5.

A.3. Proof of Theorem 2.7

(a) *Preliminaries.* Let $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$ be any solution to the noninteracting control problem, $\mathcal{R} := \bigcap_{j=1}^{\mu} \mathcal{R}_j$, and $\mathcal{Q} =$ maximal controllability subspace contained in \mathcal{R} . Let $\{\mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*\}$ be the maximal solution, $\mathcal{P}^* := \bigcap_{j=1}^{\mu} \mathcal{P}_j^*$, and $\mathcal{Q}^* =$ maximal controllability subspace contained in \mathcal{P}^* . Finally, let F be any feedback matrix rendering all of these subspaces simultaneously invariant (see Lemma 2.5) and define $\bar{A} := A + BF$.

Decompose $\mathcal{B} := \text{Im } B$ such that (see Proof of Lemma 2.5 in the appendix)

- (a) $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \dots \oplus \mathcal{B}_{\mu+1}$,
 (b) $\mathcal{R}_i \cap \text{Im } B = \sum_{j \neq i} \mathcal{B}_j$.

Further decompose \mathcal{B}_i as

- (c) $\mathcal{B}_i = \mathcal{B}_i^1 \oplus \mathcal{B}_i^2$,
 (d) $\mathcal{P}_i^* \cap \text{Im } B = \sum_{j \neq i} \mathcal{B}_j \oplus \mathcal{B}_i^1$.

This latter splitting is possible because $\mathcal{R}_i \subset \mathcal{P}_i^*$, which implies that $\mathcal{R}_i \cap \text{Im } B \subset \mathcal{P}_i^* \cap \text{Im } B$, so that $\mathcal{P}_i^* \cap \text{Im } B = \sum_{j \neq i} \mathcal{B}_j \oplus (\mathcal{P}_i^* \cap \mathcal{B}_i)$; hence $\mathcal{B}_i^1 = \mathcal{P}_i^* \cap \mathcal{B}_i$.

This then yields

- (i) $\mathcal{R}_i = \sum_{j \neq i} \langle \bar{A} | \mathcal{B}_j \rangle$,
 (ii) $\mathcal{P}_i^* = \sum_{j \neq i} \langle \bar{A} | \mathcal{B}_j \rangle + \langle \bar{A} | \mathcal{B}_i^1 \rangle$,
 (iii) $\mathcal{Q}^* = \sum_{j=1}^{\mu} \langle \bar{A} | \mathcal{B}_j^1 \rangle + \langle \bar{A} | \mathcal{B}_{\mu+1} \rangle$.

Finally, express \mathcal{Q}^* as

$$\mathcal{Q}^* = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \dots \oplus \mathcal{S}_{\mu+1},$$

where, for $1 \leq i \leq \mu$, $\mathcal{S}_i \subset \langle \bar{A} | \mathcal{B}_i^1 \rangle$ and $\mathcal{S}_{\mu+1} \subset \langle \bar{A} | \mathcal{B}_{\mu+1} \rangle$.

Lemma A.2. Let C^* denote the controllable subspace of Σ , (2.1). Then, for each i ,

$$\mathcal{R}_i + \bigcap_{j \neq i} \mathcal{R}_j = C^*.$$

Proof. Follows from $\mathcal{R}_i \cap \text{Im } B + \bigcap_{j \neq i} (\mathcal{R}_j \cap \text{Im } B) = \text{Im } B$ and the invariance of $\{\mathcal{R}_i\}_{i=1}^{\mu}$ under \bar{A} . ■

Lemma A.3. $\dim \mathcal{R} = \sum_{j=1}^{\mu} \dim \mathcal{R}_j - (\mu - 1) \dim C^*$.

Proof. Follows from repeated application of the identity: $\dim(\mathcal{S} \cap \mathcal{T}) = \dim(\mathcal{S}) + \dim(\mathcal{T}) - \dim(\mathcal{S} \cup \mathcal{T})$, in conjunction with Lemma A.2. ■

Lemma A.4. *Suppose $\mathcal{Q}^* \cap \mathcal{R} = \{0\}$. Then $\mathcal{P}^* = \mathcal{R} + \mathcal{Q}^*$.*

Proof. Since both \mathcal{R} and \mathcal{Q}^* are contained in \mathcal{P}^* , it suffices to show that $\dim \mathcal{P}^* \leq \dim \mathcal{R} + \dim \mathcal{Q}^*$. From Theorem 2.6, $\mathcal{P}_i^* = \mathcal{R}_i + \mathcal{Q}^*$, which by definition of $\{\mathcal{S}_i\}_{i=1}^{\mu+1}$ can be written as $\mathcal{P}_i^* = \mathcal{R}_i + \mathcal{S}_i$. Hence, using Lemma A.3, and $\mathcal{S}_{\mu+1} \subset \mathcal{Q} \subset \mathcal{Q}^* \cap \mathcal{R} = \{0\}$,

$$\begin{aligned} \dim \mathcal{P}^* &= \sum_{i=1}^{\mu} \dim(\mathcal{R}_i + \mathcal{S}_i) - (\mu - 1) \dim C^* \\ &\leq \sum_{i=1}^{\mu} \dim \mathcal{R}_i + \sum_{i=1}^{\mu} \dim \mathcal{S}_i - (\mu - 1) \dim C^* \\ &\leq \dim \mathcal{R} + \dim \mathcal{Q}^*, \end{aligned}$$

where Lemma A.3 has been used a second time along with the fact that $\mathcal{Q}^* = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_{\mu+1}$. ■

Lemma A.4 proves Theorem 2.7 for the special case $\mathcal{Q}^* \cap \mathcal{R} = \{0\}$. The next idea is to attain this situation ($\mathcal{Q}^* \cap \mathcal{R} = \{0\}$) by passing to the quotient “ $\Sigma | (\mathbb{R}^n / (\mathcal{Q}^* \cap \mathcal{R}))$.”

(b) *Passing to the Quotient.* Define $K := \mathcal{Q}^* \cap \mathcal{R}$. Then K is a subspace which is compatible with, and contained within, $\mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{R}, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*, \mathcal{P}^*, \mathcal{Q}^*$. Let π denote the canonical projection $\pi := \mathbb{R}^n \rightarrow \mathbb{R}^n / K$, and whenever convenient, let a $\hat{\cdot}$ (hat) above a quantity denote its projection; that is, $\hat{\mathcal{R}} := \pi(\mathcal{R})$, etc.

Let F be any feedback matrix rendering $\mathcal{R}_1, \dots, \mathcal{R}_\mu, \mathcal{P}_1^*, \dots, \mathcal{P}_\mu^*$ invariant, and, hence, also $\mathcal{R}, \mathcal{P}^*, \mathcal{Q}^*$, and K . Then the system

$$\begin{aligned} \dot{x} &= (A + BF)x + Bu, \\ y_i &= C_i x, \quad 1 \leq i \leq \mu, \end{aligned}$$

naturally projects [W] to $\hat{X} := \mathbb{R}^n / K$, as

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \\ y_i &= \hat{C}_i \hat{x}, \quad 1 \leq i \leq \mu, \end{aligned}$$

where $\hat{x} := \pi(x)$, $\hat{A}\hat{x} := \pi[(A + BF)x]$, $\hat{B} := \pi \circ B$, and $\hat{C}_i := C_i \circ \pi$. In the following, $A + \hat{B}F$ will be denoted as \hat{A} .

In order to reduce the proof of Theorem 2.7 to an application of Lemma A.4, the following points must be established:

- (i) The set of subspaces $\{\hat{\mathcal{R}}_1, \dots, \hat{\mathcal{R}}_\mu, \hat{\mathcal{R}}\}$ satisfies:
- (a) $\hat{\mathcal{R}}_i$ is a controllability subspace, $1 \leq i \leq \mu$,

- (b) $\hat{\mathcal{R}}_i \cap \hat{\mathcal{B}} + \bigcap_{j \neq i} (\hat{\mathcal{R}}_j \cap \hat{\mathcal{B}}) = \hat{\mathcal{B}}$,
 (c) $\hat{\mathcal{R}} = \bigcap_{j=1}^{\mu} \hat{\mathcal{R}}_j$.
 (ii) The set of subspaces $\{\hat{\mathcal{P}}_1^*, \dots, \hat{\mathcal{P}}_{\mu}^*, \hat{\mathcal{P}}^*, \hat{\mathcal{Q}}^*\}$ satisfies:
 (d) $\hat{\mathcal{P}}_i^*$ is the maximal controllability subspace contained in $\ker \hat{C}_i$,
 (e) $\hat{\mathcal{Q}}^*$ is the maximal controllability subspace contained in $\hat{\mathcal{P}}^*$.
 (iii) $\hat{\mathcal{Q}}^*$ and $\hat{\mathcal{R}}$ satisfy
 (f) $\hat{\mathcal{Q}}^* \cap \hat{\mathcal{R}} = \{0\}$.

Then the quotient system satisfies $\hat{\mathcal{P}}^* = \hat{\mathcal{R}} + \hat{\mathcal{Q}}^*$, which yields $\hat{\mathcal{P}}^* + K = \pi^{-1}(\hat{\mathcal{P}}^*) = \pi^{-1}(\hat{\mathcal{R}} + \hat{\mathcal{Q}}^*) = \mathcal{R} + \mathcal{Q}^* + K = \mathcal{R} + \mathcal{Q}^*$; i.e., $\mathcal{P}^* \subset \mathcal{R} + \mathcal{Q}^*$. This establishes the theorem since $\mathcal{R} + \mathcal{Q}^* \subset \mathcal{P}^*$.

Points (b), (c), and (f) follow from the following lemma, whose proof is given for the sake of completeness.

Lemma A.5. For V_1 and V_2 arbitrary subspaces of \mathbb{R}^n ,

$$\pi(V_1) \cap \pi(V_2) = \pi(V_1 + K) \cap \pi(V_2 + K) = \pi([V_1 + K] \cap [V_2 + K]).$$

Proof. The first equality follows from $\pi(K) = 0$. For the second one, note that

$$\pi([V_1 + K] \cap [V_2 + K]) \subset \pi(V_1 + K) \cap \pi(V_2 + K),$$

so to establish equality it is sufficient to prove equality of dimension. Doing the relevant computations

$$\begin{aligned} & \dim(\pi([V_1 + K] \cap [V_2 + K])) \\ &= \dim([V_1 + K] \cap [V_2 + K]) - \dim(K) \\ &= \dim[V_1 + K] + \dim[V_2 + K] - \dim[V_1 + V_2 + K] - \dim[K], \end{aligned}$$

whereas,

$$\begin{aligned} & \dim[\pi(V_1 + K) \cap \pi(V_2 + K)] \\ &= \dim[\pi(V_1 + K)] + \dim[\pi(V_2 + K)] - \dim[\pi(V_1 + V_2 + K)] \\ &= \dim[V_1 + K] + \dim[V_2 + K] - \dim[V_1 + V_2 + K] - \dim[K]. \end{aligned}$$

Hence, they are equal. ■

The following lemmas are aimed at establishing points (a), (d), and (e).

Lemma A.6. If K and V are controlled-invariant subspaces satisfying $K \subset V \subset S$, S a subspace, then $\pi(V)$ is a controlled-invariant subspace contained in $\pi(S)$.

Proof. Since $K \subset V$ and both are controlled-invariant, there exists a feedback matrix F such that $\bar{A}K \subset K$ and $\bar{A}V \subset V$ where $\bar{A} = A + BF$. From the definition of \hat{A} , and the controlled-invariance of V , one has

$$\hat{A}\pi(V) := \pi(\bar{A}V) \subset \pi[V + \text{Im } B] = \pi[V] + \text{Im } \hat{B},$$

so $\pi(V)$ is a controlled-invariant subspace of the projected system. Its containment in $\pi(S)$ is obvious. ■

Lemma A.7. *Let \mathcal{R} be a controllability subspace and $K \subset \mathcal{R}$ a controlled-invariant subspace. Then $\pi(\mathcal{R})$ is a controllability subspace.*

Proof. Let F be such that $(A + BF)\mathcal{R} \subset \mathcal{R}$ and $(A + BF)K \subset K$, so that $\langle \bar{A} | \mathcal{B} \cap \mathcal{R} \rangle = \mathcal{R} = \langle \bar{A} | \mathcal{R} \rangle$ where $\bar{A} = A + BF$. Then,

$$\begin{aligned} \langle \hat{A} | \pi(\mathcal{R}) \cap \pi(\mathcal{B}) \rangle &= \pi \langle \bar{A} | \pi^{-1}(\pi(\mathcal{R}) \cap \pi(\mathcal{B})) \rangle \\ &= \pi \langle \bar{A} | \mathcal{R} \cap (\mathcal{B} + K) \rangle \supset \pi \langle \bar{A} | \mathcal{R} \cap \mathcal{B} \rangle \\ &= \pi(\mathcal{R}). \end{aligned}$$

Also,

$$\pi \langle \bar{A} | \mathcal{R} \cap (\mathcal{B} + K) \rangle \subset \pi \langle \bar{A} | \mathcal{R} \rangle = \pi(\mathcal{R}).$$

Hence,

$$\langle \hat{A} | \pi(\mathcal{R}) \cap \pi(\mathcal{B}) \rangle = \pi(\mathcal{R}),$$

so $\pi(\mathcal{R})$ is a controllability subspace. ■

Lemma A.8. *Suppose V^* is the maximal controlled-invariant subspace contained in S , and $K \subset V^*$ is also controlled-invariant. Then $\pi(V^*)$ is the maximal controlled-invariant subspace contained in $\pi(S)$.*

Proof. By Lemma A.6, $\pi(V^*)$ is a controlled-invariant subspace contained in $\pi(S)$. To show that $\pi(V^*)$ is indeed maximal, it suffices to show that if \tilde{V} is any controlled-invariant subspace contained in $\pi(S)$, then $\pi^{-1}(\tilde{V})$ is a controlled-invariant subspace of S , and hence is contained in V^* by maximality. Let V be any subspace of \mathbb{R}^n such that $\pi(V) = \tilde{V}$. \tilde{V} being controlled-invariant means

$$\hat{A}\pi(V) \subset \pi(V) + \pi(\mathcal{B});$$

hence,

$$K + \bar{A}V \subset (V + K) + \text{Im } B$$

which yields

$$\bar{A}V \subset (V + K) + \text{Im } B.$$

Using the fact that K is controlled-invariant, one obtains

$$\bar{A}(V + K) \subset (V + K) + \text{Im } B,$$

which finally gives

$$\bar{A}\pi^{-1}(V) \subset \pi^{-1}(V) + \text{Im } B,$$

establishing that $\pi^{-1}(V)$ is controlled-invariant. ■

Lemma A.9. *Suppose V is a controlled-invariant subspace, \mathcal{R}^* is the maximal controllability subspace contained in V , and K is a controlled-invariant subspace of \mathcal{R}^* . Then $\pi(\mathcal{R}^*)$ is the maximal controllability subspace contained in $\pi(V)$.*

Proof. As always, let F be such that $(A + BF)V \subset V$ and $(A + BF)K \subset K$. By Lemma A.6, $\pi(V)$ is controlled-invariant, and hence the maximal controllability subspace contained in $\pi(V)$ is $\langle \hat{A} | \pi(V) \cap \pi(\mathcal{B}) \rangle$. The task is to show that this equals $\pi(\mathcal{R}^*)$. Proceeding

$$\begin{aligned} \langle \hat{A} | \pi(V) \cap \pi(\mathcal{B}) \rangle &= \pi \langle \bar{A} | V \cap (\mathcal{B} + K) \rangle \subset \pi \langle \bar{A} | V \cap (\mathcal{B} + \mathcal{R}^*) \rangle \\ &= \pi \langle \bar{A} | \mathcal{R}^* \rangle = \pi(\mathcal{R}^*), \end{aligned}$$

where the penultimate equality follows from the fact that $(V \cap \mathcal{B}) \subset \mathcal{R}^* \subset V$ implies $V \cap (\mathcal{B} + \mathcal{R}^*) = \mathcal{R}^*$, by the Modular Distributive Rule. For the reverse inclusion,

$$\pi \langle \bar{A} | V \cap (\mathcal{B} + K) \rangle \supset \pi \langle \bar{A} | V \cap \mathcal{B} \rangle = \pi(\mathcal{R}^*),$$

since $\mathcal{R}^* = \langle \bar{A} | V \cap \mathcal{B} \rangle$. ■

Now, point (a) follows from Lemma A.7; points (d) and (e) follow from Lemmas A.8 and A.9 taken together. Hence, the proof is complete.

A.4. Proof of Lemma 2.10

From Lemma A.2,

$$\mathcal{R}_i + \bigcap_{j \neq i} \mathcal{R}_j = C^*$$

and

$$\mathcal{P}_i^* + \bigcap_{j \neq i} \mathcal{P}_j^* = C^*;$$

by the definition of \mathcal{P}_i^* ,

$$\mathcal{R}_i \subset \mathcal{P}_i^* \subset C^*.$$

By duality, one obtains

$$\mathcal{R}_i^\perp \cap \sum_{j \neq i} \mathcal{R}_j^\perp = C^{*\perp}, \quad (\text{A.14})$$

$$\mathcal{P}_i^{*\perp} \cap \sum_{j \neq i} \mathcal{P}_j^{*\perp} = C^{*\perp}, \quad (\text{A.15})$$

and

$$C^{*\perp} \subset \mathcal{P}_i^{*\perp} \subset \mathcal{R}_i^\perp. \quad (\text{A.16})$$

Let $W_{\mu+3} := C^{*\perp}$. By (A.16), there exist subspaces W_i^a and W_i^b such that $\mathcal{P}_i^{*\perp} = W_i^a \oplus W_{\mu+3}$ and $\mathcal{R}_i^\perp = W_i^a \oplus W_i^b \oplus W_{\mu+3}$. Since $\mathcal{R} := \bigcap_{j=1}^{\mu} \mathcal{R}_j$, by duality, $\mathcal{R}^\perp = \sum_{j=1}^{\mu} \mathcal{R}_j^\perp$. This, combined with (A.14), shows that $\mathcal{R}^\perp = W_1^a \oplus W_1^b \oplus \cdots \oplus W_\mu^a \oplus W_\mu^b \oplus W_{\mu+3}$. Since $\mathcal{Q} \subset \mathcal{Q}^* \cap \mathcal{R} \subset \mathcal{R}$, by duality, $\mathcal{R}^\perp \subset (\mathcal{Q}^* \cap \mathcal{R})^\perp \subset \mathcal{R}^\perp$. Hence, there exist subspaces $W_{\mu+2}$ and $W_{\mu+1}^b$ such that $(\mathcal{Q}^* \cap \mathcal{R})^\perp = \mathcal{R}^\perp \oplus W_{\mu+2}$ and $\mathcal{Q}^\perp = \mathcal{R}^\perp \oplus W_{\mu+2} \oplus W_{\mu+1}^b$. Finally, one can choose $W_{\mu+1}^a$ such that $\mathcal{Q}^\perp \oplus W_{\mu+1}^a = (\mathbb{R}^n)^*$.

Now, defining

$$V_i^a = (W_1^a \oplus W_1^b \oplus \cdots \oplus W_{i-1}^a \oplus W_{i-1}^b \oplus W_i^b \oplus \cdots \oplus W_{\mu+3})^\perp,$$

etc., gives the desired set of subspaces.

Appendix B

B.1. Proof of Lemma 3.5

Using $\mathcal{R}_i \cap \mathcal{G} + \bigcap_{j \neq i} (\mathcal{R}_j \cap \mathcal{G}) = \mathcal{G}$ and the regularity of $\{\mathcal{R}_1, \dots, \mathcal{R}_\mu\}$, one can apply Lemma A.1 pointwise to construct a smooth decomposition of \mathcal{G} into constant dimensional (but not necessarily involutive distributions) \mathcal{G}_i such that $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_{\mu+1}$ and

$$\mathcal{G} \cap \mathcal{R}_i = \sum_{j \neq i} \mathcal{G}_j. \quad (\text{B.1})$$

Define a partition of $\{1, \dots, m\}$ by $I_1 = \{1, \dots, \dim \mathcal{G}_1\}$, $I_2 = \{1 + \dim \mathcal{G}_1, \dots, \dim(\mathcal{G}_1 + \mathcal{G}_2)\}$, etc., and choose a set of vector fields $\{\bar{g}_1, \dots, \bar{g}_m\}$ such that $\mathcal{G}_i = \text{span}\{\bar{g}_j | j \in I_i\}$.

Since \mathcal{R}_i is controlled invariant, and (B.1) holds, it follows that, for all $1 \leq k \leq m$,

$$[f, \mathcal{R}_i] \subset \mathcal{R}_i + \mathcal{G}_i, \quad (\text{B.2})$$

$$[\bar{g}_k, \mathcal{R}_i] \subset \mathcal{R}_i + \mathcal{G}_i. \quad (\text{B.3})$$

Similarly, since \mathcal{P}_i^* is controlled-invariant and $\mathcal{R}_i \subset \mathcal{P}_i^*$,

$$[f, \mathcal{P}_i^*] \subset \mathcal{P}_i^* + \mathcal{G}_i, \quad (\text{B.4})$$

$$[\bar{g}_k, \mathcal{P}_i^*] \subset \mathcal{P}_i^* + \mathcal{G}_i \quad (\text{B.5})$$

for all $1 \leq k \leq m$.

In order to simplify the notation (specifically, to avoid subscripting the subscripts), it will now be assumed that the cardinality of each index set I_i is one, and hence that $m = \mu + 1$. It will be clear that the general case follows in the same manner.

For each $1 \leq i \leq \mu$, (B.2)–(B.5) imply the existence of a regular feedback $u_i = \bar{\alpha}_i(x) + \bar{\beta}_i(x)v_i$, such that

$$[f + \bar{g}_i \bar{\alpha}_i, \mathcal{R}_i] \subset \mathcal{R}_i, \quad (\text{B.6})$$

$$[\bar{g}_i \bar{\beta}_i, \mathcal{R}_i] \subset \mathcal{R}_i, \quad (\text{B.7})$$

and

$$[f + \bar{g}_i \bar{\alpha}_i, \mathcal{P}_i^*] \subset \mathcal{P}_i^*, \quad (\text{B.8})$$

$$[\bar{g}_i \bar{\beta}_i, \mathcal{P}_i^*] \subset \mathcal{P}_i^*. \quad (\text{B.9})$$

Define $\bar{\alpha}(x) = \text{column}(\bar{\alpha}_1(x), \dots, \bar{\alpha}_{\mu+1}(x))$, where $\bar{\alpha}_{\mu+1}(x)$ is chosen so that $\bar{\alpha}(x)$ is an $m \times 1$ vector, but is otherwise arbitrary. In addition, define $\bar{\beta}(x) = \text{diag}(\bar{\beta}_1(x), \dots, \bar{\beta}_{\mu+1}(x))$, where $\bar{\beta}_{\mu+1}(x)$ is chosen so that $\bar{\beta}(x)$ is an invertible $m \times m$ matrix, but is otherwise arbitrary. Finally, select an invertible $m \times m$ matrix $\Gamma(x)$

such that

$$[\bar{g}_1(x)|\cdots|\bar{g}_m(x)] = [g_1(x)|\cdots|g_m(x)]\Gamma(x)$$

and define

$$\alpha(x) = \Gamma(x)\bar{\alpha}(x), \quad (\text{B.10})$$

$$\beta(x) = \Gamma(x)\bar{\beta}(x). \quad (\text{B.11})$$

It is now claimed that $I_1, \dots, I_{\mu+1}$ and $u = \alpha(x) + \beta(x)v$ are the desired partition and feedback, respectively. To see this, first note that $(g\beta)_j = (\bar{g}\bar{\beta})_j$ for $1 \leq j \leq m$, and that $(\bar{g}\bar{\beta})_j \in \mathcal{R}_i$ for all $j \in I_i$, by (B.1). This shows part (c) of the lemma. Consider next, $f + g\alpha = f + \bar{g}\bar{\alpha}$, which can be written as

$$f + \sum_{i=1}^{\mu+1} \sum_{j \in I_i} \bar{g}_j \bar{\alpha}_j. \quad (\text{B.12})$$

By (B.6),

$$\left[f + \sum_{j \in I_i} \bar{g}_j \bar{\alpha}_j, \mathcal{R}_i \right] \subset \mathcal{R}_i; \quad (\text{B.13})$$

from (B.1),

$$\sum_{k \neq i} \sum_{j \in I_k} \bar{g}_j \bar{\alpha}_j \in \mathcal{R}_i,$$

and so by the involutivity of \mathcal{R}_i ,

$$\left[\sum_{k \neq i} \sum_{j \in I_k} \bar{g}_j \bar{\alpha}_j, \mathcal{R}_i \right] \subset \mathcal{R}_i. \quad (\text{B.14})$$

Taking (B.13) and (B.14) together yields

$$[f + g\alpha, \mathcal{R}_i] \subset \mathcal{R}_i. \quad (\text{B.15})$$

In a similar manner, $(g\beta)_k = (\bar{g}\bar{\beta})_k$ for all $1 \leq k \leq m$. Equation (B.7) shows that

$$[(\bar{g}\bar{\beta})_i, \mathcal{R}_i] \subset \mathcal{R}_i. \quad (\text{B.16})$$

For each $j \notin I_i$, (B.1) yields that $(\bar{g}\bar{\beta})_j \in \mathcal{R}_i$.

This, combined with the involutivity of \mathcal{R}_i gives

$$[(\bar{g}\bar{\beta})_j, \mathcal{R}_i] \subset \mathcal{R}_i \quad (\text{B.17})$$

for all $j \notin I_i$. Taking (B.16) and (B.17) together yields

$$[(g\beta)_k, \mathcal{R}_i] \subset \mathcal{R}_i \quad (\text{B.18})$$

for all $1 \leq k \leq m$. Part (b) of the lemma is therefore established by (B.15) and (B.18). Part (a) follows in a similar fashion. \blacksquare

B.2. Proof of Theorem 3.2

The necessity of (3.3) and (3.4) follows from Lemma 3.1. The sufficiency argument follows the same steps as the proof of Lemma 3.5 \blacksquare

B.3. Proof of Theorem 3.7

The proof follows the same steps as those in the proof of Theorem 2.7, with appropriate modifications. For example,

$$\mathcal{R}_i = \sum_{j \neq i} \langle \bar{A} | \mathcal{B}_j \rangle$$

is replaced by

$$\mathcal{R}_i = \sum_{j \neq i} \langle \tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m | \mathcal{G}_j \rangle,$$

where $\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m$ are the vector fields of the closed-loop system obtained by applying the feedback of Lemma 3.5, $\mathcal{G} = \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_{\mu+1}$ is an appropriate decomposition of \mathcal{G} , and $\langle \tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m | \mathcal{G}_j \rangle$ denotes the smallest distribution containing \mathcal{G}_j and invariant under $\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m$. From the controllability distribution algorithm of [I, p. 159], it follows that \mathcal{R}_i so defined is the smallest distribution containing $\sum_{j \neq i} \mathcal{G}_j$ and invariant under $\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m$. When \mathcal{Q}^* is expressed as $\mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_{\mu+1}$, it is not necessary that the distributions \mathcal{S}_i be involutive; etc. ■

B.4. Proof of Lemma 3.10

Follows easily from the proof of Lemma 2.10 in combination with the techniques used, for example, in Lemma 4.1 of [IG]. ■

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