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THE DECISION PROBLEM FOR THE RESTRICTED PREDICATE CALCULUS

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Let Φ be a class of formulas. We say that Φ is *solvable* if the decision problems of the satisfiability and the finite satisfiability of the formulas of the class Φ are algorithmically solvable. We say that Φ is a *reduction class* if there exists an algorithm which sets into correspondence with each formula α of the restricted predicate calculus with the sign of equality a formula $\beta \in \Phi$ such that α and β are simultaneously satisfiable or not satisfiable and simultaneously satisfiable or not satisfiable on finite models.

A summary of the results on the decision problem are given in [1]. The corollaries formulated below follow from these results and the theorem of the present article. In this connection, F is everywhere the symbol for a two-place predicate and M denotes a formula without quantifiers.

Theorem. *The class of formulas RPC without the sign of equality of the form*

$$\forall x \exists u \forall y \exists z_1 \ldots z_n M(F; x, u, y, z_1, \ldots, z_n) \tag{1}$$

is a reduction class.

Let Π be a set of words of the alphabet $\{\forall, \exists\}$ and let σ be a collection of symbols for predicates. We denote by $\Phi^*(\Pi, \sigma)$ the class of all formulas of RPC without the sign of equality of the form

$$Q_1 x_1 \ldots Q_n x_n M(\sigma; x_1, \ldots, x_n),$$

where $Q_1 \ldots Q_n \in \Pi$. We denote by $\Phi^*(\Pi, \sigma)$ the corresponding class of formulas RPC with the sign of equality.

Corollary 1. *Either* $\Phi(\Pi, \sigma)$ *is solvable or it is a reduction class. The same applies to* $\Phi^*(\Pi, \sigma)$.

Corollary 2. $\Phi(\Pi, \sigma)$ *is solvable if and only if* $\Phi^*(\Pi, \sigma)$ *is solvable.*

Corollary 3. *Let*

$$\Pi_1 = \{\exists^m \forall^n \mid m, n = 0, 1, \ldots\}, \quad \Pi_2 = \{\exists^m \forall^i \exists^n \mid i = 1, 2; m, n = 0, 1, \ldots\}.$$

$\Phi(\Pi, \sigma)$ *is solvable if and only if at least one of the following three possibilities holds:*

a) σ *contains only symbols for one-place predicates;*

b) $\Pi \subseteq \Pi_1 \cup \Pi_2$;

c) σ *and* $\Pi \setminus (\Pi_1 \cup \Pi_2)$ *are finite.*

Corollary 4. *If* $\Phi^*(\Pi, \sigma)$ *is solvable, then, with the exception of a finite number of formulas, satisfiability for the formulas of* $\Phi^*(\Pi, \sigma)$ *coincides with satisfiability on finite models.*

Remark. The above-indicated corollaries are given, for the case of infinite σ, in the work of a group of American mathematicians headed by Hao Wang.

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We note the proof of the theorem. Let $i = \epsilon(i) + 2\delta(i)$ be the binary notation, $i = 0, 1, 2, 3$, and let

$$\varphi = \forall x \exists u \forall y \exists z_1 \ldots z_n (A_0 . A_1 . \ldots A_l . M)$$

be a formula of the form (1). We shall write simply ab instead of Fab.

$$A_0 = (\neg xx . yy \supset \neg uu . (xu \sim xy) . (ux \sim yx)) . z_1 z_1 .$$

We set $f_0^i x \sim \neg xx \cdot (1)^{\epsilon(i)} xu \cdot (-1)^{\delta(i)} \cup x$. By virtue of A_0, $f_0^i\, u$ can be considered as the abbreviation for $\neg uu \cdot (-1)^{\epsilon(i)} uz_1 \;\; (-1)^{\delta(i)} z_1 u$.

$$A_1 = [\neg xx . \neg f_0^3 x . f_0^3 y \supset \neg uu . \neg f_0^3 u .$$
$$\bigwedge_{i=0}^{2} (f_0^i u \sim (-1)^{\epsilon(i)} xy . (-1)^{\delta(i)} yx)] . f_0^3 z_2 .$$

For $i = 0, 1, 2$, we set $f_1^i x \sim \neg xx \cdot \neg f_0^3 x \cdot f_0^i u$. In this connection, by virtue of A_1, $f_1^i\, u$ can be considered as the abbreviation for $\neg uu \cdot \neg f_0^3\, u \cdot (-1)^{\epsilon(i)} uz_2 . \;\; (-1)^{\delta(i)} z_2 u$. and so forth, so that we have a large number of one-place predicates at our disposal in M. Then the theorem from [2] is used.

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BIBLIOGRAPHY

[1] V. F. Kostyrko, Algebra i Logika Sem. 3 (1964), 45.

[2] Ju. Š. Gurevič, Dokl. Akad. Nauk SSSR 166 (1966), 1028 = Soviet Math. Dokl. 7 (1966), 213.

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