Fixed Point Logics

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Abstract

We consider *fixed point logics*, i.e. extensions of first order predicate logic with operators defining fixed points. A number of such operators, generalizing inductive definitions, have been studied in the context of finite model theory, including nondeterministic and alternating operators. We review results established in finite model theory, and also consider the expressive power of the resulting logics on infinite structures. In particular, we establish the relationship between inflationary and nondeterministic fixed point logics and second order logic, and we consider questions related to the determinacy of games associated with alternating fixed points.

1 Introduction

In this paper, we are concerned with the expressive power of fixed point logics. These are logics formed by extending first order predicate logic with an operator for forming fixed points of relational operators. That is to say, by viewing formulas with free relational variables as defining maps on the space of relations, we can apply an operator that allows us to define fixed points of the map. The study of such logics has its roots in the study of inductive definability (see [15, 4]) in the context of generalized recursion theory. The kind of question considered there is this: fix a structure $\mathbb{A}$ (typically this is the structure of arithmetic, often generalized to any "acceptable" structure), and a collection of formulas $\mathcal{F}$ defining relational operators on $\mathbb{A}$; we wish to characterize the class of relations on $\mathbb{A}$ that are inductively defined by $\mathcal{F}$. The next step is to introduce an explicit construct into the logic which allows us to define the fixed

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point of a relational operator, and close the language under this construct thus obtaining a richer logic. This latter idea has its origins in finite model theory, where a variety of such *fixed point logics* have been extensively studied (see [8]). This is largely due to the fact that questions about their expressive power on finite structures have turned out to be intimately related to significant open questions in computational complexity theory. In particular, the logics LFP and IFP, which are obtained by closing first order logic under the formation of least and inflationary fixed points respectively (detailed definitions are given in the following two sections), have been shown to have expressive power equivalent, on finite ordered structures, to the computational complexity class P. This has also led to the definition of a larger variety of fixed point operators, corresponding to other computational complexity classes.

In this paper, following a survey of the known relationships of these logics on finite models, we examine their expressive power on infinite structures, where some of the questions turn out to be easier to resolve, while others appear more challenging. It is hoped that this preliminary study of these logics on infinite structures will stimulate further interest in them outside of finite model theory, and possibly cast some new light on the nature of these logics even when interpreted on finite structures.

In what follows, we define precisely the logics LFP and IFP, and summarise results on their expressive power on finite structures. We also consider three other fixed point logics—NFP, AFP and PFP—incorporating nondeterministic, alternating and partial fixed point operators respectively. On finite structures, the last two have been shown to be equivalent in expressive power, while NFP has expressive power that is intermediate between that of IFP and AFP. Moreover, on ordered finite structures, the expressive power of NFP is equivalent to the polynomial time hierarchy, and that of its alternation-free positive fragment is equivalent to the complexity class NP, while the expressive power of PFP is equivalent to the class PSPACE. Moreover, it has been shown that the problems of separating the expressive power of these logics on finite structures, even in the absence of order, are equivalent to the separation of the corresponding complexity classes, which are notoriously open problems. Alongside a brief survey of these results, we investigate the expressive power of these logics on infinite structures. In particular, we compare their expressive power to that of second order logic, showing that their relationship with the latter is quite analogous to their relationship to complexity classes in the finite case. We end with an investigation of a problem of determinacy related to the definition of AFP.

2 Preliminaries

We begin with a brief review of standard notions of inductive definability, which will permit us to fix notation. Below, when we refer to a formula, we mean a formula of some logic extending first order predicate logic. In particular, we assume that we have available the propositional connectives $\wedge$, $\vee$ and $\neg$, and the first order quantifiers $\exists$ and $\forall$.

Let $\varphi(R, \mathbf{x})$ be a formula, where R is a relation symbol, and $\mathbf{x}$ is a tuple of first order variables whose length k is the same as the arity of R. If $\mathbb{A}$ is a structure, with universe A, interpreting all symbols in φ other than those displayed, we think of φ as defining a map Φ from $\text{Pow}(A^k)$—the set of all k-ary relations on A—to $\text{Pow}(A^k)$ given by

$$\Phi(P) = \{\mathbf{a} \mid (\mathbb{A}, P, \mathbf{a}) \models \varphi\}.$$

This view of a formula defining an operator on the space of relations gives a natural formalisation of inductive definitions. For instance, if Φ is a monotone map, we can speak of the least relation R such that $R(\mathbf{x}) \leftrightarrow \varphi(R, \mathbf{x})$. We call this relation the *least fixed point* of the operator defined by φ. Even if Φ is not monotone, we can construct a fixed point of the map by taking the limit of the sequence of relations (indexed by ordinals) given by:

$$R_\alpha = \bigcup_{\beta < \alpha} \Phi(R_\beta),$$

which we call the *inflationary fixed point* of the operator Φ.

These two kinds of fixed point construction appear under various names in the literature. For instance, they are called monotone and non-monotone inductions in [4, 16]. Our aim here is not merely to study the relations definable by such inductions, but to consider the logics obtained by allowing explicit operators for forming such relations. The term "inflationary fixed point" and the logic obtained by closing first order logic under an operator for defining such fixed points were first introduced in [11].

3 Least Fixed Point Logic

In [5], Aho and Ullman proposed enriching the language of relational algebra (essentially equivalent to first order predicate logic) with an explicit syntactic construct for forming the least fixed point of monotone operators. Chandra and Harel [6] followed up this proposal in introducing a query language allowing the application of the least fixed point operator to positive formulas, providing a syntactic criterion in place of monotonicity. The result is the first of the fixed point logics we consider—LFP—which is obtained by closing first order logic simultaneously under all the formula forming operations of first order logic along with the rule:

> if R is a k-ary relation variable, $\mathbf{x}$ is a k-tuple of first order variables, $\mathbf{t}$ is a k-tuple of terms and φ is a formula in which R occurs only positively, then
>
> $$[\mathit{lf}_{R,\mathbf{x}}\, \varphi](\mathbf{t})$$
>
> is a formula, in which all occurrences of R are bound, and all occurrences of the variables in $\mathbf{x}$ excpet those occurring in $\mathbf{t}$ are bound.[1]

[1] The notation used here for the fixed point operators is introduced here for the first time, as there does not appear to be a standard accepted notation for these operators.

The intended semantics of this formula formation rule is that, for any structure $\mathbb{A}$, $\mathbb{A} \models [lfp_{R,\mathbf{x}}\ \varphi](\mathbf{t})$ if, and only if, $\mathbf{t}^{\mathbb{A}}$—the tuple of elements of $\mathbb{A}$ defined by the terms $\mathbf{t}$—is in the least fixed point of the monotone operator defined by $\varphi(R, \mathbf{x})$ on A^k.

The least fixed point can be obtained as the limit of the sequence of relations:

$$\begin{array}{rcll} R_0 & = & \emptyset & \\ R_{\alpha+1} & = & \Phi(R_\alpha) & \\ R_\alpha & = & \bigcup_{\beta<\alpha} R_\beta & \quad \text{for limit ordinals } \alpha. \end{array} \tag{1}$$

We define the *closure ordinal* of the operator Φ to be the least ordinal α such that $R_\alpha = R_{\alpha+1}$.

Example 1

1. Let φ be the formula

$$x = y \vee \exists z(E(x, z) \wedge R(z, y)),$$

   then $[lfp_{R,xy}\ \varphi](u, v)$ is a formula in two free variables (u and v) which defines the reflexive and transitive closure of E.

   The relation R_n (for finite ordinals n) would be the set of pairs of elements with an E-path of length less than n. On a finite graph G, the closure ordinal of the operator would be one greater than the maximum diameter of a connected component of G. The closure ordinal is at most ω on any structure.

2. Let φ be the formula $\forall y(y < x \rightarrow P(y))$. Then, $\psi(z) \equiv [lfp_{P,x}\ \varphi](z)$ defines the well-founded part of the binary relation $<$. In particular, if $<$ is a linear order, it defines the longest well-ordered initial segment. The closure ordinal of the operator on a linear order is the length of this initial segment. Moreover, the sentence $\forall z \psi(z)$ is true in a structure $\mathbb{A}$ if, and only if, the relation $<$ is well-founded in $\mathbb{A}$.

It is not too difficult to show that for any formula $\varphi(\mathbf{x})$ of LFP, the set $\{(\mathbb{A}, \mathbf{c}) \mid \mathbb{A} \text{ is finite and } \mathbb{A} \models \varphi[\mathbf{c}]\}$ is decidable in polynomial time [6]. The crucial point to observe is that the closure ordinal of any formula on a finite structure must be bounded by a fixed polynomial in the number of elements in the structures (a polynomial whose degree depends only on the formula, and not on the structure). This is because the sequence of relations R_α is increasing, since Φ is monotone. Thus, if the symbol R is k-ary, on a structure with n elements there is no increasing sequence of k-ary relations longer than n^k, and a fixed point must therefore be reached by that stage.

A converse to this proposition holds when we restrict ourselves to finite structures which incorporate a linear order. That is, if we distinguish a binary relation symbol $<$ and consider only structures in which this symbol is interpreted as a linear order, we have that every polynomial time decidable class of such structures is definable by a sentence of LFP (see [13, 18]).

Immerman [13] establishes a normal form for LFP on finite structures, by showing that every formula of LFP is equivalent to one of the form

$$\exists x\, [\mathbf{lfp}_{R,\mathbf{y}}\, \varphi](x, \ldots, x),$$

where φ contains no occurrences of $\mathbf{lfp}$. In particular, this requires that the natural hierarchy formed by interleaving negation with $\mathbf{lfp}$ collapses. This result crucially depends on the restriction to finite structures. To be precise, one can show the following using Immerman's methods:

Proposition 2 *For each first order formula $\varphi(R)$, there is a first order formula ψ, positive in S, such that the predicate defined by $\neg \mathbf{lfp}_{R,\mathbf{x}}\, \varphi$ is also defined by $\exists \mathbf{y}[\mathbf{lfp}_{S,\mathbf{yx}}\, \psi](\mathbf{x})$ in any structure $\mathbb{A}$ where the closure ordinal of φ is not a limit.*

The proof of this proposition relies on the construction of an inductive definition of the stage comparison relation (see [15, Section 2A]). Given the sequence of relations as given in (1), the *rank* of a tuple $\mathbf{a}$, denoted $|\mathbf{a}|$, is defined to be the least α such that $\mathbf{a} \in R_{\alpha+1}$ if there is such an α, and ∞ otherwise. The *stage comparison theorem* [15, Theorem 2A.2] guarantees the existence of a formula $\sigma(\mathbf{x}, \mathbf{y}, S)$, positive in S such that $[\mathbf{lfp}_{S,\mathbf{yx}}\, \sigma](\mathbf{u}, \mathbf{v})$ defines the relation $|\mathbf{u}| < |\mathbf{v}|$. This can then be used to define a formula $\mu(\mathbf{y})$ which defines the set of tuples of maximal rank other than ∞. By standard methods of combining inductions [15], we then obtain the desired formula, which is equivalent to $\exists \mathbf{y}(\mu(\mathbf{y}) \wedge |\mathbf{y}| < |\mathbf{x}|)$.

In contrast, it can be shown by a standard diagonalisation of the truth predicate that on the structure $(\omega, <)$—the ordering of the natural numbers—the interleaving of $\mathbf{lfp}$ with negation yields an infinite hierarchy.

4 Inflationary Fixed Point Logic

In general, an operator $F : \mathrm{Pow}(A^k) \to \mathrm{Pow}(A^k)$ is called *inflationary* if for any $S \subseteq A^k$, $S \subseteq F(S)$. Hence, associated with any operator F, there is a natural inflationary operator F' defined by $F'(S) = S \cup F(S)$ for all S. Iterating F' always yields a fixed point, which we call the *inflationary fixed point* of F. Note, there are monotone operators which are not inflationary and inflationary operators that are not monotone (see [11]). However, if an operator F is monotone, the fixed point reached by iterating the corresponding inflationary operator F' is, indeed, the least fixed point of F.

The logic IFP is defined with a syntax similar to LFP, but with an operator $\mathbf{ifp}$, which allows us to form formulas of the form $[\mathbf{ifp}_{R,\mathbf{x}}\, \varphi](\mathbf{t})$. The semantics of the formula is given by the rule that $\mathbb{A} \models [\mathbf{ifp}_{R,\mathbf{x}}\, \varphi](\mathbf{t})$ if, and only if, the tuple of elements interpreting $\mathbf{t}$ is in the union of the sequence of relations given by:

$$R_\alpha = \bigcup_{\beta<\alpha} \Phi(R_\beta). \tag{2}$$

This is equivalent to the definition given in three clauses in (1), except that at each stage, we take the union with the previous stages. In particular, this guarantees that the sequence of stages is increasing (and hence converges) even when the operator is not monotone. For this reason, we do not need to restrict the application of $\mathbf{ifp}$ to positive formulas. However, for positive formulas, the fixed point that is reached is the same as the least fixed point, as indicated above. It follows from this that the language IFP is at least as expressive as LFP.

It was shown by Gurevich and Shelah [12] that on finite structures, every formula of IFP is equivalent to a formula of LFP. The proof relies crucially on the fact that on finite structures, every inductive sequence has a last stage. This fails on infinite structures and it is not known whether the two logics are equivalent in expressive power. To be precise, there are two versions of the open question:

> **Question:** Is there a formula φ of IFP and a structure $\mathbb{A}$ such that for every formula ψ of LFP, $\mathbb{A} \not\models (\varphi \leftrightarrow \psi)$?
>
> Is there a formula φ of IFP such that for every formula ψ of LFP, there is a structure $\mathbb{A}$ such that $\mathbb{A} \not\models (\varphi \leftrightarrow \psi)$?

Clearly, a positive answer to the first would also provide a positive answer to the second.

We are in a position to resolve the relationship of IFP and LFP on one infinite structure of interest, namely $(\omega, <)$, i.e. the usual ordering of the natural numbers, and hence on the structure of arithmetic. It is known that there are first order definable operators $\varphi(R)$ such that $\mathbf{ifp}_{R,\mathbf{x}}\, \varphi$ is not equivalent to the least fixed point of any *monotone* first order formula. Aczel [4] gives an example of such a formula, where $\mathbf{ifp}_{R,\mathbf{x}}\, \varphi$ defines a well ordering whose length is greater than any recursive ordinal, while it is known that the closure ordinal of any monotone first order formula is recursive.

However, on the structure $(\omega, <)$ (and, indeed, on all structures called *acceptable* in [15]), it is still the case that every formula of IFP is equivalent to one of LFP. This follows from a general characterization of classes of inductive definitions in [16]. In particular, Theorem 15 in that paper states that, for any acceptable structure $\mathbb{A}$: "*If $\mathcal{F}$ is a typical, non-monotone class of operators on $\mathbb{A}$, then $\mathcal{F}-$IND is the smallest $\mathcal{F}$-compact Spector class on $\mathbb{A}$ such that every relation in $\mathcal{F}$ is Δ on Δ*". For the definitions of the terms used in this statement, we refer the reader to [16]. Here, we note that $\mathcal{F}-$IND is the set of relations that are obtainable as inflationary fixed points of operators in $\mathcal{F}$. In [16], the collections of operators that are generally considered are those given by quantifier-alternation fragments of higher order logics. Here, we consider classes $\mathcal{F}$ of operators definable in LFP with a fixed number of alternations of the operator $\mathbf{lfp}$ with negation. In particular, let M_0 denote the set of first-order definable operators, and M_{i+1} be the collection of operators that can be obtained by the application of $\mathbf{lfp}$ and *positive* first order operators on the negation of operators in M_i. We note that, for $n \geq 1$, M_n is a *typical, non-monotone* class of operators in the sense of [16], and that $M_{n'}$, for $n' > n$ is an M_n-compact Spector class such that every relation in M_n is Δ on Δ. It then follows that M_n−IND $\subseteq M_{n'}$. In other words, if $\varphi(R, \mathbf{x})$ is a formula of LFP defining an operator, then $\mathit{IF}_{R,\mathbf{x}}\ \varphi$ is also definable in LFP. From this, it immediately follows, by an induction on the structure of the IFP formula, that every formula of IFP is equivalent to a formula of LFP.

One consequence of this observation is that there is an LFP formula defining the non-recursive ordinal given by Aczel. However, this formula must necessarily involve nesting the operator lF inside a negation symbol.

If we do not restrict ourselves to acceptable structures, the methods of [16] do not work, but there is still something we can say about the fixed points in an arbitrary IFP formula. To state it precisely, define MFP (for *monotone fixed point* logic) to be the fragment of IFP where the IF operator can only be applied to formulas which define monotone operators. MFP is not a syntactically defined fragment of IFP. Indeed, it can be shown that it is undecidable whether a given formula $\varphi(R, \mathbf{x})$ of IFP defines a monotone operator [11]. Nonetheless, we are able to assert the following:

Proposition 3 *Every formula of IFP is equivalent to a formula of MFP.*

In the context of finite structures, Proposition 3 is weaker than the main result proved by Gurevich and Shelah in [12], namely that every IFP formula is equivalent to one of LFP. However, they also provided a separate, simpler proof of this proposition in an appendix, a proof which it turns out can be adapted to arbitrary structures. The construction shows that for every formula $\varphi(R, \mathbf{x})$ of LFP, there is a formula φ' defining a monotone operator such that $\mathit{IF}_{R,\mathbf{x}}\ \varphi'$ defines the same predicate as $\mathit{IF}_{R,\mathbf{x}}\ \varphi$. The only change that is required in adapting their construction to arbitrary structures, is where, in [12, page 277], the construction of the formula $\mathrm{Nice}(Q, x)$ contains the first order condition that the relation Q is a linear quasi-order, we need to replace it by the LFP condition that Q is a well quasi-order.

One might ask what a good candidate formula, or a candidate structure might be that might provide a positive answer to the open question on page 6. No obvious candidates spring to mind. Any well-ordered structure $\mathbb{A}$ is ruled out by the results mentioned above for acceptable structures. Moreover, if $\mathbb{A}$ is countably categorical, then the expressive power of both LFP and IFP collapses to that of first order logic on $\mathbb{A}$, as infinite inductions are not possible, thus ruling $\mathbb{A}$ out as a possible candidate. If we are to find reasonable candidates, $\mathbb{A}$ must present sufficient structure that infinite inductions are possible, but which can more readily be exploited by IFP than by LFP.

5 Second Order Logic

In this section we examine the relationship of the fixed point logics LFP and IFP with the standard second order logic—with quantification over relations.

In the context of finite structures, an early result due to Fagin [9] shows that a collection of finite structures is definable by a sentence of *existential* second order logic if, and only if, membership in the collection is decidable by a nondeterministic Turing machine operating in polynomial time. Since it is also known that every collection of finite structures definable by an IFP formula corresponds to a problem computable in polynomial time, it follows that every formula in IFP corresponds to a Δ^1_1 global predicate on finite structures. That is, it is equivalent to both a Σ^1_1 and to a Π^1_1 formula. However, this fact can also be established directly without recourse to facts about computational complexity, as we show below. If we remove the restriction to finite structures, the best that we can say is that every formula of IFP is Δ^1_2. We begin by establishing this latter fact.

Suppose then that we have a k-ary predicate symbol R, a k-tuple of variables $\mathbf{x}$ and a Δ^1_2 formula $\psi(R, \mathbf{x})$ in the vocabulary $\sigma \cup \{R\}$.[2] We wish to show that the predicate expression

$$\mathit{iF}_{R,\mathbf{x}}\, \psi$$

is also Δ^1_2.

Let $\psi_\Sigma(R, \mathbf{x})$ and $\psi_\Pi(R, \mathbf{x})$ be Σ^1_2 and Π^1_2 formulas equivalent to ψ.

Let $<$ be a new $2k$-ary relation symbol. For k-tuples of variables $\mathbf{x}$ and $\mathbf{y}$, we write $\mathbf{x} < \mathbf{y}$ instead of $< (\mathbf{x}, \mathbf{y})$. Similarly, $\mathbf{x} = \mathbf{y}$ abbreviates the formula $\bigwedge_{1 \leqslant i \leqslant k}(x_i = y_i)$, and we write $\mathbf{x} \leqslant \mathbf{y}$ for the formula $\mathbf{x} < \mathbf{y} \vee \mathbf{x} = \mathbf{y}$.

Define the formula $\mathsf{lo}(<)$ to be the conjunction of the following three formulas:

$$\begin{array}{lr} \forall \mathbf{x} \forall \mathbf{y} (\mathbf{x} < \mathbf{y} \vee \mathbf{y} < \mathbf{x} \vee \mathbf{x} = \mathbf{y}) & \text{(Linearity)} \\ \forall \mathbf{x} \forall \mathbf{y} \neg (\mathbf{x} < \mathbf{y} \wedge \mathbf{y} < \mathbf{x}) & \text{(Anti-symmetry)} \\ \forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} [(\mathbf{x} < \mathbf{y} \wedge \mathbf{y} < \mathbf{z}) \rightarrow \mathbf{x} < \mathbf{z}] & \text{(Transitivity)} \end{array}$$

That is, $\mathsf{lo}(<)$ is a first order formula that asserts that $<$ linearly orders the set of k-tuples.

Define the formula $\mathsf{wo}(<)$ to be the conjunction of $\mathsf{lo}(<)$ with the following formula, in which O is a k-ary relation symbol:

$$\forall O (\exists \mathbf{x} O(\mathbf{x}) \rightarrow (\exists \mathbf{z} O(\mathbf{z}) \wedge \forall \mathbf{y} (O(\mathbf{y}) \rightarrow \mathbf{z} \leqslant \mathbf{y}))).$$

That is, $\mathsf{wo}(<)$ is a Π^1_1 formula that asserts that $<$ is a well-order.

For any structure $\mathbb{A}$, if $(\mathbb{A}, <) \models \mathsf{wo}(<)$, we can assign to each k-tuple $\mathbf{a}$ from $\mathbb{A}$ a unique ordinal $\alpha(\mathbf{a})$ such that the well-ordered set $\{\mathbf{b} \in A^k \mid \mathbf{b} < \mathbf{a}\}$ is order-isomorphic to $\alpha(\mathbf{a})$.

Recall that on $\mathbb{A}$, the predicate defined by $\mathit{iF}_{R,\mathbf{x}}\, \psi$ is given by the union of the stages:

$$R_\alpha = \bigcup_{\beta < \alpha} \Phi(R_\beta).$$

Thus, for any structure $\mathbb{A}$ and any well-ordering $<$ of its k-tuples, there is a unique $2k$-ary relation $I^{\mathbb{A}}$ on $\mathbb{A}$ such that

$$I^{\mathbb{A}}(\mathbf{a}, \mathbf{b}) \quad \text{if, and only if,} \quad \mathbf{b} \in R_{\alpha(\mathbf{a})}$$

[2] Strictly speaking, the formula ψ is not Δ^1_2. What is meant is that the formula is equivalent to both a Σ^1_2 and a Π^1_2 formula.

In particular, if the well-ordering $<$ is long enough (at least as long as the closure ordinal of ψ), then the second projection of $I^{\mathbb{A}}$, i.e. $\{\mathbf{b} \mid I^{\mathbb{A}}(\mathbf{a},\mathbf{b}) \text{ for some } \mathbf{a}\}$ is exactly $\mathit{iF}_{R,\mathbf{x}}\,\psi$.

We will now write a Σ^1_2 formula $\theta(<, I)$ such that for any $(\mathbb{A}, <)$, $I^{\mathbb{A}}$ is the unique relation S such that

$$(\mathbb{A}, <, S) \models \theta(<, I).$$

In other words, θ defines $I^{\mathbb{A}}$ implicitly.

Before proceeding to the construction of θ, we introduce a notational convention. For any k-tuple of variables $\mathbf{z}$ not occurring in ψ, we write $\psi(I(\mathbf{z}), \mathbf{x})$ for the formula obtained from $\psi(R, \mathbf{x})$ by replacing all occurrences of $R(\mathbf{t})$ (where $\mathbf{t}$ is any k-tuple of terms) by $I(\mathbf{z}, \mathbf{t})$. Similarly, we write $\psi(\exists \mathbf{z} I(\mathbf{z}), \mathbf{x})$ for the formula obtained from $\psi(R, \mathbf{x})$ by replacing all occurrences of $R(\mathbf{t})$ by the subformula $\exists \mathbf{z} I(\mathbf{z}, \mathbf{t})$.

By the definition of R_α,

$$\mathbf{a} \in R_\alpha \text{ if, and only if, for some } \beta < \alpha \ \mathbf{a} \in \Phi_\psi(R_\beta).$$

This is now easily turned into the required formula θ given by

$$\forall \mathbf{x} \forall \mathbf{y} (I(\mathbf{x}, \mathbf{y}) \leftrightarrow \exists \mathbf{z}[\mathbf{z} < \mathbf{x} \wedge \psi(I(\mathbf{z}), \mathbf{y})]).$$

Note that the subformula ψ occurs both positively and negatively in the above formula. By replacing the positive occurrence with ψ_Σ and the negative occurrence with ψ_Π, we see that the above formula is, in fact, Σ^1_2.

The predicate $\mathit{iF}_{R,\mathbf{x}}\,\psi$ is then defined by the following:

$$\exists < \exists I[\mathsf{wo}(<) \wedge \theta \wedge \exists \mathbf{y} I(\mathbf{y}, \mathbf{x})]. \tag{3}$$

The formula is true of $\mathbf{b}$ in $\mathbb{A}$ if there is some well-ordering $<$, some corresponding $I^{\mathbb{A}}$ and some $\mathbf{a}$ such that $I^{\mathbb{A}}(\mathbf{a}, \mathbf{b})$. In other words, $\mathbf{b} \in R_{\alpha(\mathbf{a})}$. Thus, the relation defined by (3) is clearly contained in $\mathit{iF}_{R,\mathbf{x}}\,\psi$. On the other hand, since the closure ordinal of ψ has cardinality at most $\mathrm{card}(A^k)$, for every tuple in $\mathit{iF}_{R,\mathbf{x}}\,\psi$, there is some well-ordering of A^k that makes (3) true.

Note that, since $\mathsf{wo}(<)$ is Π^1_1 and θ is Σ^1_2, it follows that (3) is Σ^1_2.

The predicate $\mathit{iF}_{R,\mathbf{x}}\,\psi$ is also defined by the following:

$$\forall < \forall I[(\mathsf{wo}(<) \wedge \theta \wedge (\forall \mathbf{x}(\psi_\Pi(\exists \mathbf{z} I(\mathbf{z}), \mathbf{x}) \to \exists \mathbf{z} I(\mathbf{z}, \mathbf{x})))) \to \exists \mathbf{z} I(\mathbf{z}, \mathbf{x})]. \tag{4}$$

The formula is true of a tuple $\mathbf{b}$ in $\mathbb{A}$ if, for every well-ordering $<$, and corresponding $I^{\mathbb{A}}$, that satisfy

$$\forall \mathbf{x}(\psi_\Pi(\exists \mathbf{z} I(\mathbf{z}), \mathbf{x}) \to \exists \mathbf{z} I(\mathbf{z}, \mathbf{x})), \tag{5}$$

there is some $\mathbf{a}$ such that $I^{\mathbb{A}}(\mathbf{a}, \mathbf{b})$. The condition (5) asserts that the stages of $I^{\mathbb{A}}$ along the order $<$ reach a fixed point. That is, if we let P be the projection $\{\mathbf{b} \mid I^{\mathbb{A}}(\mathbf{a}, \mathbf{b}) \text{ for some } \mathbf{a}\}$, then (5) asserts that $\Phi(P) \subseteq P$.

The subformulas $\mathsf{wo}(<)$ and θ (which are Π^1_1 and Σ^1_2 respectively) occur negatively in (4), while ψ_Π (which is Π^1_2) occurs positively. Thus, the entire formula (4) is Π^1_2. This establishes the desired result.

A couple of refinements of the result are possible when we restrict ourselves to just infinite structures or just finite structures.

Infinite Structures If $\mathbb{A}$ is an infinite structure, the cardinality of A^k is just the same as A. Thus, since the closure ordinal of ψ has cardinality at most that of A^k, there is a well-ordering of A long enough to represent this ordinal. In other words, in the above construction, we can take $<$ to be a binary relation, and I can be $(k+1)$-ary rather than $2k$-ary. A $2k$-ary I is necessary, however, for finite structures.

Finite Structures If $\mathbb{A}$ is a finite structure, then any linear order of the elements (or the k-tuples) of $\mathbb{A}$ is a well-order. Thus, we can replace $\mathsf{wo}(<)$ in the construction of the formulas (3) and (4) by $\mathsf{lo}(<)$. This means that, if our original formula ψ is Δ^1_1, we can construct (3) and (4) to be Σ^1_1 and Π^1_1 respectively, and we get the result that, on finite structures, every IFP formula is Δ^1_1.

Note that the result for infinite structures cannot be improved to Δ^1_1, as it is known that on the structure $(\omega, <)$, every Π^1_1 formula is equivalent to the least fixed point of a positive first order formula [15]. It therefore follows that all predicates in $\Sigma^1_1 \cup \Pi^1_1$ can be expressed in LFP on this structure, and as $\Sigma^1_1 \neq \Pi^1_1$ [14], each is strictly larger than Δ^1_1

Well-Founded Quantification Call a binary relation R *well-founded* if there is no infinite chain of elements $(a_i \mid i \in \omega)$, with $R(a_{i+1}, a_i)$ for all i. We can extend this definition naturally to relations of arity $2k$ by considering them as binary relations on k-tuples. Define $W\Sigma^1_1$ to be the collection of global predicates definable by existential second order formulas, where the interpretation of the second order quantifiers is restricted to well-founded relations. Similarly, let $W\Pi^1_1$ denote the complements of $W\Sigma^1_1$ predicates, and $W\Delta^1_1$ denote those predicates that are both $W\Sigma^1_1$ and $W\Pi^1_1$. Then, it is easily checked that the argument we have given shows that every IFP formula is $W\Delta^1_1$ on all structures (finite or infinite).

6 Partial Fixed Point Logic

The logic PFP (for partial fixed point) was introduced in [2] as a variant of IFP on finite structures. Let $\varphi(R, \mathbf{x})$ be an arbitrary formula defining a (not necessarily monotone) operator Φ and consider the sequence of relations (for finite α):

$$\begin{aligned} R_0 &= \emptyset \\ R_{\alpha+1} &= \Phi(R_\alpha) \end{aligned} \tag{6}$$

This sequence is not necessarily increasing. Still, on finite structures, it either converges to a fixed point, or settles into a cycle of period greater than 1. We define the *partial fixed point* of Φ to be the fixed point that is reached in the former case, and the empty relation in the latter case. The logic PFP is obtained by closing first order logic simultaneously under the formula formation rules of first order logic and the rule that allows us to form the formula $[\mathbf{pf}_{R,\mathbf{x}}\ \varphi](\mathbf{t})$

from the formula φ. This is used to denote that $\mathbf{t}$ is a tuple in the partial fixed point of $\varphi(R, \mathbf{x})$.

The significance of PFP lies in the fact (shown in [2], based on [18]) that the properties of *ordered* finite structures definable in PFP are exactly those that are computable in polynomial space. Moreover, Abiteboul and Vianu [3] showed that the expressive power of PFP and IFP are equivalent on finite structures (without any assumption of order) if, and only if, the complexity classes P and PSPACE coincide.

This may be contrasted with the comparison of PFP and second order logic. The relations of these logics to complexity classes show that on ordered finite structures, PFP is at least as expressive as second order logic and, under reasonable complexity theoretic assumptions (namely that the polynomial hierarchy is properly contained in PSPACE) is strictly more expressive. However, without the assumption of order, it can be easily proved that there are second order definable properties that are not expressible in PFP. For instance there is no sentence of PFP that is true in exactly the finite structures of even size [2].

These comparisons do not easily carry over when we consider infinite structures. The problem that arises immediately is that the semantics of PFP does not extend to infinite structures. In defining the sequence of relations R_α in (6), we did not specify what happens at limit ordinals. There is no obvious choice. Taking the union of earlier stages does not seem sensible, as the stages are not increasing to start with. Perhaps the issue is best illustrated with an example. On finite structures, the power of PFP derives in part from the possibility of having the sequence of stages R_α be of length exponential in the size of the finite structure.

Consider the formula ψ given by:

$$[R(x) \wedge \exists y(y < x \wedge \neg R(y))] \;\vee\; [\neg R(x) \wedge \forall y(y < x \rightarrow R(y))] \;\vee\; \forall y\, R(y).$$

The operator defined by this formula on a finite structure $\mathbb{A}$ which interprets $<$ as a linear order has as its fixed point the universe A of $\mathbb{A}$. However, before this fixed point is reached, every subset of A occurs as some stage R_α of the induction. In particular, if we denote the elements of A, in order, by $0, \ldots, n-1$, then for all $a < 2^n$, R_a is the set of $i \in A$ such that bit i in the binary representation of a is 1.

When the operator defined by the formula ψ is interpreted on the infinite structure $(\omega, <)$, we find that every finite subset of ω occurs as a finite stage R_α of the induction. Now, there are a number of ways one could choose to define R_α for limit α. We could define $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$, which in this example would give us $R_\omega = \omega$, a fixed point. Alternatively, we could define $R_\alpha = \{x \mid \{\beta \mid x \in R_\beta\}$ is cofinal in $\alpha\}$, or $R_\alpha = \{x \mid \{\beta \mid x \in R_\beta\}$ is a final segment of $\alpha\}$. In the present example, the first of these would yield $R_\omega = \omega$, and the second $R_\omega = \emptyset$.

What is clear is that in order to give a semantics to the operator which would yield a sequence of stages that would include *all* subsets of ω and which would converge to a fixed point in 2^ω stages, we would require a definable well-ordering of the power-set of ω.

However, there is a fixed point logic, AFP—for alternating fixed point logic, which is equivalent in expressive power to PFP on finite structures, and which extends naturally into the infinite. In order to define it, it is easier to first consider the nondeterministic fixed point logic, NFP of which AFP is a natural extension. We turn our attention to these next.

7 Nondeterministic Fixed Points

The logic NFP of nondeterministic fixed points was introduced in [1] as another variant of IFP, with expressive power intermediate between IFP and PFP[3]. In terms of computational complexity NFP (or, strictly speaking its positive fragment) bears the same relationship to the complexity class NP that IFP bears to P and PFP bears to PSPACE. Before going into these connections, however, we present the definitions, which are somewhat more involved than for the earlier logics.

The logical operator nF forming nondeterministic fixed points is applied to a pair of formulas $\varphi_0(R, \mathbf{x})$ and $\varphi_1(R, \mathbf{x})$, where R is a k-ary relation symbol and $\mathbf{x}$ is a k-tuple of first-order variables. On any given structure $\mathbb{A}$, each formula determines an operator. We write Φ_0 and Φ_1 for the operators defined by φ_0 and φ_1 respectively. These two operators determine a sequence of relations as with inflationary inductions. However, the relations now are not indexed by ordinals but by ordinal-length binary strings. Towards this end, we introduce some notation for such binary strings.

For any ordinal α, a binary string of length α is any function $b : \alpha \to \{0, 1\}$. For two binary strings b and c, we say that b is an *initial segment* of c, written $b \preceq c$, if the length of b is less than or equal to the length of c and b is the restriction of c to the length of b. We say b is a *proper initial segment* of c, written $b \prec c$, if $b \preceq c$ and $b \neq c$. For any binary string b of length α, we write $b \cdot 0$ for the unique binary string c of length $\alpha + 1$ such that $b \prec c$ and $c(\alpha) = 0$. Similarly, $b \cdot 1$ is the unique binary string c of length $\alpha + 1$ such that $b \prec c$ and $c(\alpha) = 1$. For binary strings b and c, we also write $b \cdot c$ to denote the *concatenation* of the two strings. If b and c have length β and γ respectively, then $b \cdot c$ is defined to be the string d of length $\beta + \gamma$ such that, if $\alpha < \beta$, $d(\alpha) = b(\alpha)$, and if $\alpha = \beta + \delta$, then $d(\alpha) = c(\delta)$. We write ε for the empty string, i.e. the unique binary string of length 0.

On any structure $\mathbb{A}$, the two formulas φ_0 and φ_1, defining the operators Φ_0 and Φ_1 determine a class of relations R_b, for any binary string b. These are defined as follows:

$$\begin{array}{rcll} R_\varepsilon &=& \emptyset & \\ R_{c\cdot 0} &=& \Phi_0(R_c) \cup R_c & \\ R_{c\cdot 1} &=& \Phi_1(R_c) \cup R_c & \\ R_c &=& \bigcup_{b \prec c} R_b & \text{if length}(c) \text{ is a limit ordinal.} \end{array} \tag{7}$$

[3]Note that the logic we call NFP was not so called in [1]. There, the name NFP was used for what we call PFP.

Note that R_b is defined for binary strings of arbitrary ordinal length, irrespective of the cardinality of A. This is analogous to the case of deterministic inflationary inductions, where the stages R_α are defined for all ordinals α, irrespective of the cardinality of A.

Note also that the stages are increasing, in the sense that, if $b \preceq c$, then $R_b \subseteq R_c$. This implies that each sequence must reach a fixed point: If $\text{card}(A) = \kappa$, then for every string b such that $\text{length}(b) < \kappa^+$, there is a c such that $b \preceq c$, with $\text{length}(c) < \kappa^+$, and $R_c = R_{c\cdot 0} = R_{c\cdot 1}$.[4] We define the *nondeterministic fixed point* of the pair of operators Φ_0 and Φ_1 on $\mathbb{A}$ to be the relation that is the union of these fixed points, i.e.:

$$\bigcup\{R_c \mid \text{length}(c) < \kappa^+ \text{ and } R_c = R_{c\cdot 0} = R_{c\cdot 1}\}.$$

Because of the inflationary nature of the definition, by the comments above, this is just the same as:

$$\bigcup\{R_c \mid \text{length}(c) < \kappa^+\}.$$

or even $\{\mathbf{x} \mid \mathbf{x} \in R_c \text{ for some binary string } c\}$.

We define the logic NFP, whose syntax is similar to IFP, except that the iF formation rule is replaced by the following:

- If R is a relation symbol of arity k, $\mathbf{x}$ is a tuple of variables of length k, φ_0 and φ_1 are any formulas of NFP and $\mathbf{t}$ is a tuple of terms of length k, then
$$[\mathit{nF}_{R,\mathbf{x}}\ (\varphi_0, \varphi_1)](\mathbf{t})$$
is a formula of NFP,

For the semantics, we say that the formula $[\mathit{nF}_{R,\mathbf{x}}\ (\varphi_0, \varphi_1)](\mathbf{t})$ is true in $\mathbb{A}$ if $\mathbf{t}^{\mathbb{A}}$, the interpretation of $\mathbf{t}$ in $\mathbb{A}$, is in the relation that is the nondeterministic fixed point of the pair of operators Φ_0 and Φ_1, as defined above.

If we define the *positive* fragment of NFP to consist of those formulas in which the operator nF does not occur within the scope of a negation symbol, we can obtain a normal form for this fragment on finite structures. Every positive NFP formula is equivalent to one of the form:

$$\exists x\, [\mathit{nF}_{R,\mathbf{x}}\ (\varphi_0, \varphi_1)](x, \ldots, x)$$

where φ_0 and φ_1 are first order. This can be shown by techniques analogous to those used to prove the corresponding result for IFP (see [7]). Moreover, a class of finite *ordered* structures is definable in positive NFP if, and only if, it is decidable by a nondeterministic machine in polynomial time, i.e. it is in the complexity class NP [1]. The proof of this last statement can be extended to show that a class of finite *ordered* structures is definable in NFP

[4] Here, κ^+ denotes the least *infinite* cardinal greater than κ. In particular, if κ is finite, $\kappa^+ = \omega$.

(without restriction to positive formulas) if, and only if, it is definable by a formula of second order logic. Here, we establish this by proving a more general statement, which extends beyond finite structures: NFP and second order logic are equivalent over all well-ordered structures.

One direction of this equivalence holds without the assumption of an order:

Theorem 4 *For every formula φ of* NFP*, there is a formula ψ of second order logic, such that for any structure $\mathbb{A}$, $\mathbb{A} \models \varphi$ if, and only if, $\mathbb{A} \models \psi$.*

The proof of Theorem 4 is an adaptation of the translation of IFP into second order logic given in Section 5.

To be precise, given formulas $\psi_0(R)$ and $\psi_1(R)$, the relation defined by $n\mathcal{F}_{R,\mathbf{x}}\,(\psi_0, \psi_1)$ is given by a formula like (3):

$$\exists < \exists I[\mathsf{wo}(<) \wedge \theta \wedge \exists \mathbf{y} I(\mathbf{y}, \mathbf{x})].$$

where to construct the formula θ, we introduce the notation $\lim(\mathbf{y})$ to denote the formula that states that the ordinal corresponding to $\mathbf{y}$ under $<$ is a limit, $\mathrm{zero}(\mathbf{y})$ for the formula that says $\mathbf{y}$ is the first element in the order and $\mathbf{y} = \mathrm{succ}(\mathbf{z})$ to denote that the ordinal corresponding to $\mathbf{y}$ is the successor of the ordinal corresponding to $\mathbf{z}$. The formula θ is now given by:

$$\begin{array}{ll}\forall \mathbf{y}[& \mathrm{zero}(\mathbf{y}) \qquad\quad \wedge \forall \mathbf{x}(\neg I((\mathbf{y}, \mathbf{x})) \vee \\ & \lim(\mathbf{y}) \qquad\quad\;\; \wedge \forall \mathbf{x}(I(\mathbf{y}, \mathbf{x}) \leftrightarrow \exists \mathbf{z}(\mathbf{z} < \mathbf{y} \wedge I(\mathbf{z}, \mathbf{x}))) \vee \\ & \exists \mathbf{z}(\mathbf{y} = \mathrm{succ}(\mathbf{z}) \wedge \forall \mathbf{x}(I(\mathbf{y}, \mathbf{x}) \leftrightarrow \psi_0(I(\mathbf{z}), \mathbf{x})) \vee \\ & \exists \mathbf{z}(\mathbf{y} = \mathrm{succ}(\mathbf{z}) \wedge \forall \mathbf{x}(I(\mathbf{y}, \mathbf{x}) \leftrightarrow \psi_1(I(\mathbf{z}), \mathbf{x}))]\end{array}$$

Note, however, that, unlike in the case of IFP, we cannot replace the second order existential quantifiers by universal quantifiers.

In the other direction, we wish to show

Theorem 5 *For every formula φ of second order logic, there is a formula ψ of* NFP*, such that for any structure $\mathbb{A}$ which interprets $<$ as a well-ordering of its universe, $\mathbb{A} \models \varphi$ if, and only if, $\mathbb{A} \models \psi$.*

Note that the well-ordering is essential. In the language of equality, for instance, second order logic is far more expressive than NFP.

Proof of Theorem 5: It suffices to show that if $\theta(R)$ is a formula of NFP containing a relation symbol R, there is a formula of NFP equivalent to $\exists R\theta$. We carry out the construction for unary R below, giving an indication at the end how to generalize it to relation symbols of arbitrary arity.

We also assume that the formula $\exists R\theta$ has one free first-order variable y, so that on any structure $\mathbb{A}$, it defines a subset of the universe of $\mathbb{A}$. The generalization to predicates of higher arity is immediate. If $\exists R\theta$ contains no free variables, we can still consider y to be free. In this case, $\exists R\theta$ defines either all of A, or the empty set, depending on whether or not $\mathbb{A} \models \exists R\theta$. The reason for doing this is because the nondeterministic induction naturally defines a predicate. Thus, we construct a pair of formulas whose nondeterministic fixed point will yield the predicate defined by $\exists R\theta$. The idea is that the sequence of choices (of applying φ_0 or φ_1) will encode the relation R. That is, for each possible choice of R, there is a string such that the relation constructed by following that string encodes within it the relation R.

Before proceeding to the description of the formulas φ_0 and φ_1, it is best to understand their operation in terms of the simultaneous inductive definition of three sets R, S and T. Thus, for any three such sets, we think of $\Phi_0(R,S,T)$ as yielding three sets R', S' and T', and similarly for Φ_1. When we actually write out the formulas, we will combine these into one relation.

We wish to define Φ_0 such that, if $\Phi_0(R,S,T) = (R',S',T')$ and a is the $<$-least element of A that is not in S (provided there is such an element), then:

$$\begin{array}{rcl} R' & = & R \\ S' & = & \begin{cases} S \cup \{a\} & \text{if } S \neq A \\ S & \text{otherwise.} \end{cases} \\ T' & = & \begin{cases} \{b \mid \mathbb{A} \models \theta[R,b]\} & \text{if } S = A \\ T & \text{otherwise.} \end{cases} \end{array}$$

Similarly, we define Φ_1 such that, if $\Phi_1(R,S,T) = (R',S',T')$ and a is the $<$-least element of A that is not in S, then:

$$\begin{array}{rcl} R' & = & \begin{cases} R \cup \{a\} & \text{if } S \neq A \\ R & \text{otherwise.} \end{cases} \\ S' & = & \begin{cases} S \cup \{a\} & \text{if } S \neq A \\ S & \text{otherwise.} \end{cases} \\ T' & = & \begin{cases} \{b \mid \mathbb{A} \models \theta[R,b]\} & \text{if } S = A \\ T & \text{otherwise.} \end{cases} \end{array}$$

Thus, each of Φ_0 and Φ_1 always extends S by exactly one element (provided that S is not already the whole universe). Furthermore, Φ_1 also adds this element to R, while Φ_0 does not. Finally, T is unchanged by either of these operators, unless S has already exhausted the universe, at which point the elements of T are determined by the formula θ and the relation R. In the following, we permit ourselves an abuse of notation, by writing R_b, S_b and T_b for the relations obtained by applications of the operators Φ_0 and Φ_1 as determined by the binary string b, starting with all three sets being empty.

Let α denote the ordinal that is the order type of $<$ in $\mathbb{A}$, and let $o : A \to \alpha$ be the order isomorphism between $(A,<)$ and α. Then, the following observations are immediate. If b is any binary string of length α or less, then

$$\begin{array}{rcl} S_b & = & \{a \in A \mid o(a) < \text{length}(b)\} \\ R_b & = & \{a \in A \mid b(o(a)) = 1\} \\ T_b & = & \emptyset \end{array}$$

In particular, if $\text{length}(b) = \alpha$, then $S_b = A$. Moreover, for any set $B \subseteq A$, there is a binary string b of length α such that $B = R_b$. The string b is given by $b(\gamma) = 1$ if, and only if, $a \in B$, where $o(a) = \gamma$.

If c is a binary string of length $\alpha + 1$, and b is the initial segment of c of length α, then we have the following:

$$\begin{aligned} S_c &= S_b = A \\ R_c &= R_b \\ T_c &= \{a \mid \mathbb{A} \models \theta[R_b, a]\} \end{aligned}$$

Moreover, for any c' such that $c \preceq c'$, $S_c = S_{c'}$, $R_c = R_{c'}$ and $T_c = T_{c'}$ and a fixed point is reached.

Thus, the union of all T_c such that c is a binary string of length $\alpha + 1$ is the set:

$$\{a \mid \mathbb{A} \models \theta[R_b, a] \text{ for some } b \text{ of length } \alpha.\}$$

By our earlier observation, this is the same as

$$\{a \mid \mathbb{A} \models \theta[B, a] \text{ for some set } B \subseteq A\},$$

which is just

$$\{a \mid \mathbb{A} \models \exists R\theta[a]\}.$$

Finally, we write our formulas φ_0 and φ_1. For this, we fold the simultaneous inductions of the three sets into one nondeterministic induction of a binary relation. We assume that A has at least three elements in it and we write a_0, a_1 and a_2 to denote the elements of A such that $o(a_o) = 0$, $o(a_1) = 1$ and $o(a_2) = 2$. We then define a nondeterministic induction of a binary relation X, whose stages at any binary string b are given by:

$$X^b = \{a_0\} \times R_b \cup \{a_1\} \times S_b \cup \{a_2\} \times T_b.$$

The formula $\varphi_0(X, x, y)$ is now given by the following:

$$[x = a_1 \wedge \forall z(z < y \rightarrow X(a_1, z))] \vee [x = a_2 \wedge \forall z(X(a_1, z)) \wedge \theta(X(a_0), y)].$$

In the above, $x = a_1$, etc. can be replaced by appropriate definitions, using $<$. Also, $\theta(X(a_0), y)$ denotes the formula obtained from θ by replacing all occurrences of subformulas of the form $R(t)$ by $X(a_0, t)$.

Similarly, the formula $\varphi_1(X, x, y)$ is given by:

$$\begin{gathered} [(x = a_0 \vee x = a_1) \wedge (\neg X(a_1, y)) \wedge \forall z(z < y \rightarrow X(a_1, z))] \vee \\ [x = a_2 \wedge \forall z(X(a_1, z)) \wedge \theta(X(a_0), y)]. \end{gathered}$$

By the argument we have given, it follows that the predicate defined by $nF_{X,x,y}\ (\varphi_0, \varphi_1)$ on $\mathbb{A}$ is the set:

$$\{a_0\} \times A \cup \{a_1\} \times A \cup \{a_2\} \times \{a \mid \mathbb{A} \models \exists R\theta[a]\}.$$

Thus, the formula $\exists R\theta$ is equivalent on $\mathbb{A}$ to $[nF_{X,x,y}\ (\varphi_0, \varphi_1)](a_2, y)$.

Finally, we note that if $\exists R\theta$ contains more than one free variable (say k), then the relation T in the above has to be k-ary. For this, we make X $k+1$-ary, and pad appropriately its projections defining R and S. Similarly, if the symbol R has arity k greater than 1, we must choose X to be $k+1$-ary and use a well-ordering of k-tuples in the definition. Such a well-ordering is easily definable from $<$ by taking the lexicographical order. ■

The logic NFP was introduced in the context of finite model theory as a natural counterpart to IFP. It bears the same relationship to nondeterministic computation that IFP has to deterministic computation. What we have shown is that one natural way of extending its scope to infinite structures (i.e. by defining its semantics in terms of ordinal length binary strings) extends some of these relationships into the infinite. The correspondence between NFP, as we defined it, and second order logic on well-ordered structures appears as a natural generalisation of that between the two on ordered finite structures. However, there may be other natural ways of introducing a nondeterministic induction principle, and we invite the reader to consider alternatives.

8 Alternating Fixed Point Logic

The previous section demonstrated the close relationship between nondeterministic fixed point constructions and second order logic in the presence of a well-ordering. The correspondence is precise in that the alternation of negation with the $n\mathcal{F}$ operator matches the alternation of second order quantifiers. The logic AFP of *alternating fixed points* can be seen as extending NFP to allow an unbounded number of alternations. This logic was introduced in [1], where it was established that AFP has the same expressive power on finite structures as PFP. While alternating fixed points (and, indeed, nondeterministic fixed points) were defined in [1] in terms of trees, for the purpose of defining such constructions on infinite structures, we describe them in terms of games.

Suppose we have two formulas $\varphi_0(R, \mathbf{x})$ and $\varphi_1(R, \mathbf{x})$ which define a pair of operators Φ_0 and Φ_1 on a structure $\mathbb{A}$. Then, as in the previous section, we obtain a series of relations R_b, for ordinal-length binary strings b. While the nondeterministic fixed point of Φ_0 and Φ_1 would be obtained by taking the union of all these relations, the *alternating* fixed point is obtained by a rather more involved method of combining these relations. This method is best described in terms of a two player game, which we call the *string construction game*.

Let α be an ordinal. A play of length α between players P-I and P-II will produce a binary string of length α. An ordinal $\beta < \alpha$ is said to be *P-I's turn* if $\beta = 0$, or β is a limit ordinal or the predecessor of β is P-II's turn. β is said to be *P-II's turn* if its predecessor is P-I's turn. At any stage $\beta < \alpha$ of the game, a value in $\{0, 1\}$ has been assigned to every ordinal $\gamma < \beta$. At this stage, if β is P-I's turn, then P-I assigns either 0 or 1 to the ordinal β. Otherwise, it is P-II's turn, and P-II assigns a value of either 0 or 1 to β. As can be seen, the result of a play of length α is a binary string b of length α.

Given a tuple $\mathbf{a} \in \mathbb{A}$, we say that P-I wins a play of the string construction game, if $\mathbf{a} \in R_b$, where b is the binary string constructed in the play. We say that P-I has a winning strategy on $\mathbf{a}$ if, no matter how P-II plays, P-I can always force the construction of a string b such that $\mathbf{a} \in R_b$. Now, we are ready to define the *alternating fixed point* of the two operators Φ_0 and Φ_1: it is the relation consisting of all tuples $\mathbf{a}$ for which P-I has a winning strategy in the string construction game on $\mathbf{a}$.

Finally, we define the logic AFP as similar to NFP, except that the $n\mathcal{F}$ formula formation rule is replaced by the rule

- If R is a relation symbol of arity k, $\mathbf{x}$ is a tuple of variables of length k, $\mathbf{t}$ is a tuple of terms of length k and φ_0 and φ_1 are any formulas of AFP, then

$$[a\mathcal{F}_{R,\mathbf{x}}\,(\varphi_0,\varphi_1)](\mathbf{t})$$

  is a formula of AFP of arity k.

For the semantics, we say that

$$[a\mathcal{F}_{R,\mathbf{x}}\,(\varphi_0,\varphi_1)](\mathbf{t})$$

is true in $\mathbb{A}$ if, and only if, $\mathbf{t}^{\mathbb{A}}$ is in the alternating fixed point of the pair of operators defined by $\varphi_0(R,\mathbf{x})$ and $\varphi_1(R,\mathbf{x})$.

In terms of expressive power, we can show that every formula of NFP is equivalent to a formula of AFP.

Lemma 6 *Every formula of NFP is equivalent to one of AFP.*

Proof: It suffices to show that the predicate $[n\mathcal{F}_{R,\mathbf{x}}\,(\psi_0,\psi_1)]$ is definable by means of the $a\mathcal{F}$ operator. To do this, we construct two operators Φ_0 and Φ_1, which we think of as operating on a pair of relations (R,P). As in the proof of Theorem 5, these can be coded into a single relation for the purpose of writing the formulas. The two operators are defined by, $\Phi_0(R,P)=(R',P')$ where:

$$\begin{aligned} R' &= \begin{cases} \psi_0(R) & \text{if } R=P \\ R & \text{otherwise.} \end{cases} \\ P' &= R \end{aligned}$$

and $\Phi_1(R,P)=(R',P')$ where:

$$\begin{aligned} R' &= \begin{cases} \psi_1(R) & \text{if } R=P \\ R & \text{otherwise.} \end{cases} \\ P' &= R \end{aligned}$$

This ensures that in the string construction game played with these two operators, whenever it is P-II's turn, the result of applying the operation is merely to copy the relation R into P. However, when it is P-I's turn, either the formula ψ_0 or ψ_1 is evaluated, depending on which operator P-I chooses to play. This ensures that for any binary string b, P-I has a strategy for defining the relation R_b determined by b and the two formulas ψ_0 and ψ_1. It follows that the alternating fixed point of the two operators Φ_0 and Φ_1 is exactly $[n\mathcal{F}_{R,\mathbf{x}}\,(\psi_0,\psi_1)]$. ∎

One kind of question that immediately arises from the definition of alternating fixed points is whether the string construction game is always determined. That is, for any pair of formulas φ_0 and φ_1 defining a pair of operators Φ_0 and Φ_1 on a structure $\mathbb{A}$, and any tuple $\mathbf{a} \in \mathbb{A}$, is it necessarily the case that one of P-I or P-II has a winning strategy in the string construction game on $\mathbf{a}$? As we shall see below, the game is not determined for all $\mathbb{A}$. However, in order to classify some structures (and formulas) on which it is determined, it is useful to introduce some terminology.

For fixed $\mathbb{A}, \Phi_0$ and Φ_1, we say that a binary string b is *closed* if $R_b = R_{b\cdot 0} = R_{b\cdot 1}$. For an ordinal α, we say that the induction of Φ_0 and Φ_1 on $\mathbb{A}$ is *closed at* α if all strings of length α are closed. This allows us to state the following result:

Theorem 7 *If the induction of Φ_0 and Φ_1 on $\mathbb{A}$ is* closed at ω, *then, for any tuple* $\mathbf{a}$ *from $\mathbb{A}$, the corresponding string construction game is determined.*

Proof: To establish the result, it suffices to prove that the set of strings $\{b \mid \mathbf{a} \in R_b\}$ is an open set in the standard product topology on ${}^{\omega}2$ derived from the discrete topology on 2. This is because, by the theorem of Gale and Stewart [10], if the set of winning strings is an open set, then the game is determined.

To see that the set $\{b \mid \mathbf{a} \in R_b\}$ is an open set, we observe that by the inflationary nature of the operators, if $\mathbf{a} \in R_b$, then there is a finite string $c \prec b$ such that $\mathbf{a} \in R_c$, and therefore, $\mathbf{a} \in R_{b'}$ for all ω-length strings b' such that $c \prec b'$. ■

We also have, from Gale and Stewart [10], a set W of ω-length strings such that if W is the set of winning strings for P-I in an ω-length string construction game, then the game is indeterminate. We next see that this yields a structure (albeit an uncountable one), where the game we have defined is indeterminate.

Let $\mathbb{A}$ be a structure whose universe A is the union of three disjoint sets, $O = \{0\} \times \omega$, $P = \{1\} \times {}^{\omega}2$, and a singleton $\{a\}$. $\mathbb{A}$ interprets two binary relations: $\ll = \{((0,m),(0,n)) \mid m < n\}$, the standard ordering on O; and $E = \{((0,m),(1,s)) \mid s(m) = 1\}$. $\mathbb{A}$ also interprets a unary relation symbol S as the subset of P encoding the undetermined winning set W of Gale and Stewart.

Now, it is easy to construct the desired first order definable operators Φ_0 and Φ_1. It is easiest to think of them as defining, simultaneously, a pair of unary relations R and S (as in the proof of Theorem 5). If there is an element of O that is not in S, let x be the least such element (according to the order $\ll$) and let $\Phi_0(R,S) = (R, S \cup \{x\})$ and $\Phi_1(R,S) = (R \cup \{x\}, S \cup \{x\})$; if $O \subseteq S$ and $W(p_R)$, where $p_R \in P$ is the element $(1, \{m \mid (0,m) \in R\})$, then $\Phi_0(R,S) = \Phi_1(R,S) = (R \cup \{a\}, S)$; otherwise $\Phi_0(R,S) = \Phi_1(R,S) = (R,S)$.

It can be easily checked that the induction of Φ_0 and Φ_1 on $\mathbb{A}$ closes at $\omega+1$, and neither P-I nor P-II has a winning strategy on a.

This leaves open the question of whether the game is determined on every countable structure. What we can show is that there is a pair of formulas (even first order formulas) for which determinacy of the string construction game is independent of ZFC. We establish this below.

Theorem 8 *There are two first order definable operators Φ_0 and Φ_1 such that the determinacy of the game they define on $(\omega, <)$ is independent of ZFC.*

Proof: We first define a pair of operators which have the required property on the structure of arithmetic $(\omega, <, +, \times)$, and then note at the end how they can be modified to obtain operators that work on the order $(\omega, <)$. We use the fact that there is a Σ^1_1 subset W of ${}^{\omega}2$ such that the question whether P-I has a winning strategy to construct an ω-length string in W is independent of ZFC (see [17, 6A.12 and 6G.7]).

Since W is Σ^1_1, there is a first order formula $\theta(X, R)$, where X and R are unary relation symbols, such that $(\omega, <, +, \times, R) \models \exists X \theta$ if, and only if, $b_R \in W$, where $b_R : \omega \to \{0, 1\}$ is the string given by $b_R(n) = 1$ if, and only if, $n \in R$.

We define the operators Φ_0 and Φ_1 so that they construct three relations X, R and T, where T is only changed at the last stage, and 0 is included in T if, and only if, $(\omega, <, +, \times)$ satisfies $\theta(X, R)$. R is constructed in the first ω moves of the game, using an auxiliary relation S. In the next ω moves, we construct X, using an auxiliary relation Y, as well as a relation Z that guarantees that the construction of X is entirely under the control of player P-I, as in the proof of Lemma 6. Thus, the operators are defined in terms of the simultaneous definition of six sets, as follows

First, we define Φ_0 such that $\Phi_0(X, Y, Z, R, S, T) = (X', Y', Z', R', S', T')$, where:

$$
\begin{aligned}
R' &= R \\
S' &= \begin{cases} S \cup \{x\} & \text{if } S \neq \omega \\ S & \text{otherwise.} \end{cases} \\
&\quad \text{where } x \text{ is the least element not in } S \\
X' &= X \\
Y' &= \begin{cases} Y & \text{if } S \neq \omega \text{ or } Y \neq Z \text{ or } Y = \omega \\ Y \cup \{y\} & \text{otherwise} \end{cases} \\
&\quad \text{where } y \text{ is the least element not in } Y \\
Z' &= Y \\
T' &= \begin{cases} T \cup \{0\} & \text{if } S = Y = \omega \text{ and } \theta(X, R) \\ T & \text{otherwise.} \end{cases}
\end{aligned}
$$

Similarly, we define Φ_1 such that, if $\Phi_1(R,S,T) = (R',S',T')$:

$$
\begin{array}{rcl}
R' & = & \left\{ \begin{array}{ll} R \cup \{x\} & \text{if } S \neq \omega \\ R & \text{otherwise.} \end{array} \right. \\
& & \text{where } x \text{ is the least element not in } S \\
S' & = & \left\{ \begin{array}{ll} S \cup \{x\} & \text{if } S \neq \omega \\ S & \text{otherwise.} \end{array} \right. \\
& & \text{where } x \text{ is the least element not in } S \\
X' & = & \left\{ \begin{array}{ll} X & \text{if } S \neq \omega \text{ or } Y \neq Z \text{ or } Y = \omega \\ X \cup \{y\} & \text{otherwise} \end{array} \right. \\
& & \text{where } y \text{ is the least element not in } Y \\
Y' & = & \left\{ \begin{array}{ll} Y & \text{if } S \neq \omega \text{ or } Y \neq Z \text{ or } Y = \omega \\ Y \cup \{y\} & \text{otherwise} \end{array} \right. \\
& & \text{where } y \text{ is the least element not in } Y \\
Z' & = & Y \\
T' & = & \left\{ \begin{array}{ll} T \cup \{0\} & \text{if } S = Y = \omega \text{ and } \theta(X,R) \\ T & \text{otherwise.} \end{array} \right.
\end{array}
$$

To complete the argument, note that after ω moves of the game, a binary string b_R has been constructed, and the relation R is a subset of ω. S is used as a counter for these first ω moves. In the next ω moves, a set X is built up, with Y used as a counter, and Z used to ensure that X can only be changed at stages that are P-I's turn. Finally, at stage $\omega + \omega + 1$, 0 is included in T if, and only if, $\theta(X,R)$ holds. Since, after ω moves, P-I is entirely in control of constructing X, he has a winning strategy for the inclusion of 0 in T if, and only if, $(\omega, <, R) \models \exists X\theta$. But, this is true if, and only if, $b_R \in W$. Thus, P-I can ensure that $0 \in T$ after $\omega+\omega+1$ steps if, and only if, he can ensure that at stage ω, $b_R \in W$. It follows that the existence of a winning strategy for either player is independent of ZFC.

Finally, note that the relations $+$ and $\times$ are inductively definable (in IFP, for instance) in the structure $(\omega, <)$. Thus, the operators we have defined could be modified, so that in the first ω stages, these two relations are defined. Indeed, since their interpretation is fixed in advance, they can be constructed simultaneously with R, and we obtain the required operators in the language of order. Moreover, the induction still closes at stage $\omega + \omega + 1$. ■

As indicated in the proof, the induction defined by the two operators constructed closes at $\omega + \omega + 1$. By similar techniques, we can define two AFP formulas whose induction closes at $\omega + 1$, which have the same property. Essentially, the additional ω steps used to simulate the existential second order quantifier can be built into the definition of the operator. This should be compared with Theorem 7.

Finally, we note that the closure ordinals considered above are small, compared to what is possible. This is established by the following result.

Theorem 9 *There are two first order definable operators Φ_0 and Φ_1 such that the induction of Φ_0 and Φ_1 on $(\omega, <)$ is not closed at any countable ordinal.*

Proof: It is again easiest to describe the operators in terms of the simultaneous iterative construction, this time, of a pair of binary relations R and S and a unary relation T.

Choose any first-order definable order on $\omega \times \omega$, of order-type ω. Let (x, y) be the least pair in this order that is not in the relation S, if there is such a pair. We wish to define Φ_0 such that if $\Phi_0(R, S, T) = (R', S', T')$ then:

$$\begin{aligned} R' &= R \\ S' &= \begin{cases} S \cup \{(x,y)\} & \text{if } S \neq \omega \times \omega \\ S & \text{otherwise.} \end{cases} \\ T' &= \begin{cases} \{b \mid \forall a\, (R(a,b) \to T(a))\} & \text{if } S = \omega \times \omega \\ T & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, we define Φ_1 such that, if $\Phi_1(R, S, T) = (R', S', T')$ and (x, y) is the least pair that is not in S, then:

$$\begin{aligned} R' &= \begin{cases} R \cup \{(x,y)\} & \text{if } S \neq \omega \times \omega \\ R & \text{otherwise.} \end{cases} \\ S' &= \begin{cases} S \cup \{(x,y)\} & \text{if } S \neq \omega \times \omega \\ S & \text{otherwise.} \end{cases} \\ T' &= \begin{cases} \{b \mid \forall a\, (R(a,b) \to T(a))\} & \text{if } S = \omega \times \omega \\ T & \text{otherwise.} \end{cases} \end{aligned}$$

We then have, that for any binary string b of length ω, $S_b = \omega \times \omega$. Moreover, for any binary relation R on ω, there is a string b of length ω, such that $R_b = R$. In particular, for any countable ordinal α, there is a string b such that R_b is a well-ordering of length α.

After stage ω, neither R nor S change. Moreover, both operators act in the same way on T. The induction reaches a fixed point when T contains the well-founded part of R (compare Example 1.2). The length of this induction is the same as the height of the well-founded part of R. If R_b is a well-ordering of length α, and c is a string of length α, then $T_{b \cdot c} \neq T_{b \cdot c \cdot 0} = T_{b \cdot c \cdot 1}$, and closure is not reached until $\omega + \alpha$.[5] Since R can be *Any* binary relation on ω, we conclude that there is no countable ordinal at which the induction of Φ_0 and Φ_1 is closed.
■

We believe that this strongly suggests that one might be able to find a countable structure and two AFP (or even first order) definable operators for which one can prove (in ZFC) that the string construction game is indeterminate.

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