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SEMI-CONSERVATIVE REDUCTION*

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Section 1

Let L be the set of formulas of an ordinary version of the first order predicate calculus. The term "class" is used for "subset of L".

Let K_1 and K_2 be classes. A *conservative reduction* (see [3]) of K_1 to K_2 is an algorithm f: $K_1 \to K_2$ such that the following four conditions hold (where A runs through K_1):

(a) if A is satisfiable then fA is so,
(b) if A is finitely satisfiable then fA is so,
(c) if fA is satisfiable then A is so,
(d) if fA is finitely satisfiable then A is so.

This notion is widely used in the decision problem. A class K is called *conservative* if there exists a conservative reduction of L to K.

It is easy to check that no one of the conditions $(a)-(d)$ can be missed. But often K_1 is conservative and we are interested only in the fact that a conservative reduction of K_1 to K_2 can be constructed.

Definition. A *semi-conservative reduction* of K_1 to K_2 is an algotithm f: $K_1 \to K_2$ such that the conditions (b) and (c) hold.

Theorem 1. If a conservative class K_1 is semi-conservatively reducible to a class K_2 then K_2 is conservative too.

Turing's thesis is used here. We shall construct a recursive function, realizing a desired reduction, from given recursive and partially recursive functions. It is interesting if it is possible to eliminate Turing's thesis and to construct desired reductions from given ones without the technique of the theory of recursive functions.

Section 2

We are using book [1]. P denotes the set of positive integers. Usually the set of non-negative integers serves as the basic set in the theory of recursive functions. Smullyan has chosen P to the purpose. It makes no difference.

Below x and y range over P, and X, Y, U, V are subsets of P. For a function f: $P \to P$ let

$$fX = \{fx : x \in X\} \quad \text{and} \quad f^{-1}X = \{x : fx \in X\}.$$

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All recursively enumerable (r.e.) sets are arranged in the sequence $R_1, R_2, \ldots$ such that the predicate $y \in R_x$ is r.e.
The following definitions are used (cf. [1]). X and Y are *effectively inseparable* (e.i.) *under a number function* $G(x, y)$ iff they are disjoint, and G is recursive, and for every disjoint supersets R_x and R_y (supersets of X and Y respectively), $G(x, y)$ is outside both R_x and R_y. X and Y are *doubly creative* (d.c.) *under a number function* $G(x, y)$ iff they are disjoint, and G is recursive, and for every R_x, R_y which are disjoint and respectively disjoint from X and Y (i.e. R_x is disjoint from X and R_y is disjoint from Y), $G(x, y)$ is outside all four sets X, Y, R_x and R_y. X and Y are *e.i.* iff they are e.i. under some function G.

Lemma 1. Let
X and Y be e.i.,
U and V be disjoint,
$f(x)$ be a recursive function, and
fX be included in U, fY be included in V.
Then there exists a recursive function $G(x, y)$ such that U and V are e.i. under G and Rng G is included in Rng f.

Proof. See the proof of Proposition 4 in Chapter 5 of [1].

Lemma 2. If U and V are r.e. and e.i. under $G(x, y)$ then there exists a recursive function $H(x, y)$ such that U and V are d.c. under H and Rng H is included in Rng G.

Proof. See the proof of Theorem 12 (ibid.).

Lemma 3. Let
U and V be d.c. under $H(x, y)$, and
X and Y be r.e. and disjoint.
Then there exists a recursive function $h(x)$ such that $X=h^{-1}U$, $Y=h^{-1}V$ and Rng h is included in Rng H.

Proof. See the proof of Theorem 15 (ibid,).

Section 3

Fix an effective $1-1$-enumeration of the formulas by positive integers. We say that a class K is recursive or r.e. iff the corresponding number set is so, a function f: $L \to L$ is recursive iff the corresponding number function is so, and so on.
Let N be the set of logically false formulas, and F be the set of finitely satisfiable formulas. For a class K let NK (resp., FK) be the intersection of N (resp., F) and K.
In [2] it is proved the following.

Proposition 1. N and F are e.i.

Proof of Theorem 1. Without loss of generality, $K_1 = L$. Suppose that a recursive function f semi-conservatively reduces L to K_2. Then fN, fF, fL are included in N, F, K_2 respectively. Let $K = fL$. It is enough to prove that K is conservative. Clearly NK and FK are r.e.
By Lemma 1 and Proposition 1, NK and FK are e.i. under a function G such that Rng G is included in K. By Lemmas 2 and 3, there exists a recursive function $h(x)$ such that $N = h^{-1}NK$, $F = h^{-1}FK$ and Rng h is included in Rng G. Clearly h is a conservative reduction of L to K. Q.E.D.

Corollary. Let K be a recursive class. Then K is conservative iff NK and FK are e.i.

Proof. If K is conservative then NK and FK are e.i. according to Lemma 1.
Let NK and FK be e.i. Then there exists a formula A outside both NK and FK. NK and FK are r.e. since K is recursive. By Lemmas 2 and 3, there exists a recursive function $h(x)$ such that $N = h^{-1}NK$ and $F = h^{-1}FK$. Let $g(x) = h(x)$ if $h(x)$ belongs to K, and $g(x) = A$ in the other case. Clearly g is recursive and conservatively reduces L to K. Q.E.D.

Note. Let K be a r.e. class. It is easy to deduce from Lemmas 1–3 that K is conservative iff NK and FK are e.i. under a function G such that Rng G is included in K. The proviso "Rng G is included in K" is essential. For example, let K be the union of N and F. Then K is r.e. and NK, FK are e.i. but K is not conservative.

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