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Intuitionistic Logic with Strong Negation

Introduction

A Kripke model of intuitionistic predicate logic can be described (see [1]) as a quadruple $\mathscr{K} = \langle M, \leqslant, \delta, \tau\rangle$ where $\langle M, \leqslant\rangle$ is a poset (partially ordered set), δ is a non-decreasing function associating a set of individual constants with each $X \in M$, and for each $X \in M$ and each formula A, $\tau_X A$ is equal to either "true" or "uncertain". The details can be found in § 3 below. In particular, $\tau_X(\neg A)$ = "true" iff for each $Y \geqslant X$, $\tau_Y A \neq$ "true".

In the spirit of Grzegorczyk's paper [2] $\mathscr{K}$ may be interpreted as a scheme of a scientific research. Elements of M are the stages of the research, $\leqslant$ is the precedence relation, $\delta(X)$ is the set of objects involved in the research at stage X. For an atomic formula A, $\tau_X A$ is a product of experiment. "The compound sentences are not a product of experiment" – writes Grzegorczyk – "They arise from reasoning. This concerns also negations: we see that the lemon is yellow, we do not see that it is not blue".

This paper is a reaction for this remark of Grzegorczyk. In many cases the falsehood of a simple scientific sentence can be ascertained as directly (or indirectly) as its truth. An example: a litmus-paper is used to verify sentence "The solution is acid". We regard a generalizations of Kripke models when $\tau_X A$ can be equal to "false", "uncertain" or "true". That gives rise to a conservative extension of the intuitionistic logic which is nicer at least in one aspect: it is more symmetric, it satisfies very natural duality laws.

We use the strong negation to formalize the arising logic. The propositional intuitionistic logic with strong negation was regarded in [5], [7], [8] and [10]. We use here Vorob'ev's calculus in [10]. Thomason developed in [9] semantics, which is very close to ours, and the corresponding calculus ***CF***. Unfortunately ***CF*** is not a conservative extension of the ordinary intuitionistic logic. For example formula $\forall x(A \vee C) \supset (\forall x A \vee C)$, where x does not occur in C, is provable in ***CF***. It seems that even in the propositional case the duality laws of intuitionistic logic with strong negation were not mentioned before.

In § 1 we introduce a Hilbert-type calculus $\overline{\boldsymbol{H}}$ formalizing intuitionistic logic with strong negation. In § 2 $\overline{\boldsymbol{H}}$ is interpreted in the ordinary intuitionistic calculus $\boldsymbol{H}$ and it is proved that $\overline{\boldsymbol{H}}$ extends $\boldsymbol{H}$ conservatively. In § 3 Kripke models of $\overline{\boldsymbol{H}}$ are defined. In § 4 the completeness theorem is proved. In § 5 duality laws are proved. In § 6 complete and independent systems of logical operators for $\boldsymbol{H}$ and for the propositional part of $\overline{\boldsymbol{H}}$ are presented. In § 7 a 3-valued logic associated with $\overline{\boldsymbol{H}}$ is discussed. In § 8 a Gentzen-type calculus corresponding to $\overline{\boldsymbol{H}}$ is considered.

The paper was written in Russian in 1972 and translated into English in 1976. Discussions with Leo Esakia were very usefull to the author.

Note: Metalogic of this paper is classic.

§ 1. Predicate calculus

In this section we define a calculus $\overline{\boldsymbol{H}}$ formalizing intuitionistic logic with strong negation.

Let $\boldsymbol{H}$ be the intuitionistic predicate calculus of [3] enriched by a denumerable list of individual constants. Recall that $A \sim B$ abbreviates $(A \supset B) \mathbin{\&} (B \supset A)$. $\overline{\boldsymbol{H}}$ is obtained from $\boldsymbol{H}$ by adding a new unary propositional connective "$-$" (called strong negation or minus) and the following axiom schemata:

$\bar{1}$. $-(A \supset B) \sim A \mathbin{\&} -B$,

$\bar{2}$. $-(A \mathbin{\&} B) \sim -A \vee -B$,

$\bar{3}$. $-(A \vee B) \sim -A \mathbin{\&} -B$,

$\bar{4}$. $-\neg A \sim A$,

$\bar{5}$. $--A \sim A$,

$\bar{6}$. $-\exists x A \sim \forall x -A$,

$\bar{7}$. $-\forall x A \sim \exists x -A$,

$\bar{8}$. (for atomic A's only) $-A \supset \neg A$.

Here and below minus and other unary logical operators bind closer than any binary connective. Here and below A and B range over the $\overline{\boldsymbol{H}}$-formulae if the contrary is not said explicitly. In this section the sign $\vdash$ means provability and deducibility on $\overline{\boldsymbol{H}}$.

Clearly $\overline{\boldsymbol{H}}$ satisfies the Deduction Theorem.

THEOREM 1.1. $\vdash -A \supset (A \supset B)$.

PROOF. It is enough to deduce B from A and $-A$ without variation of variables. If A is atomic use $\bar{8}$. In the other cases use $\bar{1}, \ldots, \bar{7}$ respectively. #

Hence $\vdash -A \supset \neg A$ for all A's, not only for atomic one's.

In order to prove replacement theorems fix an $\overline{\boldsymbol{H}}$-formula C and a propositional letter p. Let C_A be the result of replacing all occurrences of p in C by A. Let V_A be the set of individual variables x such that x

occurs free in A and some occurrence of p in C lies in the scope of $\exists x$ or $\forall x$.

THEOREM 1.2. (Replacement property of equivalence). *If the minus does not occur in C then $A \sim B \vdash C_A \sim C_B$ where the variables of $V_A \cup V_B$ are varied.*

PROOF. See proof of the Replacement Theorem (Theorem 14) in [3]. #

Let $A \equiv B$ abbreviate $(A \sim B) \& (-A \sim -B)$ (the strong equivalence).

LEMMA 1.3.

(i) $A \equiv B \vdash, \quad -A \equiv -B$, *and the same for* $\neg$;

(ii) $A_1 \equiv B_1, \quad A_2 \equiv B_2 \vdash A_1 \supset A_2 \equiv B_1 \supset B_2$, *and the same for* & *and* $\vee$;

(iii) $A \equiv B \vdash \forall x A \equiv \forall x B$ *where x is varied, and the same for* $\exists$.

PROOF is clear. #

THEOREM 1.4. (Replacement property of the strong equivalence).

$A \equiv B \vdash C_A \equiv C_B$ *where the variables of $V_A \cup V_B$ are varied.*

PROOF by induction on C. The induction step uses Lemma 1.3. #

§ 2. Reduced formulae

Here we interpret $\bar{\boldsymbol{H}}$ in $\boldsymbol{H}$ and prove that $\bar{\boldsymbol{H}}$ extends $\boldsymbol{H}$ conservatively.

A is called *reduced* (cf. [10]) iff the scope of each occurrence of minus in A is an atomic formula. An $\bar{\boldsymbol{H}}$-proof $(A_1, \ldots, A_n)$ is called *reduced* iff the formulas $A_1, \ldots, A_n$ are reduced. The *reduction operation* r is defined inductively:

$rA = A$ and $r(-A) = -A$ if A is atomic;
$r(\neg A) = r(A)$;
$r(A \supset B) = r(A) \supset r(B)$, and the same for & and $\vee$;
$r(\forall x A) = \forall x(rA)$, and the same for $\exists$;
$r\big(-(A \supset B)\big) = rA \ \& \ (-rB)$;
$r\big(-(A \ \& B)\big) = (-rA) \vee (-rB)$ and
$r\big(-(A \vee B)\big) = (-rA) \& (r - B)$;
$r(-\forall x A) = \exists x(-rA)$ and
$r(-\exists x A) = \forall x(-rA)$;
$r(-\neg A) = r(--A) = rA$.

Theorems 2.1 and 2.2 below generalize the analoguous results in [10].

THEOREM 2.1. *rA is reduced and the formula $A \sim r(A)$ is provable in $\bar{\boldsymbol{H}}$.*

PROOF. Easy induction on A. #

THEOREM 2.2. *If A is provable in $\bar{\boldsymbol{H}}$ then there exists a reduced proof of rA in $\bar{\boldsymbol{H}}$.*

PROOF. Easy induction on the length of an $\overline{\boldsymbol{H}}$-proof of A. #

THEOREM 2.3. *Let $P_1, \ldots, P_k, Q_1, \ldots, Q_k$ be different predicate letters where P_i and Q_i are m_i-ary;* $E = \bigwedge_i \forall x_1 \ldots x_{m_i}(Q_i x_1 \ldots x_{m_i} \supset \neg P x_1 \ldots x_{m_i})$; *$A$ range over the $\overline{\boldsymbol{H}}$-formulae with predicate letters among $P_1, \ldots, P_k$; qA be the result of replacing $-P_1, \ldots, -P_k$ in rA by $Q_1, \ldots, Q_k$.*

Then A is provable in $\overline{\boldsymbol{H}}$ iff $E \supset qA$ is provable in $\boldsymbol{H}$.

PROOF. Suppose that A is provable in $\overline{\boldsymbol{H}}$. By Theorem 2.2 there exists a reduced proof $(A_1, \ldots, A_n)$ of rA. Without loss of generality all predicate letters occurring in this proof are among $P_1, \ldots, P_k$. Then $(qA_1, \ldots, qA_n)$ is a deduction of qA in $\boldsymbol{H}$ from hypothesises of the form $Q_i t_1 \ldots t_{m_i} \supset \neg P_i t_1 \ldots t_{m_i}$ which are deducible in $\boldsymbol{H}$ from E.

Now let $(B_1, \ldots, B_n)$ be an $\boldsymbol{H}$-proof of $E \supset qA$. Without loss of generality all predicate letters occurring in this proof are among $P_1, \ldots, P_k, Q_1, \ldots, Q_k$. Let $A_1, \ldots, A_n$ and D be the results of replacing $Q_1, \ldots, Q_k$ by $-P_1, \ldots, -P_k$ in $B_1, \ldots, B_n$ and E respectively. Then $(A_1, \ldots, A_n)$ is an $\overline{\boldsymbol{H}}$-proof of $D \supset rA$ and D is provable in $\overline{\boldsymbol{H}}$. By Theorem 2.1, A is provable in $\overline{\boldsymbol{H}}$. #

Let wA be the result of replacing the strong negation in A by the ordinary one.

LEMMA 2.4. *Let A be reduced. If A is provable in $\overline{\boldsymbol{H}}$ then wA is provable in $\boldsymbol{H}$.*

PROOF. Easy induction on the length of a reduced $\overline{\boldsymbol{H}}$-proof of A. #

THEOREM 2.5. *Let A be an $\boldsymbol{H}$-formula. If A is provable in $\overline{\boldsymbol{H}}$ then A is provable in $\boldsymbol{H}$.*

PROOF. Use Lemma 2.4. #

§ 3. Kripke models

We recall here Kripke models of $\boldsymbol{H}$ (according to [1]) and define Kripke models of $\overline{\boldsymbol{H}}$.

Let $Cn(A)$ be the set of individual constants occuriing in A. A Kripke model of $\boldsymbol{H}$ (respectively $\overline{\boldsymbol{H}}$) is a quadruple $\mathscr{K} = \langle M, \leqslant, \delta, \tau\rangle$ where: $\langle M, \leqslant\rangle$ is a poset (the poset of stages of $\mathscr{K}$); δ associates a set of individual constants with each stage in such a way that $X \leqslant Y$ implies $\delta X \subseteq \delta Y$; and τ: $M \times$ (the set of H-sentences)$\to\{0, 1\}$ (resp. τ: $M \times$ (the set of $\boldsymbol{H}$-sentences)$\to\{-1, 0, 1\}$) satisfies the following conditions 3.1-3.7 (resp. 3.1-3.14):

3.1 If A is atomic, $\tau_X A \neq 0$ and $X \leqslant Y$ then $Cn(A) \subseteq \delta X$ and $\tau_Y A = \tau_X A$;

3.2 $\tau_X(A \,\&\, B) = 1$ iff $\min\{\tau_X A, \tau_X B\} = 1$;

3.3 $\tau_X(A \vee B) = 1$ iff $Cn(A \vee B) \subseteq \delta X$ and $\max\{\tau_X A, \tau_X B\} = 1$;

3.4 $\tau_X(\neg A) = 1$ iff $Cn(A) \subseteq \delta X$ and for each $Y \geqslant X$, $\tau_Y A < 1$;
3.5 $\tau_X(A \supset B) = 1$ iff $Cn(A \supset B) \subseteq \delta X$ and for each $Y \geqslant X$, if $\tau_Y A = 1$ then $\tau_Y B = 1$;
3.6 $\tau_X \forall x A(x) = 1$ iff for every $Y \geqslant X$ and $c \in \delta Y$, $\tau_Y A(c) = 1$;
3.7 $\tau_X \exists A(x) = 1$ iff there exists $c \in \delta X$ such that $\tau_X A(c) = 1$;
3.8 $\tau_X(A \,\&\, B) = -1$ iff $Cn(A \,\&\, B) \subseteq \delta X$ and $\min\{\tau_X A, \tau_X B\} = -1$;
3.9 $\tau_X(A \vee B) = -1$ iff $\max\{\tau_X A, \tau_X B\} = -1$;
3.10 $\tau_X(\neg A) = -1$ iff $\tau_X A = 1$;
3.11 $\tau_X(A \supset B) = -1$ iff $\tau_X A = 1$ and $\tau_X B = -1$;
3.12 $\tau_X \forall x A(x) = -1$ iff there exists $c \in \delta X$ such that $\tau_X A(c) = -1$;
3.13 $\tau_X \exists x A(x) = -1$ iff for every $Y \geqslant X$ and $c \in \delta Y$, $\tau_Y A(c) = -1$;
3.14 $\tau_X(-A) = -\tau_X A$.

LEMMA 3.1. *For each formula A, if $\tau_X A \neq 0$ and $X \leqslant Y$ then* $Cn(F) \subseteq \delta X$ and $\tau_Y A = \tau_X A$.

PROOF. Easy induction on A. #

LEMMA 3.2. *Let M, $\leqslant$ and δ be as above and τ^0: $M \times$ (the set of atomic $\boldsymbol{H}$-formulae)$\to\{0, 1\}$ (resp. τ^0: $M \times$ (the set of atomic $\overline{\boldsymbol{H}}$-formulae)$\to\{-1, 0, 1\}$) satisfy condition 3.1 then. Then there exists a unique extension τ of τ^0 such that $\mathscr{K} = \langle M, \leqslant, \delta, \tau\rangle$ is a Kripke model of $\boldsymbol{H}$ (resp. of $\overline{\boldsymbol{H}}$).*

PROOF is clear. #

DEFINITION. Let $\mathscr{K} = \langle M, \leqslant, \delta, \tau\rangle$ be a Kripke model of $\boldsymbol{H}$ (resp. $\overline{\boldsymbol{H}}$). A is *defined* (resp. *true*) at stage X iff $Cn(A) \subseteq \delta X$ (resp. $\tau_X A = 1$). A is *defined* in $\mathscr{K}$ iff A is defined at some stage of $\mathscr{K}$. A is *true* in $\mathscr{K}$ iff A is defined in $\mathscr{K}$ and it is true at each stage of $\mathscr{K}$ where it is defined.

THEOREM 3.3. (Correctness Theorem). *If A is provable in $\boldsymbol{H}$ (resp in $\overline{\boldsymbol{H}}$) then it is true in all Kripke models of $\boldsymbol{H}$ (resp. of $\overline{\boldsymbol{H}}$) where it is defined.*

PROOF. Easy induction on the length of a formal proof of A. #

COROLLARY 3.4. *Let $\mathscr{K}$ be a model of $\overline{\boldsymbol{H}}$ and X be a stage of $\mathscr{K}$. Suppose that $Cn(A) = Cn(B)$. If $(A \sim B)$ is provable in $\overline{\boldsymbol{H}}$ then $\tau_X A = 1$ iff $\tau_X B = 1$. If $(A \equiv B)$ is provable in $\overline{\boldsymbol{H}}$ then $\tau_X A = \tau_X B$.*

THEOREM 3.5. (Completeness Theorem for $\boldsymbol{H}$, see [1] or [4]). *If an $\boldsymbol{H}$-sentence A is not provable in $\boldsymbol{H}$ then there exists a Kripke model of $\boldsymbol{H}$ where A is defined but not true.*

§ 4. Completeness Theorem

THEOREM 4.1. *Let A be an $\overline{\boldsymbol{H}}$-sentence. If A is not provable in $\overline{\boldsymbol{H}}$ then there exists a Kripke model of $\overline{\boldsymbol{H}}$ where A is defined but not true.*

PROOF. Without loss of generality A is reduced in the sense of § 2, see Theorem 2.1 and Corollary 3.4. We use notation of Theorem 2.3.

Suppose that A is not provable in $\overline{\boldsymbol{H}}$. By Theorem 2.3, $E \supset qA$ is not provable in $\boldsymbol{H}$. By Theorem 3.5 there exists a Kripke model $\mathscr{K} = \langle M, \leqslant, \delta, \sigma\rangle$ of $\boldsymbol{H}$ where A is defined but not true. Without loss of generality E is true in $\mathscr{K}$. For, $(E \supset qA)$ is defined but not true at some stage X of $\mathscr{K}$. Hence there exists $Y \geqslant X$ such that $\tau_Y E = 1 \neq \tau_Y(qA)$. Take the submodel of $\mathscr{K}$ with the set $\{Z \in M\colon Z \geqslant Y\}$ of stages.

For every formula $P_i c_1 \ldots c_{m_i}$ and stage X of $\mathscr{K}$ define:

$$\tau_X^0 P_i c_1 \ldots c_{m_i} = \begin{cases} 1 & \text{if} \quad \sigma_X P_i c_1 \ldots c_{m_i} = 1, \\ -1 & \text{if} \quad \sigma_X Q_i c_1 \ldots c_{m_i} = 1, \\ 0 & \text{otherwise} \end{cases}$$

The definition is correct since E is true in $\mathscr{K}$. By Lemma 3.2 there exists a unique model $\mathscr{L} = \langle M, \leqslant, \delta, \tau\rangle$ of $\overline{\boldsymbol{H}}$ such that τ extends τ^0. By induction on a subformula B of A it is easy to check that at each stage X, $\tau_X B = 1$ implies $\sigma_X(qB) = 1$. Hence A is not true in $\mathscr{L}$. #

COROLLARY 4.2. (Adequacy Theorem for $\overline{\boldsymbol{H}}$). *$A$ is provable in $\overline{\boldsymbol{H}}$ iff it is true in each model of $\overline{\boldsymbol{H}}$ where it is defined.*

§ 5. Duality

Here we prove that an inessential extension of $\overline{\boldsymbol{H}}$ satisfies very natural duality laws.

Let calculus ***H1*** be obtained from $\overline{\boldsymbol{H}}$ by adding a unary propositional connective α, binary propositional connective β and the following axiom schemas:

$$\alpha A \equiv -\neg -A,$$
$$(A\beta B) \equiv -(-A \supset -B).$$

H1-formulae can be regarded as abbreviations of $\overline{\boldsymbol{H}}$-formulae.

Now we define duality of the logical operators:

$\&$	is dual to	$\vee$	and vice versa,
$\forall$	is dual to	$\exists$	and vice versa,
$\neg$	is dual to	α	and vice versa,
$\supset$	is dual to	β	and vice versa,
$-$	is dual to itself.		

Below in this section s range over the logical operators of ***H1***, $\bar{s}$ is the operator dual to s, A and B range over the ***H1***-formulas and the sign $\vdash$ means provability and deducibility in ***H1***.

LEMMA 5.1.

$\vdash --A \equiv A;$

$\vdash -sA \equiv \bar{s}(-A)$ *where s is $\neg$ or α;*

$\vdash -(AsB) \equiv (-A)\bar{s}(-B)$ *where* s *is* $\&, \vee, \supset$ *or* β;
$\vdash -sxA \equiv \bar{s}x(-A)$ *where* s *is* $\forall$ *or* $\exists$.

Proof is clear. #

Corollary 5.2. *There exists an algorithm* $A \Rightarrow A^\circ$ *which associates a formula* A° *with each formula* A *in such a way that* $A \equiv A^\circ$, *and the scope of each occurrence of minus in* A° *is atomic, and a logical operator* s *occurs in* A° *iff* s *or* $\bar{s}$ *occurs in* A.

Below in this section:

A' is the result of replacing each logical operator in A by the dual operator; $\bar{A}$ is the result of replacing each atomic formula in A by its strong negation; $A^* = \bar{A}'$.

Theorem 5.3. $\vdash A^* \equiv A$.

Proof by induction on A. #

Let $A \rightarrow B$ abbreviates $(A \supset B) \& (-B \supset -A)$ (the strong implication).

Theorem 5.4. *If* $\vdash A \rightarrow B$ *then* $\vdash B' \rightarrow A'$.

Proof. Let $(A_1, \ldots, A_n)$ be a proof of $A \rightarrow B$. Then $(\bar{A}_1, \ldots, \bar{A}_n)$ is a proof of $\bar{A} \rightarrow \bar{B}$. Now $\bar{A} \rightarrow \bar{B} \vdash -\bar{B} \rightarrow -\bar{A} \vdash \bar{B}^* \rightarrow \bar{A}^* \vdash B' \rightarrow A'$. #

Corollary 5.5. *If* $\vdash A \equiv B$ *then* $\vdash A' \equiv B'$.

Note. Calculus ***H1*** can be conservatively extended in such a way that the duality statements 5.3-5.5 remain true. For example, ***H1*** can be enriched by the connective $\rightarrow$ and the connective dual to $\rightarrow$.

§6. Propositional logic

Here we prove Adequacy Theorem for the propositional part of $\bar{\boldsymbol{H}}$ and present complete and independent systems of logical operators for $\bar{\boldsymbol{H}}$ and its propositional part.

Let $\boldsymbol{P}\bar{\boldsymbol{H}}$ be the propositional part of $\bar{\boldsymbol{H}}$. Formulae of $\boldsymbol{P}\bar{\boldsymbol{H}}$ are those of $\bar{\boldsymbol{H}}$ built from propositional letters by propositional connectives. Axioms of $\boldsymbol{P}\bar{\boldsymbol{H}}$ are those of $\bar{\boldsymbol{H}}$ which are $\boldsymbol{P}\bar{\boldsymbol{H}}$-formulae. Modus ponens is the only inference rule of $\boldsymbol{P}\bar{\boldsymbol{H}}$.

In this section A, B range over the $\boldsymbol{P}\bar{\boldsymbol{H}}$-formulae, F range over the $\bar{\boldsymbol{H}}$-formulae.

Lemma 6.1. *Let* $(F_1, \ldots, F_n)$ *be an* $\bar{\boldsymbol{H}}$*-proof,* p *be a propositional letter and for each* $i = 1, \ldots, n$, A_i *be obtained from* F_i *by (i) omitting all* $\forall x$ *and* $\exists x$, *and (ii) replacing all atomic formulae which are not propositional letters by* p. *Then* $(A_1, \ldots, A_n)$ *is a* $\boldsymbol{P}\bar{\boldsymbol{H}}$*-proof.*

Proof is clear. #

Hence a $\boldsymbol{P}\bar{\boldsymbol{H}}$-formula provable in $\bar{\boldsymbol{H}}$ is provable in $\boldsymbol{P}\bar{\boldsymbol{H}}$.

A triple $\mathscr{K} = \langle M, \leqslant, \tau\rangle$ will be called a Kripke model of $\boldsymbol{P\bar{H}}$ iff $\langle M, \leqslant\rangle$ is a poset and $\tau\colon M\times$ (the set of $\boldsymbol{P\bar{H}}$-sentences)$\to\{-1, 0, 1\}$ satisfies the relevant conditions among 3.1-3.14. A is true in $\mathscr{K}$ iff for every $X\in M$, $\tau_X A = 1$.

LEMMA 6.2. *Let M and $\leqslant$ be as above and $\tau^0\colon M\times$ (the set of atomic $\boldsymbol{P\bar{H}}$-sentences)$\to\{-1, 0, 1\}$ satisfies condition 3.1. Then there exists a unique extension τ of τ^0 such that $\langle M, \leqslant, \tau\rangle$ is a Kripke model of $\boldsymbol{P\bar{H}}$.*

PROOF is clear. #

From Adequacy Theorems for $\boldsymbol{\bar{H}}$ follows

THEOREM 6.3. (Adequacy Theorem for $\boldsymbol{P\bar{H}}$). *A is provable in $\boldsymbol{P\bar{H}}$ iff it is true in all Kripke models of $\boldsymbol{P\bar{H}}$.*

Formulae F_1 and F_2 are called strongly equivalent iff $(F_1 \equiv F_2)$ is provable in $\boldsymbol{\bar{H}}$. According to Corollary 3.4 strongly equivalent formulae can be considered to have the same meaning. It is worth while to study formulae modulo strong equivalence.

Formula $\neg A \equiv (A\supset -A)$ is provable in $\boldsymbol{P\bar{H}}$ (see [10]). This fact can be easily checked. Conjuction and disjunction are mutually expressible using minus (Lemma 5.1). So we proved

THEOREM 6.4. *$\{-, \&, \supset\}$ and $\{-, \vee, \supset\}$ are complete systems of connectives of $\boldsymbol{P\bar{H}}$.*

LEMMA 6.5. *System $\{\neg, \&, \vee, \supset\}$ is not complete in $\boldsymbol{P\bar{H}}$.*

PROOF by induction to absurdity.

Suppose that $(\neg p \equiv A)$ is provable in $\boldsymbol{P\bar{H}}$ where minus does not occur in A. Without loss of generality p is the only propositional letter of A. Let $\mathscr{K}$ be a one-stage model of $\boldsymbol{P\bar{H}}$. Then $(A\sim p)$, or $(A\sim\neg p)$, or $\big(A\sim(p\ \&\neg p)\big)$, or $\big(A\sim(p\vee\neg p)\big)$ is true in $\mathscr{K}$. If p is true in $\mathscr{K}$ then formulae $(-p\equiv p)$ and $\big(-p\equiv(p\vee\neg p)\big)$ are not true in $\mathscr{K}$. If p is uncertain in $\mathscr{K}$ then formulae $(-p\equiv\neg p)$ and $\big(-p\equiv(p\ \&\neg p)\big)$ are not true in $\mathscr{K}$. #

LEMMA 6.6. *System $\{-, \supset\}$ is incomplete in $\boldsymbol{P\bar{H}}$.*

PROOF. Consider the Kripke model of Figure 1 where $1<2, 1<3$, $\tau_1 p = 0$, $\tau_2 p = 1$ and $\tau_3 p = -1$. Let A be built from p using connectives- and $\supset$ only.

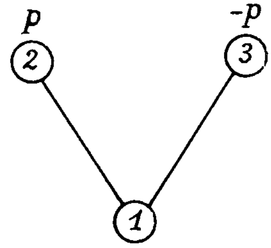

Figure 1

If $\tau_2 A = \tau_3 A$ then $\tau_1 A = \tau_2 A$. We prove this fact by induction. Cases $A = p$ and $A = -B$ are clear. Let $A = B \supset C$. If $\tau_2 A = \tau_3 A = \varepsilon < 1$ then $\tau_2 B = \tau_3 B = 1 = \tau_1 B$, $\tau_2 C = \tau_3 C = \varepsilon = \tau_1 C$ and $\tau_1 A = \varepsilon$. Let $\tau_2 A = \tau_3 A = 1$. We have to prove that $\tau_X B = 1$ implies $\tau_X C = 1$ for each $X \geqslant 1$. It is clear for $X = 2, 3$. If $\tau_1 B = 1$ then $\tau_2 B = \tau_3 B = 1 = \tau_2 C = \tau_3 C$ and by the induction hypothesis $\tau_1 C = 1$.

Now check that $\tau_2(p \vee -p) = \tau_3(p \vee -p) = 1$ but $\tau_1(p \vee -p) = 0$. #

LEMMA 6.7. *System* $\{-, \neg, \&, \vee\}$ *is not complete in* $\bar{\boldsymbol{PH}}$.

PROOF. Suppose that $p \supset q$ is strong equivalent in $\bar{\boldsymbol{PH}}$ to a formula A built from p and q without $\supset$. Then $p \supset q$ in equivalent to rA (see §2) and $\supset$ does not occur in rA. Let B be obtained from rA by replacing $-$ by $\neg$. By Lemma 2.4 formula $(p \supset q) \sim B$ is provable in $\bar{\boldsymbol{H}}$ which contradicts [6]. #

THEOREM 6.8. *Systems* $\{-, \&, \supset\}$ *and* $\{-, \vee, \supset\}$ *are complete and independent in* $\bar{\boldsymbol{PH}}$. *Moreover, they are the only complete and independent systems of connectives in* $\bar{\boldsymbol{PH}}$.

PROOF. See Theorem 6.4 and Lemmae 6.5-6.7. #

COROLLARY 6.9. $\{-, \vee, \supset, \exists\}$ *is a complete and independent system of logical operators in* $\bar{\boldsymbol{H}}$.

PROOF. The completeness follows from Theorem 6.4 and the fact that $\vdash \forall x A \equiv -(\exists x -A)$ in $\bar{\boldsymbol{H}}$. Independence of $\exists$ is clear. If $-p$ is strongly equivalent in $\bar{\boldsymbol{H}}$ to a formula A built without minus then according to Lemma 6.1 minus is expressible through $\vee$ and $\supset$ in $\bar{\boldsymbol{PH}}$ which contradicts to Lemma 6.5. Independence of $\vee$ and $\supset$ is proved analogously. #

All other complete and independent systems of logical operators of $\bar{\boldsymbol{H}}$ can be obtained from the system of Lemma 6.9 by changing $\vee$ for & and/or changing $\exists$ by $\forall$.

§ 7. A 3-valued logic

Let $\bar{\boldsymbol{C}}$ be the calculus obtained from $\bar{\boldsymbol{H}}$ by adding a new axiom schema $\neg\neg A \supset A$. A function τ associating $-1, 0$ or 1 with each sentence of $\bar{\boldsymbol{C}}$ will be called a *model* of $\bar{\boldsymbol{C}}$ iff there exists a one-stage Kripke model $\langle\{0\}, \leqslant, \delta, \sigma\rangle$ of $\bar{\boldsymbol{H}}$ such that $\delta 0$ is the set of all individual constants and $\sigma_0 = \tau$. Formula A is *true* in τ iff $\tau A = 1$. From the Adequacy Theorem for $\bar{\boldsymbol{H}}$ follows

THEOREM 7.1. *A sentence A is provable in* $\bar{\boldsymbol{C}}$ *iff it is true in all models of* $\bar{\boldsymbol{C}}$.

It is not difficult to check that $\{-, \supset, \exists\}$ and $\{-, \supset\}$ are complete and independent systems of logical operators for $\bar{\boldsymbol{C}}$ and the propositional part of $\bar{\boldsymbol{C}}$ respectively.

§ 8. Gentzen-type calculus

A Gentzen-type intuitionistic predicate calculus ***G1*** is described in [3]. Let calculus $\bar{\boldsymbol{G}}$ be obtained from ***G1*** by the following changes. Remove logical operators $\neg$, $\&$, $\forall$ and the correspondent logical rules of inference, and add minus (the strong negation) and the following rules (in notation of [3]):

$$\frac{A,\Gamma\to\Theta}{-(A\supset B),\Gamma\to\Theta}\qquad\frac{-B,\Gamma\to\Theta}{-(A\supset B),\Gamma\to\Theta}\qquad\frac{\Gamma\to A\quad\Gamma\to -B}{\Gamma\to -(A\supset B)}$$

$$\frac{-A,\Gamma\to\Theta}{-(A\vee B),\Gamma\to\Theta}\qquad\frac{-B,\Gamma\to\Theta}{-(A\vee B),\Gamma\to\Theta}\qquad\frac{\Gamma\to -A;\ \Gamma\to -B}{\Gamma\to -(A\vee B)}$$

$$\frac{-A(t),\Gamma\to\Theta}{-\exists x A(x),\Gamma\to\Theta}\qquad\frac{\Gamma\to -A(y)}{\Gamma\to -\exists x A(x)}$$

(y does not occur in $A(x)$)

$$\frac{A,\Gamma\to\Gamma}{--A,\Gamma\to\Theta}\qquad\frac{\Gamma\to A}{\Gamma\to --A}\qquad\frac{\Gamma\to A}{-A,\Gamma\to}$$

THEOREM 8.1. *If $\Gamma\vdash E$ in $\bar{\bar{\boldsymbol{H}}}$ with all variables held constant than $\vdash\Gamma\to E$ in $\bar{\boldsymbol{G}}$, and vice versa.*

PROOF imitates the corresponding proof in [3]. #

THEOREM 8.2. *Given a proof in $\bar{\boldsymbol{G}}$ of a sequent in which no variable occurs both free and bound, another proof in $\bar{\boldsymbol{G}}$ of the same sequent can be found which contains no cut.*

PROOF imitates the corresponding proof in [3]. #

COROLLARY 8.3. *In $\bar{\bar{\boldsymbol{H}}}$*

(i) *if* $\vdash A\vee B$ *then* $\vdash A$ *or* $\vdash B$,

(ii) *if* $\vdash -(A\ \&\ B)$ *then* $\vdash -A$ *or* $\vdash -B$,

(iii) *if* $\vdash\exists x A(x)$ *then* $\vdash\forall x A(x)$ *or* $\vdash A(c)$ *for some individual constant* c,

(iv) *if* $\vdash -\forall x A(x)$ *then* $\vdash\forall x - A(x)$ *or* $\vdash -A(c)$ *for some individual constant* c.

(One can read "$\vdash -A$" as "A is logically false". So (ii) states that if A & G is logically false then either A or B is logically false).

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