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EXPANDED THEORY OF ORDERED ABELIAN GROUPS

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The theory of ordered Abelian groups with quantification over convex subgroups is studied. An elimination of the elementary quantifiers is presented and a primitively recursive decision procedure for this theory is constructed.

0. Introduction

For the sake of brevity the terms "group", "o-group" and "chain" will be used for "Abelian group", "linearly ordered Abelian group" and "linearly ordered set" respectively.

Algebraically speaking an o-group G is a group and a chain, and for every $x, y, z \in G$, $x < y$ implies $x + z < y + z$.

Let G be an o-group. It is easy to check that G is torsion free, has neither minimal nor maximal element and is either discretely or densely ordered. A subset $X \subseteq G$ is called a convex subgroup of G iff X is a subgroup and X is convex (the latter means that for every $x_1, x_2 \in X$ and each $y \in G$, if $x_1 < y < x_2$ then $y \in X$). Convex subgroups play a fundamental role in non-formalized theory of o-groups (see [3]). It is easy to check that convex subgroups of G are linearly ordered by inclusion.

Let us review the history.

G is called Archimedean iff for every positive $x, y \in G$ there exists a natural n such that $x < ny$. G is Archimedean iff $\{0\}$ and G are the only convex subgroups of G. Archimedean o-group is embeddable into the naturally ordered additive group of reals (see Holder's Theorem in [3]). The elementary theory of Archimedean o-groups was studied by Robinson and Zakon (see [15]). Here are their main results. G is called n-regular iff for every $x_1, \ldots, x_n \in G$ there exists $y \in G$ such that $x_1 < \cdots < x_n$ implies $x_1 \leq ny \leq x_n$. G is called regular iff it is n-regular for each positive integer n. Each Archimedean o-group is regular. Each regular o-group is elementarily equivalent to some Archimedean o-group. Two descrete (respectively dense) regular o group are elementarily equivalent iff they are elementarily equivalent as groups

The main result of Kargapolov's paper [10] is a classification of the o-groups of finite rank by their elementary properties (a torsion free group has a finite rank iff it is embeddable in a finite dimensional vector space over the rational field) According to [4], every two o-groups are universally equivalent

In [5], all o-groups were classified by their elementary properties, and the elementary theory of o-groups was algorithmically reduced to the elementarily theory of chains Together with [11] it gives a decision procedure for the elementary theory of o-groups An elmination of quantifiers in the elementary theory of o-groups was presented in [6] Together with [12] it gives a primitively recursive decision procedure for the elementary theory of o-groups The part of [6] relating to o-groups was never published

Here we study the theory of o-groups with quantification over convex subgroups We eliminate the elementary quantifiers, and construct a primitively recursive decision procedure for this theory The main results of the present paper were announced in [7] An earlier version of a part of this paper may be found in the Soviet Institute of Scientific and Technical Information (Moscow), number 6708–73, [8] is the corresponding abstract

Let us summarize the contents of the present paper

Part 1 of the present paper is purely algebraic The key notion here is the functor $F(s, x)$ (called the s-fundament of x)

In Part 2 we define the Expanded Theory of o-groups, and eliminate the elementary quantifiers The Expanded Theory is the theory of o-groups with quantification over convex subgroup enriched by some definable predicates The elimination of elementary quantifiers reduces the Expanded Theory to so-called Convex Subgroups Theory The latter is an elementary theory of the chains of convex subgroups with some surplus one-place predicates (each o-group provides us with a model of the Convex Subgroups Theory)

In Part 3 we axiomatize the Convex Subgroups Theory in the elementary theory of complete chains with surplus one-place predicates in such a way that for each sentence α in the language of the Convex Subgroups Theory one can easily select a finite number of axioms deciding α

In the Appendix we prove that the weak monadic second order theory of complete chains with surplus one-place predicates is primitively recursive (This strengthens the result of [13], but was obtained simultaneously and independently) Together with the previous parts it gives a primitively recursive decision procedure for the Expanded Theory of o-groups

Some words about possible generalizations The theorem about elimination of elementary quantifiers can be easily generalized by enriching the part of the language concerning convex subgroups Generalizations of the decidability result are restricted by undecidability results in theory of chains For example allowing quantification over arbitrary subsets of convex subgroups leads to undecidable theory if the Continuum Hypothesis holds. This follows from undecidability of the monadic theory of the real line, see Shelah's paper [16]. One of the possible generalizations is obtained by allowing quantification over finite subsets of convex subgroups. The elementary quantifiers can be eliminated, and the enriched theory remains primitively recursive. One can have some generalizations of the form: this specific theory of o-group is recursive modulo that specific theory of chains. About decision problem for lattice ordered Abelian groups see [9].

Some words about notation. If a group H is the (internal) direct sum of its subgroups H_i, $i \in I$, we write $H = \Sigma\{H_i : i \in I\}$. In this case each $h \in H$ is equal to some finite sum $h_{i_1} + \dots + h_{i_n}$ where $h_{i_1} \in H_{i_1}, \dots, h_{i_n} \in H_{i_n}$. The external direct sum of groups H_i, $i \in I$, is also denoted by $\Sigma\{H_i : i \in I\}$. The elements of the external direct sum are functions $f : I \to \bigcup\{H_i : i \in I\}$ such that $f(i) \in H_i$ and $\{i : f(i) \neq 0\}$ is finite. We can write $f = \Sigma f(i)$. Now let I be a chain and H_i's ($i \in I$) be o-groups. By $L\Sigma\{H_i : i \in I\}$ we denote the lexicographic (or ω-lexicographic) sum of H_i's. It is the direct sum ΣH_i ordered as follows: $\Sigma h_i > 0$ iff $\Sigma h_i \neq 0$ and $h_j > 0$ where $j = \max\{i : h_i \neq 0\}$. The lexicographic multiple $H \cdot I$ of an o-group H is $L\Sigma\{H_i : i \in I$ and $H_i = H\}$.

"w.l.o.g." is an abbreviation for "without loss of generality".

A.I. Kokorin persuaded me (after [5] was published) to continue to work on algorithmic problems for ordered groups. (I have returned to o-groups after Cohen's preprint [1] demonstrating potentialities of the method of elimination of quantifiers.) Jonathan Levin corrected the English of an earlier version of this paper. The referee found some places which had to be corrected. I am grateful to all these people.

PART 1. ALGEBRA

1. Fundamental subgroups

Throughout this section G is an o-group, $x, y, z \in G$ and X, Y are convex subgroups of G. Here and below p is a prime number.

Definition 1.1 ([5]). For an integer $s \neq 0$ we define

$$F(s, x) = \bigcup\{X : \forall y\ (x + sy \notin X)\}, \qquad F(p, i, x) = F(p^i, x).$$

$F(s, x)$ is called the *s-fundament* of x.

Corollary 1.1. *$F(s, x)$ is a convex subgroup or $\emptyset$, $F(s, x) = \emptyset$ iff $x \equiv 0 \pmod s$.*

Corollary 1.2. *Let $a, b \neq 0$ be integers. Then $F(ab, bx) = F(a, x)$, $F(a, x) \subseteq F(ab, x)$, $F(a, bx) \subseteq F(a, x)$. If a and b are relatively prime, then $F(a, bx) = F(a, x)$ and $F(ab, x) = F(a, x) \cup F(b, x)$.*

Corollary 1.3. *Let $a = p^i b$ where $b \not\equiv 0 \pmod p$ Then $F(p, k, x) = F(p, k + i, ax)$*

Corollary 1.4. *Let $s = p_1^{i_1} \cdots p_k^{i_k}$ Then $F(s, x) = F(p_1, i_1, x) \cup \cdots \cup F(p_k, i_k, x)$*

Corollary 1.5. *$F(s, x + y) \subseteq F(s, x) \cup F(s, y)$ and if $F(s, x) \neq F(s, y)$, then $F(s, x + y) = F(s, x) \cup F(s, y)$*

A proof is easy For example, we will show that $F(ab, x) \subseteq F(a, x) \cup F(b, x)$ if a and b are relatively prime Suppose it is not true, then $F(ab, x)$ contains some $x + ay$ and $x + bz$ Because a and b are relatively prime, $ca + db = 1$ for some integers c and d, and so

$$ca(x + bz) + db(x + ay) = x + ab(cz + dy) \in F(ab, x)$$

However, this is impossible

Definition 1.2. $\Gamma_1(s, X)$ is the subgroup $\{x \; F(s, x) \subset X\}$

(ii) $\Gamma_2(s, X)$ is the subgroup $\{x \; F(s, x) \subseteq X\}$,

(iii) $\Gamma(s, X)$ is the factor group $\Gamma_2(s, X)/\Gamma_1(s, X)$,

(iv) $\Gamma(p, k, X) = \Gamma(p^k, X)$ and the same for Γ_1 and Γ_2

It is more precise to write $\Gamma(s, X, G)$ instead of $\Gamma(s, X)$ and the same for F, Γ_1, Γ_2 This more precise notation is used in the following two lemmas

Lemma 1.1. *Let a bar denote the natural homomorphism $G \to G/X$ Then $\Gamma(s, X, G)$ is isomorphic to $\Gamma(s, \bar{X}, \bar{G})$*

Proof. For every $Y \supseteq X$, $F(s, x) \subseteq Y$ iff $F(s, \bar{x}) \subseteq \bar{Y}$ Now it is easy to check that the correspondence $x + \Gamma_1(s, X) \to \bar{x} + \Gamma_1(s, \bar{X})$ is a required isomorphism

Lemma 1.2. *Let $X \subset Y$ Then $\Gamma(s, X, G)$ is isomorphic to $\Gamma(s, X, Y)$*

Proof. The correspondence $x + \Gamma_1(s, X, Y) \to x + \Gamma_1(s, X, G)$ is an isomorphism from $\Gamma(s, X, Y)$ onto $\Gamma(s, X, G)$

A group $\Gamma(s, X)$ satisfies the axiom $\forall v\ (sv = 0)$ and so it has a representation as a direct sum of cyclic groups

Definition 1.3. $\rho(s, k, X)$ is the cardinal number of cyclic direct summands of the order p^k in a representation of $\Gamma(p, s, X)$ as a direct sum of cyclic groups

Definition 1.4 (cf, [17]) Elements $v_1, \ldots, v_n$ of a group H are *independent* (*strongly independent*) modulo p^k if, for every integer $a_1, \ldots, a_n$, $\Sigma a_i v_i = 0$ ($\Sigma a_i v_i \equiv 0 \pmod{p^k}$) implies $a_1 \equiv \cdots \equiv a_n \equiv 0 \pmod{p^k}$ A subset $M \subseteq H$ is *inde-*

pendent (*strongly independent*) modulo p^k if every finite subset of M is so $\rho^1(p,k,H)$ ($\rho^3(p,k,H)$) is the power of a maximal independent (strongly independent) modulo p^k subset $M \subseteq H$ such that every element of M has the order p^k

Corollary 1.6. *Let $H = \Gamma(p,s,X)$ and $1 \leq k \leq s$ Then $\rho^3(p,k,H) = p(s,k,X)$ and $\rho^1(p,k,H) = \Sigma\{p(s,i,X)\ \ k \leq i \leq s\}$*

A proof is easy

Lemma 1.3. *Let $F(p,s,x) = \{0\}$ and $\bar{x} = x + \Gamma_1(p,s,\{0\}) \in \Gamma(p,s,\{0\})$*

(1) *The order of $\bar{x}$ is p^k iff $\emptyset = F(p,s,p^k x) \subset F(p,s,p^{k-1}x) = \{0\}$*

(2) *If the order of $\bar{x}$ is p^k then $x \equiv 0 \pmod{p^{s-k}}$*

A proof is easy

Theorem 1.1. *$p(s,k,X) = p(s+1,k,X)$ for $k < s$ and $p(s,s,X) = p(s+1,s,X) + p(s+1\ s+1,X)$*

Proof. W l o g $X = \{0\}$ (see Lemma 1 1) To simplify notation we sometimes omit p and $\{0\}$ Let a bar (respectively prime) denote the natural homomorphism from $\Gamma_2(s)$ onto $\Gamma(s)$ (resp from $\Gamma_2(s+1)$ onto $\Gamma(s+1)$) Note that

(i) $x \in \Gamma_2(s)$ iff $px \in \Gamma_2(s+1)$ and

(ii) if $x \in \Gamma_2(s)$, then $\bar{x}$ and $(px)'$ have the same order

Case $k < s$ Let U (resp V) be the family of strongly independent modulo p^k subsets of $\Gamma(s)$ (resp $\Gamma(s+1)$) consisting of elements of order p^k U and V are partially ordered by inclusion Let $\{\bar{x}_i\ \ i \in I\}$ be maximal in U It is enough to check that $\{(px_i)'\ \ i \in I\}$ belongs to V and is maximal there First we check the strong independence Suppose that $\Sigma a_i(px_i)' = p^k u'$ i e $\Sigma a_i p x_i = p^k u + p^{s+1}v$ for some v Then $\Sigma a_i p\bar{x}_i = p^k \bar{u}$ hence there exist b_i's such that $a_i p = p^k b_i$ By Lemma 1 3, $x_i = p^{s-k} y_i$ for some y_i So $p^k u = \Sigma p^k b_i p^{s-k} y_i - p^{s+1} v$ and $u = pw$ for some w Then $\Sigma a_i p x_i = p^k p w + p^{s+1} v$ and $\Sigma a_i \bar{x}_i = p^k \bar{w}$ which implies $a_i \equiv 0 \pmod{p^k}$ for every i Now we check the maximality Let y' be of order p^k By Lemma 1 3, $y = pz$ for some z There exist a_i's, b and u such that $b \not\equiv 0 \pmod{p^k}$ and $b\bar{z} = \Sigma a_i \bar{x}_i + p^k \bar{u}$, since $\{\bar{x}_i\ \ i \in I\}$ is maximal Then $bz = \Sigma a_i x_i + p^k u \pmod{p^s}$, $by = \Sigma a_i p x_i + p^k pu \pmod{p^{s+1}}$ and $by' = \Sigma a_i (px_i)' + p^k (pu)'$

Case $k = s$ By Corollary 1 6 it is enough to check that $\rho^1(p,s,\Gamma(s)) = \rho^1(p,s,\Gamma(s+1))$ Let U (resp V) be the family of independent moduls p^k subsets of $\Gamma(s)$ (resp $\Gamma(s+1)$) consisting of elements of order p^k Let $\{\bar{x}_i\ \ i \in I\}$ be maximal in U We check that $\{(px_i)'\ \ i \in I\}$ belongs to V and is maximal there If $\Sigma a_i (px_i)' = 0$ then $\Sigma a_i p x_i \equiv 0 \pmod{p^{s+1}}$, $\Sigma a_i x_i \equiv 0 \pmod{p^s}$, $\Sigma a_i \bar{x}_i = 0$, and $a_i \equiv 0 \pmod{p^k}$ for every i The maximality is checked as above (but $u = 0$)

Definition 1.5. (i) $A(x) = \bigcup\{X : x \notin X\}$,
(ii) $X^+ = \bigcap\{Y : X \subset Y\}$ if $X \neq G$ and $X^+ = G$ if $X = G$

Corollary 1.7. *$A(x)$ is a convex subgroup or $\emptyset$, $A(x) = \emptyset$ iff $x = 0$*

Corollary 1.8. *For any integer c, $A(cx) \subseteq A(x)$ If $c \neq 0$, then $A(cx) = A(x)$*

Corollary 1.9. *$A(x+y) \subseteq A(x) \cup A(y)$ If $A(x) \neq A(y)$, then $A(x+y) = A(x) \cup A(y)$*

Corollary 1.10. *$X \subset X^+$ iff $\exists x\,(X = A(x))$*

A proof is easy
Note Cf the definition of $A(x)$ and Definition 1 1

Definition 1.6. Let a bar denote the natural homomorphism $G \to G/X$, $R \in \{<, \leqslant, =, \geqslant, >\}$ and $a \neq 0$ be an integer

(1) $xR0 \,(\mathrm{mod}\, X) \equiv \bar{x}R0$,
(2) $[x = 1 \,(\mathrm{mod}\, X)] \equiv 0 < \bar{x}$ and $\neg\exists y\,(0 < \bar{y} < \bar{x})$,
(3) $E(X) \equiv \exists x\,(x = 1 \,(\mathrm{mod}\, X))$,
(4) $[x = a \,(\mathrm{mod}\, X)] \equiv \exists y\,(y = 1 \,(\mathrm{mod}\, X)$ and $\bar{x} = a\bar{y})$,
(5) $xRy \,(\mathrm{mod}\, X) \equiv \bar{x}R\bar{y}$

Let a and b be integers and $b > 0$ Then

Corollary 1.11. $xRy \,(\mathrm{mod}\, X) \equiv bxRby \,(\mathrm{mod}\, X) \equiv (-by)R(-bx) (\mathrm{mod}\, X)$

Corollary 1.12. $xRa \,(\mathrm{mod}\, X) \equiv bxRba \,(\mathrm{mod}\, X) \equiv (-ba)R(-bx) (\mathrm{mod}\, X)$

Theorem 1.2. *Let $X \subset Y$, $k < s$ and*

$$\forall Z\,(X \subset Z \subset Y \text{ implies } \wedge\{p(s,i,Z) = 0 : i \leqslant s\})$$

Then $p(s,k,X) = 0$

Proof By Lemmas 1 1 and 1 2 it can be assumed that $X = \{0\}$ and $Y = G$ Let a bar denote the natural homomorphism $G \to G/p^sG$ Clearly every $F(p,s,x) \subseteq \{0\}$ Let (reductio ad absurdum) $p(s,k,\{0\}) > 0$ Then there exists x such that $\bar{x} \neq 0 \,(\mathrm{mod}\, p)$ and $p^k\bar{x} = \bar{0}$ The last means $p^kx = p^sy$ for some y But $F(p,s,y) \subseteq \{0\}$ and $\bar{y} \in F(p,s,\{0\})$ and $\bar{x} = p^{s-k}\bar{y}$ which contradicts $\bar{x} \neq 0 \,(\mathrm{mod}\, p)$

Theorem 1.3. *$E(X)$ implies $X \subset X^+$ and $p(s,s,X) = 1$*

Proof. Let $E(X)$ $X \subset X^+$ follows clearly from the definition of E In order to prove that $p(s, s\ X) = 1$ it can be assumed that $X = \{0\}$ and $X = G$, see Lemmas 1.1 and 1.2. Then G is isomorphic to the naturally ordered additive group of natural numbers and G/p^sG is the cyclic group of the order p^s.

Definition 1.7.

$$D(p,s,k,x) \equiv [x \equiv 0 \pmod{p^s} \vee \exists y\, [F(p,s,x-p^ky) \subset F(p,s,x) = F(p,s,y)]]$$

Corollary 1.13. *Let* $\emptyset \neq F(p,s,x) = X$ *and a bar denote the natural homomorphism* $\Gamma_2(p,s,X) \to \Gamma(p,s,X)$. *Then* $D(p,s,k,x)$ *is equivalent to* $\bar{x} \equiv 0 \pmod{p^k}$.

A proof is easy.

Lemma 1.4. *Let* $F(p,s,x) = \{0\}$. *Then* $D(p,s,k,x) \equiv \exists yz\, [F(p,s,y) = \{0\}$ *and* $x = p^ky + p^sz]$.

A proof is easy.

Lemma 1.5. *Let* $\emptyset \neq F(p,s,x) = X$ *and* $c = p^id$ *where* $d \not\equiv 0 \pmod{p}$. *Then* $D(p,s,k,x) \equiv [F(p,k,x) \subset X$ *and* $D(p,s+i,k,cx)]$.

Proof. According to Corollary 1.13 and Lemma 1.1 it can be assumed that $d = 1$ and $X = \{0\}$. We also use Lemma 1.4.

(1) Let $F(p,s,y) = \{0\}$ and $x = p^ky + p^sz$. Then $F(p^k, x) = \emptyset$ and $F(p, s+i, p^iy) = \{0\}$ and $p^ix = p^k(p^iy) + p^{s+i}z$.

(2) Let $F(p,k,x) = \emptyset$, $F(p,s+i,y) = \{0\}$ and $p^ix = p^ky + p^{s+i}z$. Then $x \equiv 0 \pmod{p^k}$, $y = p^iy'$ for some y', $F(p,s,y') = F(p,s+i,y) = \{0\}$ and $x = p^ky' + p^sz$.

Definition 1.8. For any integer c, $E(p,s,c,x) \equiv \exists X\, \exists y\, [X = F(p,s,x)$ and $y \equiv 1 \pmod{X}$ and $F(p,s,x-cy) \subset X]$.

Corollary 1.14. *Let* $X = F(p,s,x)$, $y \equiv 1 \pmod{X}$ *and a bar denote the natural isomorphism* $\Gamma_2(p,s,X) \to \Gamma(p,s,X)$. *Then* $E(p,s,c,x)$ *is equivalent to* $\bar{x} = c\bar{y}$.

Corollary 1.15. *If* $x \equiv 0 \pmod{p\ }$, *or* $X = F(p,s,x)$ *and* $\neg E(X)$, *or* $c = 0$, *then* $\neg E(p,s,c,x)$.

Corollary 1.16. *If* $c \equiv d \pmod{p\ }$, *then* $E(p,s,c,x) \equiv E(p,s,d,x)$.

A proof is easy.

Lemma 1.6. *Let* $c = p^id$ *where* $d \not\equiv 0 \pmod{p}$. *Then* $E(p,s,k,x) \equiv E(p,s+i,ck,cx)$.

Proof. $F(p, s, x - ky) = F(p, s + i, cx - cky)$

2. First special o-group

Let H be an o-group, $h \in H$ and $h \not\equiv 0 \pmod p$ in H Let k and s be integers and $1 \leqslant k \leqslant s$ $\omega(\omega^*)$ is the chain (the inverse chain) of naturals

Lemma 2.1. *There exists a subgroup $H' \subset H$ such that $p^k h \in H'$ and $p^k h \not\equiv 0 \pmod p$ in H' and the factor group H/H' has the power p^k*

Proof. By reasons of induction it is enough to prove the lemma for $k = 1$ The factor group H/pH is a vector space over the field of power p Let S be a maximal subspace in H/pH such that $h + pH \not\in S$ The full pre-image of S in H is a required subgroup

For every $n \in \omega$ let f_n $H_n \to H$ be an o-group isomorphism, $H'_n = f^{-1}(H')$ and $h_n = f^{-1}(h)$ Let $G_0 = L\Sigma\{H'_n \ n \in \omega^*\}$, $G_1 = L\Sigma\{H_n \ n \in \omega^*\}$ and G be the least subgroup of G_1 containing $G_0 \cup \{h_n - h_{n+1} \ n \in \omega\}$ Let X_m be the subgroup $L\Sigma\{H_n \ n \geqslant m\}$ of G_1

Lemma 2.2. *Every factor group $X_m \cap G/X_{m+1} \cap G$ is o-isomorphic to H*

A proof is easy

Lemma 2.3. $p^k h_0 \not\equiv 0 \pmod p$ *in* G

Proof. Let f $G_1 \to H$ be defined as follows $f(x_n + \quad + x_0) = f_n x_n + \quad + f_0 x_0$ It is easy to check that f is a group isomorphism, $fG = fG_0 = H'$ and $f(p^k h_0) = p^k h$ But $p^k h \not\equiv 0 \pmod p$ in H'

Lemma 2.4. $F(p, k, p^k h_0) \subseteq \{0\}$ *in* G

Proof. It is enough to prove that every $X_m \cap G \supset F(p, k, p^k h_0)$ But $p^k h_0 \equiv p^k h_m \pmod{p^k}$ in G and $p^k h_m \in X_m \cap G$

Lemma 2.5. $F(p, 1, p^k h_0) = \quad = F(p, k, p^k h_0) = \{0\}$ *in* G

Proof. $\{0\} \subseteq$ (by the Lemma 2 3) $F(p, 1, p^k h_0) \subseteq \quad \subseteq F(p, k, p^k h_0) \subseteq$ (by the Lemma 4) $\{0\}$

Let an asterisk denote the natural group homomorphism $G \to G/p^s G$ and Γ be the group $\Gamma(p, s, \{0\})$ of G Clearly $\Gamma \subseteq G^*$ and $(p^s h_0)^* \in \Gamma$

Lemma 2.6. *$(p^s h_0)^*$ has the order p^k in Γ.*

Proof. $F(p^s, p^k p^s h_0) = F(1, p^k h_0) = \emptyset$ in G and $F(p^s, p^{k-1} p^s h_0) = F(p, p^k h_0) = \{0\}$ in G by Lemma 2.5. By Lemma 1.3 in Section 1 the order of $(p^s h_0)^*$ in Γ is p^k.

Lemma 2.7. *Every $x^* \in \Gamma$ is a multiple of $(p^s h_0)^*$.*

Proof. Let $x^* \in \Gamma$. In G_1, $x = x_n + \cdots + x_0$ for some n and $x_i \in H_i$. It is easy to see that $x \equiv 0 \pmod{p^s}$ in G_1. Let $x_i = p^s y_i$ and $y_i = a_i h_i + h_i'$ where $h_i' \in H_i'$. Then,

$$X = \sum p^s (a_i h_i + h_i') \equiv \sum p^s a_i h_i \equiv \left(\sum a_i\right) p^s h_0 \pmod{p^{s+k}} \text{ in } G.$$

Lemma 2.8. *In G, $p(s, k, \{0\}) = 1$ and $p(s, i, \{0\}) = 0$ for every $i \neq k$.*

Proof. See Lemmas 2.6 and 2.7.

3. Second special o-group

Let $\mathbf{Q}$ be the naturally ordered additive group of rational numbers.

Lemma 3.1. *Every $p(s, k, \{0\}) = 0$ in $\mathbf{Q}$.*

Proof. Clear.

Fix an integer $s \geqslant 1$ and a prime p. Let $\mathbf{Q}_p$ be the least subgroup of $\mathbf{Q}$ containing all quotients a/b where a and b are integers and $b \not\equiv 0 \pmod{p}$.

Lemma 3.2. *In $\mathbf{Q}_p$, $p(s, s, \{0\}) = 1$, and $p(s, k, \{0\}) = 0$ for $k < s$.*

Proof. Clear.

For every $n \in \omega$ let $f_n : H_n \to \mathbf{Q}$ be an isomorphism of o-groups and $H_n' = f_n^{-1}(\mathbf{Q}_p)$ and $h_n = f_n^{-1}(1)$. Let $G_0 = L\Sigma\{H_n' : n \in \omega^*\}$ and $G_1 = L\Sigma\{H_n : n \in \omega^*\}$ where ω^* is the inverse ordered set of natural numbers. Let G be the least subgroup of G_1 containing $G_0 \cup \{(h_m - h_{m+1})/p^n : m, n \in \omega\}$, and X_m be the subgroup $L\Sigma\{H_n : n \geqslant m\}$ of G_1.

Lemma 3.3. *For every m, $G/X_m \cap G$ is divisible and $X_m \cap G/X_{m+1} \cap G$ is isomorphic to $\mathbf{Q}$.*

Proof. Clear.

Lemma 3.4. *$h_0 \not\equiv 0 \pmod p$ in G*

Proof. Let $f: G_1 \to \mathbf{Q}$ be defined as follows $f(x_n + \cdots + x_0) = f_n x_n + \cdots + f_0 x_0$ It is easy to check that f is a group isomorphism, $fG = fG_0 = \mathbf{Q}_p$ and $fh_0 = 1$ But $1 \not\equiv 0 \pmod p$ in $\mathbf{Q}_p$

Lemma 3.5. *$F(p, 1, h_0) = \cdots = F(p, s, h_0) = \{0\}$ in G*

Proof. See the proofs of Lemmas 2 4 and 2 5 in Section 2

Let an asterisk denote the natural homomorphism $G \to G/p^sG$ and Γ be the group $\Gamma(p, s, \{0\})$ of G It is easy to see that Γ is a subgroup of G^* and $h_0^* \in \Gamma$

Lemma 3.6. *h_0^* has order p^s in Γ*

Proof. We need only prove that $p^{s-1}h_0^* \neq 0^*$ But, $F(p, s, p^{s-1}h_0) = F(p, 1, h_0) \neq \emptyset$ by Lemma 3 5

Lemma 3.7. *Every $x^* \in \Gamma$ is a multiple of h_0^**

Proof. Let $x^* \in \Gamma$ In $G_1, x = x_n + \cdots + x_0$ for some n and $x_i \in H_i$ There exist a natural number m, integers $a_n, \ldots, a_0$ and elements $y_m \in H'_n, \ldots, y_0 \in H'_0$ such that

$$p^m x_n = a_n h_n + p^{m+s} y_n, \quad \ldots, p^m x_0 = a_0 h_0 + p^{m+s} y_0$$

Now we count in G $p^m x \equiv \Sigma a_i h_i \equiv (\Sigma a_i) h_0 \pmod{p^{m+s}}$ By Lemma 3 4, $\Sigma a_i = p^m b$ for some integer b Then $x \equiv bh_0 \pmod{p^s}$, i e $x^* = bh_0^*$

Lemma 3.8. *In G, $p(s, s, \{0\}) = 1$ and $p(s, k, \{0\}) = 0$ for every $k < s$*

Proof. See Lemmas 3 6 and 3 7.

4. Third special o-group

Definition 4.1. A *succession* is a function $\alpha: \omega \to \omega$ such that for some n, $\alpha n \neq 0$ and $(\forall i > n)\ \alpha i = 0$. That n is called the *length* of succession α

In this section α, β, γ and δ are successions The restriction of α to $n = \{i \in \omega \ i < n\}$ is denoted by $\alpha \mid n$

Definition 4.2. S is the set of successions ordered as follows $\alpha < \beta$ if there exists n such that $\alpha \mid n = \beta \mid n$ and $\alpha n > \beta n$

Corollary 4.1. *The chain S is dense and has neither maximal nor minimal succession*

Corollary 4.2. *If $\alpha < \beta < \gamma$ and $\alpha \mid n = \gamma \mid n$, then $\beta \mid n = \alpha \mid n$*

Corollary 4.3. *If the length of α is n and $\alpha \mid n + 1 = \beta \mid n + 1$, then $\beta \leq \alpha$*

Definition 4.3. $\beta = \phi(\alpha, n)$ if $\beta \mid n = \alpha \mid n$, $\beta n = (\alpha n) + 1$, $0 = \beta(n+1) = \beta(n+2) =$

Corollary 4.4. $\forall \alpha \, \forall n \, \exists \beta \, [\beta = \phi(\alpha, n)]$ *and* $\forall \beta \, \exists \alpha \, \exists n \, [\beta = \phi(\alpha, n)]$

Corollary 4.5. $\phi(\alpha, n) = \inf\{\gamma \ \gamma \mid n + 1 = \alpha \mid n + 1\}$ *and* $\alpha = \sup\{\phi(\alpha, n) \ n \in \omega\}$

Corollary 4.6. $\phi(\alpha, m) = \phi(\beta, n)$ *is equivalent to* $m = n$ *and* $\alpha \mid n + 1 = \beta \mid n + 1$

Fix a natural $k > 0$ About $\mathbf{Q}$ and $\mathbf{Q}_p$ see Section 3 For every succession α let $f_\alpha \ H_\alpha \to Q$ be an isomorphism of o-groups and $f_\alpha^{-1}(\mathbf{Q}_p) = H'_\alpha$, $f^{-1}(1) = h_\alpha$ Let $G_0 = L\Sigma\{H'_\alpha \ \alpha \in S\}$, $G_1 = L\Sigma\{H_\alpha \ \alpha \in S\}$ and G be the least subgroup of G_1 containing G_0 and such that $\alpha \mid n = \beta \mid n$ always implies $(h_\alpha - h_\beta)/p^{nk} \in G$ In other words if $\alpha \mid n = \beta \mid n$ then $h_\alpha \equiv h_\beta \pmod{p^{nk}}$ in G Let X_α be the subgroup $G \cap L\Sigma\{H_\beta \ \beta \leq \alpha\}$ and $Y_\alpha = X_\alpha^+$ in G

It is evident that $X_\alpha = A(h_\alpha)$ in G and that for every $x \in G$ there exists α such that $A(x) = X_\alpha$ in G

Lemma 4.1. *Y_α / X_α is isomorphic to $\mathbf{Q}$*

Proof. Clear

Lemma 4.2. *If $\beta = \phi(\alpha, n)$ then $Y_\beta \subseteq F(p, nk + 1 \ h_\alpha)$*

Proof. Let $f \ G_1 \to \mathbf{Q}$ be defined as follows $f(\Sigma a_\gamma h_\gamma) = \Sigma\{a_\gamma \ \beta < \gamma\}$ Clearly f is a group homomorphism and $fY_\beta = \{0\}$, $fG_0 = \mathbf{Q}_p$, $fh_\alpha = 1$ If $F(p, nk + 1, h_\alpha) \subset Y_\beta$ then $fh_\alpha \equiv 0 \pmod{p^{nk+1}}$ in fG So it is enough to prove that if $x \in G$ then fx is a multiple of $1/p^{nk}$ G is constructed from G_0 and elements $(h_\gamma - h_\delta)/p^{ik}$ where $\gamma \mid i = \delta \mid i$ If $x \in G_0$ then fx is a multiple of 1 Let $x = (h_\gamma - h_\delta)/p^{ik}$ where

$\gamma \mid i = \delta \mid i$ and let $\gamma < \delta$ If $\beta < \gamma$ or $\delta \leq \beta$ then $fx = 0$ Let $\gamma \leq \beta < \delta$ Then $fx = 1/p^{ik}$ By Corollary 4 2, $\beta \mid i = \delta \mid$ By Corollary 4 3, $i \leq n$ So fx is a multiple of $1/p^{nk}$

Lemma 4.3. *If $\beta = \phi(\alpha, n)$, then $F(p, nk + k, h_\alpha) \subseteq Y_\beta$ in G*

Proof. If $\gamma \mid n+1 = \alpha \mid n+1$ then $h_\alpha \equiv h_\gamma \pmod{p^{nk+k}}$ and $F(p, nk+k, h_\alpha) = F(p, nk+k, h_\gamma) \subseteq X_\gamma$ Let $\beta = \phi(\alpha, n)$ By Corollary 4 5, $Y_\beta = \bigcap\{X_\gamma \quad \gamma \mid n+1 = \alpha \mid n+1\}$ So $F(p, nk+k, h_\alpha) \subseteq Y_\beta$

Corollary 4.7. *If $\beta = \phi(\alpha, n)$, then $F(p, nk+1, h_\alpha) = \quad = F(p, nk + k \; h_\alpha) = Y_\beta$ in G*

Proof. See Lemma 4 2 and Lemma 4 3

Lemma 4.4. *In G, if $F(p, m, h_\alpha) = F(p, n, h_\beta)$, then $h_\alpha \equiv h_\beta \pmod{p^n}$*

Proof. The case $m = 0$ or $n = 0$ is trivial Let $ik < m \leq ik + k$, $jk < n \leq jk + k$ and $F(p, m, h_\alpha) = F(p, n, h_\beta)$ By Corollary 4 7, $\phi(\alpha, i) = \phi(\beta, j)$ By Corollary 4 6, $i = j$ and $\alpha \mid i+1 = \beta \mid i+1$ and so $h_\alpha \equiv h_\beta \pmod{p^{ik+k}}$

Lemma 4.5. *In G, if $F(p, m, ah_\alpha) = F(p, n, bh_\beta)$, then $bh_\alpha \equiv bh_\beta \pmod{p^n}$*

Proof. Let $a = p^i c$ and $b = p^j d$ where $c, d \not\equiv 0 \pmod p$ and let $F(p, m, ah_\alpha) = F(p, n, bh_\beta)$ Then $F(p, m-i \; h_\alpha) = F(p, n-j, h_\beta)$ and by Lemma 4 4, $h_\alpha \equiv h_\beta \pmod{p^{n-j}}$, i e , $bh_\alpha \equiv bh_\beta \pmod{p^n}$

Lemma 4.6. *In G, if $\emptyset \neq F(p, s, x) = Y$, then $p^n x = ah_\gamma + y$ and $F(p, s+n, y) \subset Y$ for some n, a, γ and y*

Proof. Let $\emptyset \neq F(p, s, x) = Y$ and $p^n x = \Sigma a_\alpha h_\alpha$ Let an asterisk denote the natural homomorphism $G \to G/\Gamma_1(p, s+n, Y)$ By Lemma 4 5 it it can be assumed that $(p^n x)^* = \Sigma(b_\alpha h_\alpha)^*$ where $\alpha \neq \beta$ and $a_\alpha \neq 0$ and $a_\beta \neq 0$ implies $F(p, s+n, b_\alpha h_\alpha) \neq F(p, s+n, b_\beta h_\beta)$ Let $F(p, s+n, b_\gamma h_\gamma) = \max_\alpha F(p, s+n, b_\alpha h_\alpha)$ Then $(p^n x)^* = (b_\gamma h_\gamma)^*$

Corollary 4.8. *If $\emptyset \neq F(p, s, x) = Y$ in G, then $Y = Y_\beta$ for some β*

Proof. See Lemma 4 6 and Corollary 4 7

Fix $\beta = \phi(\alpha, n)$ and $s \geq k$ Let an asterisk denote the natural homomorphism $G \to G/\Gamma_1(p, s, Y_\beta)$ and $\Gamma = \Gamma(p, s, Y_\beta)$ Clearly Γ is a subgroup of G^*.

Let $g = p^{s-k}(h_\alpha - h_\beta)/p^{nk}$

Lemma 4.7. *g^* is an element of Γ of order p^k*

Proof. $F(p, s, g^*) =$ (by Corollary 1 3 in Section 1) $F(p, k + nk\, h^*_\alpha - h^*_\beta) = F(p, k + nk. h^*_\alpha) = Y^*_\beta$ by Corollary 4 7 So $g^* \in \Gamma$

Clearly, $F(p, s, p^k g^*) = \emptyset$ But

$$F(p, s, p^{k-1} g^*) = F(p, 1 + nk, h^*_\alpha - h^*_\beta) = F(p\ 1 + nk, h^*_\alpha) = Y^*_\beta$$

by Corollary 4 7 So the order of g^* is p^k by Lemma 1 3 in Section 1

Lemma 4.8. *Every $x^* \in \Gamma$ is a multiple of g^**

Proof. By Lemma 4 6 $p^n x = ah_\gamma + y$ and $F(p, s + n, y) \subset Y$ for some n, a, γ and y Let $a = p^i b$ where $b \not\equiv 0 \pmod p$ By Lemma 4 5 $ah_\gamma \equiv ah_\alpha \pmod{p^{s+n}}$ W l o g $\gamma = \alpha$ By Corollary 4 7 $nk + 1 \leqslant s + n - i \leqslant nk + k$ Let $m = (nk + k) - (s + n - i)$ So $p^n x^* = p^i bh^*_\alpha = p^i b(h_\alpha - h_\beta)^*$ and $p^{n+s-k} x^* = p^i bp^{s-k}(h_\alpha - h_\beta)^* = p^i bp^{nk} g^*$ and $x^* = p^m bg^*$

Corollary 4.9. *$p(s, k, Y_\beta) = 1$ and $p(s, i, Y_\beta) = 0$ for $i \neq k$ in G*

5. Gluing and interlacement

Definition 5.1. $\Delta^* G$ is the set of all convex subgroups of an o-group G ordered by inclusion

Lemma 5.1. *Let H be a subgroup of an o-group G and $\sigma\ \Delta^* H \to \Delta^* G$ be defined as follows $\sigma Y = \bigcup\{X \in \Delta^* G\ \ X \cap H \subseteq Y\}$ Then (1) $\sigma Y \cap H = Y$ and (2) σ is a monomorphism*

Proof. (1) Clearly $\sigma Y \cap H \subseteq Y$ Let $h \in Y$ and $Z = A(h, G)$ The latter means that Z is $A(h)$ calculated in G Then $Z^+ \cap H \subseteq Y$ and $h \in Z^+ \subseteq \sigma Y$

(2) $Y_1 \subset Y_2$ implies $\sigma Y_1 \subset \sigma Y_2$

Indeed, $\sigma Y_2 - \sigma Y_1 \supseteq (\sigma Y_2 - \sigma Y_1) \cap H = Y_2 - Y_1$

Definition 5.2. The monomorphism σ of Lemma 5 1 will be called *canonical*

Theorem 5.1. *Let a direct sum $G = \Sigma H_i$ be linearly ordered and every $\sigma_i\ \ \Delta^* H_i \to \Delta^* G$ be the canonical monomorphism Let $X \in \Delta^* G$ and $Y_i = X \cap H_i$ Then every*

$$p(s, k\ X, G) = \Sigma\{p(s, k, Y_i, H_i)\ \ \sigma_i Y_i = X\}$$

Proof. Let $r = p^s$

(1) $\Gamma_1(r, X) = \Sigma \Gamma_1(r, Y_i, H_i)$

Indeed, let $h \in \Gamma_1(r, Y_i, H_i)$, i e some $h + rh' \in Y_i$ Then $h + rh' \in X$ and $h \in \Gamma_1(r, X, G)$

Conversely, let $\Sigma h_i \in \Gamma_1(r, X)$, i e some $\Sigma h_i + r\Sigma h'_i \in X$ Then $h_i + rh'_i \in Y_i$ and $h_i \in \Gamma_1(r, Y_i, H_i)$

(2) $\Gamma_2(r\ X) = \Sigma\{\Gamma_1(r, Y_i, H_i)\ \ X \subset \sigma_i Y\} + \Sigma\{\Gamma_2(r, Y_i, H_i)\ \ X = \sigma_i Y_i\}$

Indeed, let $h \in \Gamma_2(r, Y_i, H_i)$, i e for every $Y_i \subset Z_i \in \Delta^* H_i$ some $h + rh' \in Z_i$ And let $\sigma_i Y_i = X \subset Z \in \Delta^* G$ Then $Y_i \subset Z \cap H_i$ and some $h + rh' \in Z \cap H_i$, i e $h \in \Gamma_2(r, X)$

Conversely, let $\Sigma h_i \in \Gamma_2(r, X)$, i e for every $X \subset Z \in \Delta^* G$ some $\Sigma h_i + r\Sigma h'_i \in Z$

Case 1. Let $X \subset \sigma_i Y_i = Z$ Then some $\Sigma h_i + r\Sigma h'_i \in Z$ and $h_i + rh'_i \in Z \cap H_i = Y_i$, i e $h_i \in \Gamma_1(r, Y\ H_i)$

Case 2 Let $X = \sigma_i Y_i$ and $Y_i \subset Z_i \in \Delta^* H_i$ and $\sigma_i Z_i = Z$ Then $X \subset Z$ and some $\Sigma h_i + r\Sigma h'_i \in Z$, $h_i + rh'_i \in Z'_i$, i e $h_i \in \Gamma_2(r, Y_i, H_i)$

(3) $\Gamma(r, X) \cong \Sigma\{\Gamma(r, Y_i, H_i)\ \ X = \sigma_i Y_i\}$

Statement (3) follows from statements (1) and (2) and implies the statement of the Theorem 5 1

Definition 5.3. *A* chain *C* is *compact* if

$$(\forall X \subseteq C)(\exists y, z \in C)(X \neq \emptyset \supset y = \inf X \text{ and } z = \sup X)$$

Definition 5.4. Let C be a chain and $x \in C$ Then

$$x^+ = \begin{cases} \inf\{y\ \ x < y\}, & \text{if } x \neq \max C \quad \text{and} \\ x, & \text{if } x = \max C \end{cases}$$

Definition 5.5. Let H be an o-group and C be a compact chain Monomorphism $\sigma\ \ \Delta^* H \to C$ is *regular* if

(1) $Y \subset Y^+$ implies $\sigma Y < (\sigma Y)^+$ and

(2) every $\sigma Y = \inf\{\sigma Z\ \ Y \subseteq Z \subset Z^+\}$

Theorem 5.2 (Interlacement Theorem) *Let*

(1) *C be a compact chain and* $C \models \forall x\ \forall y\ \exists z\ (x < y \supset x \leqslant z < z^+ \leqslant y)$,

(2) $\{H_i\ \ i \in I\}$ *be a family of o-groups and* $\psi_i\ \ \Delta^* H_i \to C$ *be a regular monomorphism and*

(3) $(C \models x < x^+)$ *implies* $(\exists! i)(x \in \operatorname{rng} \psi_i)$

Then there exist an o-group G and an isomorphism $\phi\ \ C \to \Delta^* G$ *such that every*

$$p(s, k, X, G) = \Sigma\{p(s, k\ \ Y_i, H_i)\ \ \phi\psi_i Y_i = X\}$$

and

$$H_i \models E(Y) \text{ implies } G \models E(\phi\psi_i Y).$$

Proof. Let $\psi_i h_i$ be an abbreviation for $\psi_i A(h_i, H_i)$

Lemma 5.2. *$h_i, h_j \neq 0$ and $\psi_i h_i = \psi_j h_j$ implies $i = j$*

Proof. Let h_i $h_j \neq 0$ and $x = \psi_i h_i = \psi_j h_j$ Then $x < x^+$ because of regularity of ψ Now use (3) from Theorem 5 2

Let G be the direct sum ΣH_i ordered as follows If $g = \Sigma h_i$ and $\psi_j h_j = \max\{\psi_i h_i \; h_i \neq 0\}$ then $g > 0$ iff $h_j > 0$

Let $\phi x = \{\Sigma h_i \;$ every $\psi_i h_i < x\}$

Lemma 5 3. *ϕ is an isomorphism from C onto Δ^*G*

Proof. (1) Evidently $\phi x \in \Delta^*G$

(2) Let $x < y$ Then (see (1) Theorem 5 2) $x \leq z < z^+ \leq y$ for some (see (3) Theorem 5 2) $z = \psi_i h_i$ and $h_i \in \phi y - \phi x$

(3) Let $X \in \Delta^*G$ and $x = \sup\{(\psi_i h_i)^+ \; h_i \in X\}$

We state that $\phi x = X$ It is enough to prove that always $X \cap H_i = \phi x \cap H_i$ Clearly $X \cap H_i \subseteq \phi x$ Conversely, let $h_i \in \phi x$ Then $\psi_i h_i < (\psi_j h_j)^+$ for some $h_j \in X$ Therefore, $\psi_i h_i \leq \psi_j h_j$ and $|h_i| < n|h_j|$ in G Because X is convex, $h_i \in X$

Lemma 5.4. *The monomorphism $\phi\psi_i \; \Delta^*H_i \to \Delta^*G$ is canonical*

Proof. (1) $\phi\psi_i Y \cap H_i = Y$ Indeed,

$h \in Y \equiv A(h, H_i) \subset Y \equiv \psi_i h < \psi_i Y \equiv h \in \phi\psi_i Y$

(2) $X \in \Delta^*G$ and $X \cap H_i \subseteq Y$ imply $X \subseteq \phi\psi_i Y$

Indeed, let $X = \phi x \in \Delta^*G$ and $X \cap H_i \subseteq Y$ And let (reductio ad absurdum) $\phi\psi_i Y \subset \phi x$, i e $\psi_i Y < x$ Because of regularity of ψ_i, there exists $h \in H_i$ such that $\psi_i Y \leq \psi_i h < x$ Then $h \in (\phi x \cap H_i) - Y$ which contradicts $X \cap H_i \subseteq Y$

Proof of Theorem 5.2. Now the first statement of Theorem 5 2 follows from Theorem 5 1 The second statement is evident Theorem 5 2 is proved

Lemma 5.5. *Let H_1 and H_2 be countable Archemedean o-groups Then there exists an Archemedean ordering of the direct sum $H_1 + H_2$ preserving the orderings of the summands*

Proof. By Holder's Theorem (see [3]) it can be assumed that H_1 and H_2 are subgroups of the naturally ordered additive group **R** of reals For any real $r \neq 0$, H_i is isomorphic to the subgroup $\{rx \;\; x \in H_i\}$ So it can be assumed that $H_1 \cap H_2 = \{0\}$ But in that case the statement of the Lemma 5 5 is clear

Let G be an o-group, $x \in G$ and $X = A(x)$ in G Then $X \subset X^+$ and the Archemedean o-group X^+/X is called an Archemedean factor of G

Theorem 5.3 (Gluing Theorem) *Let G_i be an o-group and all Archemedean factors of G_i are countable $i = 1,2$ Let ψ $\Delta^*G_1 \to \Delta^*G_2$ be an isomorphism of the chains Then there exists an o-group G_0 (a gluing of G_1 and G_2) and chain isomorphisms ϕ_i $\Delta^*G_0 \to \Delta^*G_i$ such that $\phi_2 = \psi\phi_1$ and every $p(s,k,X,G_0) = p(s,k,\phi_1 X, G_1) + p(s,k,\phi_2 X, G_2)$*

Proof. For $x \in G_i$, let $H_i(x)$ be the Archemedean factor of G_i corresponding to x If $H_i(x_i) = Y_i^+/Y_i$ and $\psi Y_1 = Y_2$, fix an Archemedean order of $H_1(x_1) + H_2(x_2)$ preserving the orders of the summands

Let G_0 be direct sum $G_1 + G_2$ ordered as follows Let $g_0 = g_1 + g_2 \neq 0$ and $H_i(g_i) = Y_i^+/Y$ If $\psi Y_1 \subset Y_2$ (respectively $Y_2 \subset \psi Y_1$) then $g_0 > 0$ iff $g_2 > 0$ (respectively $g_1 > 0$) If $\psi Y_1 = Y_2$ then $g_0 > 0$ iff the element $(g_1 + Y_1) + (g_2 + Y_2)$ of the Archemedean o-group $H_1(g_1) + H_2(g_2)$ is positive G_0 is a desired o-group

6. Fourth special o-group

Let us fix p, s and k where $1 \leq k \leq s$ Let $\mathbf{Q}$ and $\mathbf{Q}_p$ be as in Section 3 We build a countable o-group G satisfying the following conditions

(i) if $x \neq 0$ then $A^+(x)/A(x)$ is isomorphic to $\mathbf{Q}$,

(ii) the chain $(\{A(x)\ x \neq 0\}, \subset)$ is order isomorphic to $\mathbf{Q}$,

(iii) G is q-divisible for each prime $q \neq p$,

(iv) if $p(s,i,X) = r > 0$ then $i = k$, $r = 1$ and X is different from any $A(x)$,

(v) for each x. $p(s,k,A^+(x)) = 0$,

(vi) if $X^+ \subset Y$ then $\exists Z\,(X^+ \subset Z \subset Y$ and $p(s,k,Z) = 1)$

Here is the idea of the construction Let G' be a copy of the third special o-group G' satisfies conditions (i)–(iv) and (vi) For each non-zero $x \in G'$, "shove" another copy of the third special o-group "between' $A^+(x)$ and $G'/A^+(x)$ Do the same for the new copies of the third special o-group Repeat the process

Now we construct the desired o-group Let α, β range over the successions of Section 4 and S be the chain of successions ${}^{\omega}S$ is the set of functions t $n \to S$ where $n \in \omega$ We order ${}^{\omega}S$ as follows $t_1 < t_2$ iff $t_1 \subset t_2$ or $\exists m\,(t_1|m = t_2|m$ and $t_1(m) < t_2(m))$ We imagine elements of ${}^{\omega}S$ as sequences, hence it is clear what $t^\frown\alpha$ means

For each $t \in {}^{\omega}S$ let H_t be an o-group, isomorphic to $\mathbf{Q}, f_t$ $H_t \to \mathbf{Q}$ be an isomorphism, $H'_t = f_t^{-1}(\mathbf{Q}_p)$ and $h_t = f_t^{-1}(1)$ Let $U_t = L\Sigma\{h'_{t^\frown\alpha}\ \alpha \in S\}$, $W_t = L\Sigma\{H_{t^\frown\alpha}\ \alpha \in S\}$, $V_t = \{(h_{t^\frown\alpha} - h_{t^\frown\beta})/p^{nk}\ \alpha\,|\,n = \beta\,|\,n\}$ and G_t be the least subgroup of W_t containing $U_t \cup V_t$ Clearly, G_t is a copy of the third special o-group Let $W = L\Sigma\{H_t\ t \subset {}^{\omega}S\}$ and G be the least subgroup of W containing $\bigcup\{G_t\ t \neq 0\}$ It is not difficult to check that G is the desired o-group

PART 2. ELIMINATION OF QUANTIFIERS

7. Elimination theorem

The elementary language of o-groups ELL is the first order language with an equality sign whose non-logical constants are the individual constant "0" the symbol "$-$" of one-place operation the symbol "$+$" of two-place operation and the symbol "$<$" of two-place predicate The elementary theory of o-groups ELT is given in ELL by axioms of (Abelian) groups, axioms of chains (i e linear order axioms) and by the following axiom

$$\forall x\, \forall y\, \forall z\, (x < y \supset x + z < y + z)$$

Terms of ELL are called *elementary terms* $t_1 - t_2$ is the abbreviation for $t_1 + (-t_2)$

An Expanded Theory of o-groups EXT is now defined Let L2 be the monadic second order language corresponding to ELL Every o-group G gives us a natural model of L2 by the following definition second order variables range over the set $\Delta^*G \cup \{\emptyset\}$ (the convex subgroups of G and the empty set) Let T2 be the set of L2-formulas which are true in all these natural models We shall essentially be studying the theory T2 but in order to eliminate quantifiers some inessential extension of T2 is more conveniently used An Expanded Language of o-groups EXL is obtained from L2 by adding some non-logical constants

Definition 7.1 (of second order terms (*superterms*) of EXL)

(1) Second order variables of EXL (i e second order variables of L2) are superterms,

(2) $\emptyset$ is a superterm, and for each elementary term t, $A(t)$ is a superterm,

(3) $F(p, s, t)$ is a superterm for every elementary term t, prime p and natural $s \geq 1$,

(4) if T is a superterm, then so is T^+

Definition 7.2 (Of atoms (atom formulas) of EXL Here t is an elementary term, T, T_1, T_2 are superterms, p is a prime number, k, s, r are naturals and $1 \leq k \leq s$ and l is an integer)

(1) $D(p, s, k, t)$, $E(p, s, l\ t)$ are atoms,

(2) $T_1 = T_2$, $T_1 \subset T_2$, $E(T)$ and $p(s, k, T) > r$ are atoms,

(3) $t \in T$ is an atom and

(4) $t = l\,(\mathrm{mod}\ T)$, $t < l\,(\mathrm{mod}\ T)$, $t > l\,(\mathrm{mod}\ T)$ are atoms

A natural model of EXL is obtained from a natural model of L2 by means of definitions of Section 1 and the following definition

Definition 7.3. (1) $\emptyset^+$ is the zero-subgroup $\{0\}$,

(2) $T_1 = T_2$ and $T_1 \subset T_2$ are defined naturally,

(3) $E(\emptyset)$ is false,

(4) $p(s, k, \emptyset) > r$ is always false and $t = l\,(\mathrm{mod}\,\emptyset)$, $t < l\,(\mathrm{mod}\,\emptyset)$, $t > l\,(\mathrm{mod}\,\emptyset)$ are always false

So every o-group gives us one natural model of EXL An Expanded Theory of o-groups EXT is the set of EXL-formulas which are true in all these natural models

The atoms of T2 are expressible in EXT

$$[t_1 = t_2] \equiv [t_1 - t_2 = 0\,(\mathrm{mod}\,\emptyset^+)],$$

$$[t_1 < t_2] \equiv [t_1 - t_2 < 0\,(\mathrm{mod}\,\emptyset^+)]$$

The inverse statement is also true but we do not need it and we do not prove it

Theorem 7.1 (Elimination Theorem) *For every* EXL-*formula* α *there exists an* EXL-*formula* α^* *such that* α^* *has no bound elementary variables and* $\alpha \equiv \alpha^*$ *in* EXT

The Elimination Theorem is the object of Part 2 (Sections 7–10) The proof below gives a primitively recursive procedure for building α^* from α And of course α^* has the same free variables as α

The Convex Subgroups Theory, CST, is defined in Section 11 As a corollary of Theorem 7 1 we have the following

Theorem 7.2. *There exists a primitively recursive algorithm which for every EXL-sentence* α *builds a* CSL-*sentence* α^* *such that* $\alpha \in \mathrm{EXT}$ *iff* $\alpha^* \in \mathrm{CST}$

Proof. Let α be an EXL-sentence α does not contain free elementary variables By Theorem 7 1, α does not contain elementary variables at all

W l o g the individual constant 0 does not occur in α Indeed $D(p, s, k, 0)$ is always true, $E(p, s, k, 0)$ is always false, $0 \in T \equiv \emptyset \subset T$ and it is easy to eliminate 0 from atoms $0 = l\,(\mathrm{mod}\,T)$, $0 > l\,(\mathrm{mod}\,T)$, $0 < l\,(\mathrm{mod}\,T)$

Further,

$$(Y = X^+) \equiv (X \subset Y \;\&\; \neg\exists Z\,(X \subset Z \subset Y))$$

$$\vee\,(Y = X \;\&\; (\forall U \supset X)\exists Z\,(X \subset Z \subset U))$$

So it can be assumed that every superterm of α is a variable or $\emptyset$ We also admit a new individual constant U which denotes the maximal (non-proper) convex subgroup

W l o g all quantifications in α are restricted by $\emptyset \subset X \subset U$ Indeed, $\exists X \beta(X) \equiv \beta(\emptyset) \vee \beta(U) \vee \exists X\,(\beta(X) \;\&\; (\emptyset \subset X \subset U))$

W.l.o.g. the individual constants $\emptyset$ and U do not occur in α. Indeed, $E(\emptyset)$, $E(U)$, $p(s,k,\emptyset)>r$, $p(s,k,U)>r$, $U\subset\emptyset$ are false; $\emptyset=\emptyset$, $\emptyset\subset U$, $U=U$ are true. And because every variable in α is bounded by the open interval $(\emptyset, U)$, we can replace $\emptyset=X$ by the propositional constant "false", $\emptyset\subset X$ by the "true" and so on.

As a matter of fact we now have a desired fromula α^*.

An EXL-formula α is called open if α has no bound elementary variables. Below, we write "$\alpha\equiv\beta$" instead of "$\alpha\equiv\beta$ in EXT", "α implies β" instead of "α implies β in EXT" and so on.

The Elimination Theorem is proved by an induction on α. The only non-trivial case is the following.

Lemma 7.1 (Main Lemma). *For every open* EXL-*formula* $\alpha(x)$ *there exists an open* EXL-*formula* α^* *such that* $\exists x\,\alpha(x)\equiv\alpha^*$.

The following simple statements are used often.

Lemma 7.2 (Cases Lemma). *If* α *implies* $\bigvee\beta_i$, *then* $\exists x\,\alpha\equiv\bigvee\exists x\,(\alpha\ \&\ \beta_i)$.

Lemma 7.3. *Let* α *be an* EXL-*formula and* β *be a subformula of* α *such that any free occurrence in* β *of any variable is never bound in* α. *Let* α_t *(respectively* α_f*) be obtained from* α *by replacing* β *by the propositional constant* "true" *(respectively* "false"*). Then* $\exists x\,\alpha\equiv\exists x\,(\beta\ \&\ \alpha_t)\vee\exists x\,(\neg\beta\ \&\ \alpha_f)$.

Lemma 7.4. *Let* α *be an* EXL-*formula and* T *be a superterm in* α *such that any occurrence in* T *of any variable is never bound in* α. *Let* α' *be obtained from* α *by replacing* T *by a new second order variable* X. *Then* $\exists x\,\alpha\equiv\exists X\,\exists x\,(X=T\ \&\ \alpha')$.

8. Primary case

An EXL-formula $\alpha(x)$ is called a p-formula if x can occur in $\alpha(x)$ only through $F(p,r,t)$, $D(p,r,c,t)$ or $E(p,r,c,t)$. In other words a p-formula contains neither $A(t)$, $tRc \pmod T$ nor $F(q,r,t)$, $D(q,r,c,t)$, $E(q,r,c,t)$ where $q\neq p$.

Theorem 8.1. *Let* $\alpha(x)$ *be an open* p-*formula. There exists an open* p-*formula* α^* *such that* $\exists x\,\alpha(x)\equiv\alpha^*$.

Theorem 8.1 is the object of this section. Let R be the set of numbers r occurring in α through $F(p,r,t(x))$, $D(p,r,c,t(x))$ or $E(p,r,c,t(x))$. Let $s=\max R$. It can be assumed that s is the only element of R, see Corollary 1.3 and Lemmas 1.5 and 1.6 in Section 1. Below we write $F(t)$, $D(c,t)$, $E(c,t)$ instead of $F(p,s,t)$, $D(p,s,c,t)$, $E(p,s,c,t)$ respectively. $t_1\equiv t_2$ is an abreviation for $F(t_1-t_2)=\emptyset$.

Note that $x \equiv y$ implies $\alpha(x) \equiv \alpha(y)$ Every elementary term t of α can be represented in a form $ax + b_1 y_1 + \cdots + b_m y_m$ where $0 \leq a, b_i < p^s$ Moreover, it can be assumed that $a = p^k$, see Corollary 1 3 and Lemmas 1 5 and 1 6 in Section 1 Below τ is an elementary term without x and M is the set of terms τ occurring in α through $F(ax + \tau)$, $D(c, ax + \tau)$ or $E(c, ax + \tau)$ It can be assumed that if $\tau \in M$ then $p\tau \in M$

By the Cases Lemma it can be assumed that for $k = 1, \ldots, s$, α has conjuncts $F(p^{s-k}x + \tau_k) \subseteq F(p^{s-k}x + \tau)$ for some τ_k and every $\tau \in M$ Let $t_k = p^{s-k}x + \tau_k$ and $t_0 = 0$ Then α implies $F(t_0) \subseteq F(t_1) \subseteq \cdots \subseteq F(t_s)$ Indeed, $F(t_k) \subseteq F(p^{s-k}x + p\tau_{k+1}) = F(p \cdot t_{k+1}) \subseteq F(t_{k+1})$

By the Cases Lemma 7 2 it can be assumed that $(F(t_k) \subset F(t_{k+1}))$ or $(F(t_k) = F(t_{k+1}))$ is a conjunct in α In order to avoid using indices we assume that

$$\emptyset = F(t_0) = \cdots = F(t_i) \subset F(t_{i+1}) = \cdots = F(t_j) \subset F(t_{j+1}) = \cdots = F(t_s)$$

Evidently

$$p^{s-k}x \equiv \begin{cases} p^{i-k}p^{s-i}x \equiv p^{i-k}(t_i - \tau_i) \equiv -p^{i-k}\tau_i, & \text{if } k \leq i, \\ p^{j-k}p^{s-j}x \equiv p^{j-k}(t_j - \tau_j), & \text{if } i < k \leq j, \\ p^{s-k}x \equiv p^{s-k}(t_s - \tau_s), & \text{if } j < k \leq s \end{cases}$$

Let $\alpha'(x_j, x_s)$ be a formula such that $\alpha(x) \equiv \alpha'(t_j, t_s)$

Lemma 8.1.

$$\exists x\, \alpha(x) \equiv \exists x_s \exists x_j [\alpha'(x_j, x_s) \,\&\, p^{j-i}(x_j - \tau_j) = -\tau_i \,\&\, p^{s-j}(x_s - \tau_s) = x_j - \tau_j]$$

Proof. $\alpha(x)$ implies $\alpha'(t_j, t_s)$ Conversely $\alpha'(x_j, x_s)$ implies $\alpha(x_s - \tau_s)$

Corollary 8.1. *It is enough to prove Theorem 8 1 for $\alpha(x)$ such that $\alpha(x)$ has conjuncts $p^k x = \tau_0$ and $\emptyset \subset F(x) \subseteq F(p^i x + \tau)$ for some $1 \leq k < s$ and τ_0 and every $0 \leq i < k$ and $\tau \in M$*

Let $N = \{p^i x + \tau \;\; 0 \leq i < k \,\&\, \tau \in M\}$ W l o g N is the set of all elementary terms $t(x)$ in α

W l o g $(X = F(x))$ is a conjunct in α, see Lemma 7 4 in Section 7

Lemma 8.2 *W l o g α has conjuncts $F(\tau) \subseteq X$, $\tau \in M$*

Proof. Let $\tau \in M$ By the Cases Lemma it can be assumed that $F(\tau) \subseteq X$ or $X \subset F(\tau)$ is a conjunct of α But $F(x) = X \subset F(\tau)$ implies $F(p^i x + \tau) = F(\tau)$, $D(c, p^i x + \tau) \equiv D(c, \tau)$ and $E(c, p^i x + \tau) \equiv E(c, \tau)$ So we can cancel τ from M

Lemma 8.3. *The conjunct $p^k x = \tau_0$ can be replaced in α by $p^{s-k}\tau_0 \equiv 0$ & $F(p^k x - \tau_0) \subset X$*

Proof. Let $\alpha'(x)$ be the result of the replacement. $\alpha(x)$ implies $\alpha'(x)$. Conversely, suppose $\alpha'(x)$. Then $p^k x - \tau_0 \equiv y$ for some $y \in X$ and $p^{s-k} y \equiv 0$, i.e. $y = p^k z$ for some z. It is easy to check that $\alpha(x - z)$ is true.

Evidently α implies $F(t) = X$ for $t \in N$.

Corollary 8.2. *W.l.o.g.* $\alpha = \alpha_0 \,\&\, \alpha_1(x) \,\&\, \beta(x)$ *where* α_0 *is* $p^{s-k}\tau_0 \equiv 0 \,\&\, \emptyset \subset X \,\&$ $\bigwedge\{F(\tau) \subseteq X \mid \tau \in M\}$, α_1 *is* $F(p^k x - \tau_0) \subset X \,\&\, \bigwedge\{F(t) = X \mid t \in N\}$ *and* x *can occur in* $\beta(x)$ *only through atoms* $D(c, t)$, $E(c, t)$.

Lemma 8.4. *Let* α_0 *and* $E(X)$ *imply* $\exists x\, (\alpha_1 \,\&\, \beta) \equiv \gamma$ *and* α_0 *and* $\neg E(X)$ *imply* $\exists x\, (\alpha_1 \,\&\, \beta) \equiv \delta$. *Then,* $\exists x\, \alpha \equiv \alpha_0 \,\&\, E(X) \,\&\, \gamma \vee \alpha_0 \,\&\, \neg E(X) \,\&\, \delta$.

Proof. Clear.

So it is enough to find the corresponding γ and δ.

Suppose α_0 and $E(X)$. By the Cases Lemma it can be assumed that β has a conjunct $E(a, x)$, $1 \leq a < p^s$. Therefore we can replace $D(j, t)$ by $\{E(b \cdot p^j, t) \mid 1 \leq b < p^{s-j}\}$, $E(b, p^j x + \tau)$ by $E(b - a \cdot p^j, \tau)$, $F(p^j x + \tau) = X$ by $\neg E(-ap^j \cdot \tau)$.

As a result $\alpha_1(x) \,\&\, \beta(x) \equiv F(x) = X \,\&\, E(a, x) \,\&\, \alpha'$ for some open α' without x. And,

$$\exists x\, (\alpha_1 \,\&\, \beta) \equiv \alpha' \,\&\, \exists x\, (F(x) = X \,\&\, E(a, x)) \equiv \alpha' \,\&\, E(X)$$

Suppose α_0 and $\neg E(X)$.

W.l.o.g. every atom in β has a form $D(j, t(x))$. Indeed, let β_0 be an atom in $\beta(x)$. If $\beta_0 = E(j, t(x))$ then β_0 can be replaced by "false". Let β_0 not contain x. By the Cases Lemma it can be assumed that β_0 or $\neg\beta_0$ is a conjunct of $\beta(x)$. It can be assumed that β_0 occurs only once in $\beta(x)$. Let $\alpha_1 \,\&\, \beta = \pm\beta_0 \,\&\, \alpha'$. Then $\exists x\, (\alpha_1 \,\&\, \beta) \equiv \pm\beta_0 \,\&\, \exists x \alpha'(x)$.

Let a bar denote the natural isomorphism $\Gamma_2(p, s, X) \to \Gamma(p, s, X)$. Let α'_1 be $p^k\bar{x} = \bar{\tau}_0 \,\&\, \bigwedge\{\bar{t} \neq 0 \mid t \in N\}$ and β' be obtained from β by replacement of $D(j, t)$ by $\bar{t} \equiv 0 \pmod{p^j}$. Evidently $\exists x\, (\alpha_1(x) \,\&\, \beta(x)) \equiv \exists\bar{x}\, (\alpha'_1(\bar{x}) \,\&\, \beta'(\bar{x}))$.

Let $K(p, s)$ be the class of (Abelian) groups satisfying the axiom $p^s \cdot v = 0$. Let a first order language $L(p, s)$ be obtained from the elementary language of groups by adding the atoms $t \equiv 0 \pmod{p^j}$. Let $T(p, s)$ be the theory of $K(p, s)$ in $L(p, s)$.

Lemma 8.5. $T(p, s)$ *admits a quantifier elimination.*

Proof. It is easy to check Lemma 8.5 with the aid of [17] or even without it.

According to Lemma 8.5 the formula $\exists\bar{x}\, (\alpha'_1(\bar{x}) \,\&\, \beta'(\bar{x}))$ is equivalent in $T(p, s)$ to some Boolean combination of atoms $\bar{\tau} = 0$ and $\bar{\tau} \equiv 0 \pmod{p^j}$. Then

$\exists x(\alpha_1(x) \& \beta(x))$ is equivalent to the corresponding Boolean combination of atoms $F(\tau) \subset X$ and $D(j, \tau)$ Theorem 8 1 is proved

9. Without exiles

Superterms $A(t(x))$ and atoms $t(x)Rk \pmod T$ will be called exiles

Theorem 9.1. *Let $\alpha(x)$ be an open EXL-formula without exiles There exists an open EXL-formula α^* such that $\exists x\alpha(x) \equiv \alpha^*$*

Proof. Let σ be the set of pairs (p, r) occurring in α through $F(p, r, t(x))$, $D(p, r, c, t(x))$, $E(p, r, c, t(x))$ Let $\pi = \{p \ \exists r\,(p, r) \in \sigma\}$ and $s_p = \max\{p \ (p, r) \in \sigma\}$

Lemma 9.1. *W l o g $\alpha(x) = \beta \ \& \ \wedge\{\alpha_p(x) \ p \in \pi\}$ where every $\alpha_p(x)$ is a p-formula and β does not contain x*

Proof. See Lemmas 7 3 and 7 4

Lemma 9.2. $\exists x\alpha(x) \equiv \beta \ \& \ \wedge\{\exists x\alpha_p(x) \ p \in \pi\}$

Proof. Suppose β and $\alpha_p(x_p)$, $p \in \pi$ There exist integers a_p such that $a_p \equiv 1 \pmod{s_p}$ and $a_p \equiv 0 \pmod{s_q}$ for $q \in \pi - \{p\}$ It is easy to check that $\alpha(\Sigma a_p x_p)$ holds

Now Theorem 8 1 implies Theorem 9 1

10. Banishment

Superterms $A(t(x))$ and atoms $t(x)Rk \pmod T$ are called *exiles*

Theorem 10.1. *Let $\alpha(x)$ be an open EXL-formula There exists an open EXL-formula $\alpha^*(x)$ without exiles such that $\exists x\alpha(x) \equiv \exists x\alpha^*(x)$*

Theorem 10 1 is the object of this section The Main Lemma of Section 7 follows from Theorem 10 1 and Theorem 9 1

Below τ is an elementary term without x

Lemma 10.1. *W l.o.g. every elementary term $t(x)$ in α has a form $x + \tau$*

Proof. Every $t(x)$ can be represented in a form $ax + \tau$ for some integer a Let

$S = \{a \neq 0\ \ ax + \tau$ occurs in $\alpha\}$ Let b be the least common multiple of numbers in S It can be assumed that b is the only element of S, see Corollaries 1 2, 1 8, 1 12 and Lemmas 1 5, 1 6 Let $\alpha'(x)$ be the formula such that $\alpha(x) = \alpha'(bx)$ Then $\exists x\, \alpha(x) \equiv \exists x\, [\alpha'(x)\ \&\ F(b, x) = \emptyset]$ Now Corollary 1 4 is used

Lemma 10.2. *W l o g* $\alpha = (A(x) = X)\ \&\ \beta\ \&\ \gamma$ *where*

(1) β *is a conjunction of exile atoms* $(x + \tau)Rk \pmod X$,

(2) β *has no conjuncts* $x + \tau = 0 \pmod X$,

(3) γ *has no exiles at all*,

(4) *for every conjunct* $(x + \tau)Rk \pmod X$ *in* β *there exists a conjunct* $A(\tau) \subseteq X$ *in* γ *and*

(5) $X \neq \emptyset$ *is a conjunct in* γ

Proof. Let M be the set of terms τ occurring in α through $A(x + \tau)$ or $(x + \tau)Rk \pmod T$ By the Cases Lemma it can be assumed that $(A(x + \tau_0) = A(x + \tau))$ or $(A(x + \tau_0) \subset A(x + \tau))$ is a conjunct in α for some fixed τ_0 and every $\tau \in M$ Because of $\exists x\, \alpha(x) \equiv \exists x\, \alpha(x - \tau_0)$ it can be assumed $\tau_0 = 0$ Moreover it can be assumed that $(A(x) = A(x + \tau))$ is a conjunct in α, $\tau \in M$ Indeed, $A(x) \subset A(x + \tau)$ is equivalent to $A(x) \subset A(\tau)$ and implies $(x + \tau)Rk \pmod T) \equiv \tau Rk \pmod T$ So $\alpha = \alpha_1\ \&\ \alpha_2$ where $\alpha_1 = \bigwedge \{A(x) = A(x + \tau)\ \ \tau \in M\}$

W l o g $\alpha_2 = (A(x) = X)\ \&\ \alpha_3$, see Lemma 7 4 W l o g α_3 has no exile superterms Let $\beta = (x + \tau)Rk \pmod T$ be an exile atom in α_3 W l o g $T = X$ Indeed, it can be assumed that $T \subset X$, $T = X$ or $X \subset T$ is a conjunct in α_3 In the case $X \subset T$ we can replace β by $0Rk \pmod T$ In the case $T \subset X$ and $k \neq 0$ we can replace β by $E(T)\ \&\ (x + \tau)R0 \pmod X$ In the case $T \subset X$ and $k = 0$ we can replace β by $(x + \tau)R0 \pmod X$ W l o g β or $\neg\beta$ is a conjunct in α_3, see Lemma 7 3 It can be assumed that β is a conjunct because of

$$\neg(x + \tau < k \pmod X) \equiv (x + \tau = k \pmod X) \vee (x + \tau > k \pmod X)$$

and similarly for other cases

A conjunct $A(x) = A(x + \tau)$ in α_1 can be replaced by

$A(\tau) \subseteq X\ \&\ x + \tau < 0 \pmod X$ or

$A(\tau) \subseteq X\ \&\ x + \tau > 0 \pmod X$

If α_3 has a conjunct $x + \tau = 0 \pmod X$ then α is false

By the Cases Lemma $X = \emptyset$ or $X \neq \emptyset$ is a conjunct in α In the case $X = \emptyset$ $\alpha(x) \equiv \alpha(0)$ Q E D

Let σ be the set of pairs (p, r) occurring in γ through $F(p, r, t(x))$, $D(p, r, c, t(x))$ or $E(p, r, c, t(x))$ Let s be the least common multiple of the numbers p^r $(p, r) \in \sigma$

Lemma 10.3. *If* $x \equiv y \pmod s$, *then* $\gamma(x) \equiv \gamma(y)$

Proof. Clear

Below $F(s,t)\subseteq X$ and $E(s,c,t)$ are used as abbreviations for
$\wedge\{F(p,r,t)\subseteq X\ \ (p,r)\in\sigma\}$ and
$\wedge\{E(p,r,c,t)\ \ (p,r)\in\sigma\}$ respectively
(cf , Corollary 1 4)

Suppose $(x+\tau_0=k\ (\mathrm{mod}\ X))$ is a conjunct in β W l o g it is the only conjunct in β Indeed it implies that

$$(x+\tau)Rl\,(\mathrm{mod}\,X)\equiv(\tau-\tau_0)R(l-k)(\mathrm{mod}\,X)$$

W l o g $\tau_0=0$ Indeed,

$$\exists x\,[A(x)=X\ \&\ x+\tau_0=k\ (\mathrm{mod}\,X)\ \&\ \gamma(x)]\equiv$$

$$\equiv\exists x\,[A(x)=X\ \&\ x=k\ (\mathrm{mod}\,X)\ \&\ \gamma(x-\tau_0)]$$

Let $\alpha_0(x)=E(X)\ \&\ \gamma(x)\ \&\ F(s,x)\subseteq X\ \&\ E(s,k,x)$

Lemma 10.4. $\exists x\,\alpha(x)\equiv\exists x\,\alpha_0(x)$

Proof. $\alpha(x)$ implies $\alpha_0(x)$ Conversely, suppose $\alpha_0(x)$ Because of $F(s,x)\subseteq X$ there exists $y\in X^+-X$ such that $y\equiv x\ (\mathrm{mod}\,s)$ Clearly $\alpha_0(y)\ \&\ A(y)=X$ holds Therefore $y=k+ns\ (\mathrm{mod}\,X)$ for some n Let $u=1\,(\mathrm{mod}\,X)$ Then $\alpha(y-nsu)$ holds

Let every conjunct in β be an inequality Note that

$$x+\tau_1<k_1(\mathrm{mod}\,X)\ \&\ x+\tau_2<k_2(\mathrm{mod}\,X)\equiv$$

$$\equiv x+\tau_1<k_1(\mathrm{mod}\,X)\ \&\ \tau_2-\tau_1\leqslant k_2-k_1(\mathrm{mod}\,X)\vee$$

$$x+\tau_2<k_2(\mathrm{mod}\,X)\ \&\ \tau_1-\tau_2\leqslant k\ -k_2(\mathrm{mod}\,X)$$

So it can be assumed that β has at most one conjunct of a form $x+\tau<k\ (\mathrm{mod}\,X)$ and (similarly) at most one conjuct of a form $x+\tau>k\ (\mathrm{mod}\,X)$ It also can be assumed that $E(X)$ or $\neg E(X)$ is a conjunct in γ

Case 1 β has at most one conjunct Let α_1 be $F(s,x)\subseteq X\ \&\ X\subset X^+\ \&\ \gamma(x)$

Lemma 10.5. $\exists x\,\alpha(x)\equiv\exists x\,\alpha_1(x)$

Proof. $\alpha(x)$ implies $\alpha_1(x)$ Conversely, suppose $\alpha_1(x)$ W l o g $A(x)=X$, see the proof of Lemma 10 4 If α has no exile-atoms then $\alpha(x)$ holds Let $\beta=x+\tau<k\,(\mathrm{mod}\,X)$ (respectively $\beta=x+\tau>k\,(\mathrm{mod}\,X)$) Let $y>0\,(\mathrm{mod}\,X)$ Then $\alpha(x-nsy)$ (respectively $\alpha(x+nsy)$) holds for sufficiently large n

Case 2 $\beta = \tau_1 + k_1 < x < \tau_2 + k_2 \pmod X$ and $\neg E(X)$ is a conjunct in γ. It can be assumed that $k_1 = k_2 = 0$. If $k_1 \neq 0$ or $k_2 \neq 0$ then α is false. Let $\alpha_2 = \alpha_1$ & $\tau_1 < \tau_2 \pmod X$.

Lemma 10.6. $\exists x\, \alpha(x) \equiv \exists x\, \alpha_2(x)$.

Proof. $\alpha(x)$ implies $\alpha_2(x)$. Conversely, suppose $\alpha_2(x)$. W.l.o.g. $A(x) = X$. The Archemedean o-group X^+/X is isomorphic to some dense ordered subgroup of the o-group of reals. So there exists $y > 0 \pmod X$ such that $sy < \tau_2 - \tau_1 \pmod X$. Then $\alpha(x + nsy)$ holds for some n.

Case 3 $\beta = \tau_1 + k_1 < x < \tau_2 + k_2 \pmod X$ and $E(X)$ is a conjunct in γ. Let δ be $\tau_1 + k_1 + 2s < \tau_2 + k_2 \pmod X$. It can be assumed that δ or $\neg\delta$ is a conjunct in γ. But in the case $\neg\delta$ we can replace β by one of the atoms $x = \tau_1 + k_1 + l$, $1 \leq l < 2s$ and use Lemma 10.4. So it can be assumed that δ is a conjunct in γ. Let $\alpha_3 = F(s, x) \subsetneq X$ & γ.

Lemma 10.7. $\exists x\, \alpha(x) \equiv \exists x\, \alpha_3(x)$.

Proof. $\alpha(x)$ implies $\alpha_3(x)$. Conversely, suppose $\alpha_3(x)$. W.l.o.g. $A(x) = X$. The Archemedean o-group X^+/X is isomorphic to the o-group of integers. Let $y = 1 \pmod X$. Then $\alpha(x + nsy)$ holds for some n.

PART 3. CONVEX SUBGROUPS THEORY

11. Decidability theorem

The Convex Subgroups Language CSL is a first order language whose non-logical constants are "$<$", the one-place predicate symbol E and the one-place predicate symbols $p(s, k) > r$ where p, s, k and r are naturals, p is prime and $1 \leq k \leq s$. Every o-group G gives the natural model ΔG of CSL as follows. Elements of ΔG are proper convex subgroups of G (a convex subgroup $X \subseteq G$ is proper if $X \neq G$). $X < Y \equiv X \subset Y$. The predicates $E(X)$ and $p(s, k, X) > r$ are defined according to Section 1. The Convex Subgroups Theory CST is the set of CSL-formulas holding in all ΔG.

Theorem 11.1. CST *is decidable.*

Let σ be a finite set of quadruples (p, s, k, r) of naturals where p is prime and $1 \leq k \leq s$. Let L_σ be a sublanguage of CSL whose non-logical symbols are $<$, E and $p(s, k) > r$ where $(p, s, k, r) \in \sigma$. Let $T_\sigma = L_\sigma \cap \text{CST}$.

Theorem 11.2. *T_σ is uniformly decidable on σ*

Clearly Theorem 11 2 implies Theorem 11 1

Let $\Delta_\sigma G$ be the corresponding L_σ projection of the natural CSL-model ΔG Evidently T_σ is the theory of all $\Delta_\sigma G$

Let σ_1 $\sigma_2, \sigma_3, \sigma_4$ be the corresponding projections of σ and $s = \max \sigma_2$ According to Theorem 1 1 it can be assumed that s is the only element of σ_2 W l o g it can be assumed also that if $p \in \sigma_1$, $1 \leq k \leq s$ and $0 \leq r \leq \max \sigma_4$, then $(p, s, k, r) \in \sigma$

The following abbreviations are used

$p(k, x) > r$ for $p(s, k, x) > r$,
$p(k, x) = 0$ for $\neg(p(k, x) > 0)$,
$p(k, x) = r + 1$ for $p(k, x) > r \;\&\; \neg(p(k, x) > r + 1)$,
$p(x) = 0$ for $\{p(k, x) = 0 \;\; 1 \leq k \leq s\}$,
$p'(x) = 0$ for $\{p(k, x) = 0 \;\; 1 \leq k < s\}$,
$y = x^+$ for $x < y \;\&\; \neg\exists z\,(x < z < y) \vee x = y \;\&\; (\forall u > x)\exists z\,(x < z < u)$

A model A of L_σ is called a σ-chain (a complete σ-chain) if A is a chain (a complete chain) The definition of complete chains is found in Section 14

K_σ is the class of complete σ-chains satisfying the following axioms (where $p \in \sigma_1$ and $r, r + 1 \in \sigma_4$)

(K1) $\exists x\, \forall y\, (x \leq y)$,

(K2) $x < y \supset \exists z\, (x \leq z < z^+ \leq y)$,

(K3) $p(k, x) > r + 1 \supset p(k, x) > r$,

(K4) $p'(x) \neq 0 \supset \exists y\, (x < y) \;\&\; (\forall y > x)\exists z\, (x < z < y \;\&\; p(z) \neq 0)$,

(K5) $E(x) \supset (x < x^+ \vee \forall y\, (y \leq x)) \;\&\; p(s, x) = 1$

ThK_σ is the theory of K_σ in L_σ

Lemma 11.1. *ThK_σ is uniformly decidable on σ*

Proof. Let C_σ be the theory of all complete σ-chains in L_σ By Theorem 15 2 C_σ is uniformly decidable on σ But ThK_σ is finitely axiomatizable in T_σ

Lemma 11.2. $\Delta_\sigma G \in K_\sigma$

Proof. Evidently $\Delta_\sigma G$ is a complete σ-chain and satisfies axioms (K1)–(K3) For axioms (K4) and (K5) see Theorems 1 2 and 1 3

In Section 12 we build a class M_σ of σ-chains such that Th$M_\sigma \subseteq$ ThK_σ According to Section 13 for every $C \in M_\sigma$ there exists an o-group G such that $\Delta_\sigma G \cong C$ So $T_\sigma \subseteq$ (according to Section 13) Th$M_\sigma \subseteq$ (according to Section 12)

$\text{Th}K_\sigma \subsetneq$ (by the Lemma 11 2) $\subsetneq T_\sigma$ So, $T_\sigma = \text{Th}K_\sigma$ and Lemma 11 1 imply Theorem 11 2

12. σ-chains

Definition 12.1. A σ-chain S is the *internal ordinal sum* $\Sigma\{A_i \ i \in I\}$ of its convex submodels A_i on a chain I if

(1) $S = \bigcup\{A_i \ i \in I\}$ and

(2) $i < j \ x \in A_i, \ y \in A_j$ imply $x < y$

Definition 12 2. An *external ordinal sum* $S = \Sigma\{A_i \ i \in I\}$ of σ-chains A_i on a chain I is defined as follows Elements of S are pairs (i, x) where $i \in I$ and $x \in A_i$ $(i, x) < (j, y)$ iff $i < j$ or $i = j$ and $x < y$ And for every one-place predicate symbol P in L_σ, $S \models P(i, x)$ iff $A_i \models P(x)$ An ordinal multiple $A \ I = \Sigma\{A_i \ i \in I$ and $A_i = A\}$

Notations. Let A and B be σ-chains, B is one-element and $b \in B$ The following abbreviations are used

$p(k, B) = r$ for $B \models p(k, b) = r$

$E(B)$ for $B \models E(b)$,

$p'A = 0$ for $\wedge\{p(k \ a) = 0 \ a \in A$ and $k < s\}$,

$pA = 0$ for $\wedge\{p(k \ a) = 0 \ a \in A$ and $k \leqslant s\}$

Let U_σ be the class of such one-element σ-chains B that $\neg E(B)$ Let 0_σ denote every σ-chain $B \in U_\sigma$ such that $(\forall p \in \sigma_1)pB = 0$ Let ω (respectively ω^*) be the naturally (resp inversely) ordered set of natural numbers Let $\mathbf{R}$ be the chain of reals

Definition 12.3. Let F be a finite set of σ-chains An ordinal sum $\Sigma\{A_i \ i \in I\}$ is called *F-dense* if

(1) $\forall i \, (\exists B \in F) A_i \cong B$,

(2) $(\forall B \in F)(\{i \ A_i \cong B\}$ is dense in $I)$ and

(3) I has neither minimal nor maximal elements

Lemma 12.1. *Every two F-dense σ-chains are elementary equivalent*

Proof. By the Ehrenfeucht Criterion [2]

Definition 12.4. An ordinal sum $S = \Sigma\{A_r \ r \in \mathbf{R}\}$ is called an *F-shuffling* and is denoted by τF if

(1) S is F-dense,

(2) $(\exists B \in F)B$ is not one-element,

(3) $(\exists B \in F) \{r \ A_r \not\cong B\}$ is countable and

(4) if $\{r \quad A_r \not\equiv B\}$ is countable and B is one-element then $B = 0_\sigma$

Definition 12.5. Let M_σ be the least class of σ-chains such that

(1) If σ-chain A is one-element and $p'A = 0$ then $A \in M_\sigma$,

(2) If $A, B \in M_\sigma$ then $A + B \in M_\sigma$,

(3) If $A \in M_\sigma$ then $A \quad \omega \in M_\sigma$,

(4) if $A \in M_\sigma$, $B \in U_\sigma$ and $(\forall p \in \sigma_1)(pA = 0 \supset p'B = 0)$, then $B + A \quad \omega^* \in M_\sigma$,

(5) $C + \tau F \in M_\sigma$ if $C \in U_\sigma$ and finite $F = F_1 \cup F_2$ where non-zero $F_1 \subset \{A + B \quad A \in M_\sigma$ and $B \in U_\sigma\}$ and $F_2 \subset U_\sigma$, and

$$(\forall p \in \sigma_1)[((\forall D \in F)pD = 0) \supset p'C = 0]$$

Theorem 12.1. $\mathrm{Th} M_\sigma \subseteq \mathrm{Th} K_\sigma$

Proof. It is enough to prove that for every $n = 1, 2,$ every $A \in K_\sigma$ is n-equivalent to some $B \in M_\sigma$ Fix n

Definition 12.6. σ chain A will be called *good* if it satisfies one of the following requirements

(G1) A is n-equivalent to some $B \in M_\sigma$,

(G2) A does not have the minimal element and $B + A$ satisfies (G1) for every $B \in U_\sigma$ such that, for every $p \in \sigma_1$ and $a \in A$, if $p'B \neq 0$ then $(\exists c \in A)(c < a$ and $pc \neq 0)$,

(G3) A is one-element and $(\exists p \in \sigma_1)p'A \neq 0$ and

(G4) $A \cong A' + B$ where A' satisfies (G1) or (G2) and B satisfies (G3)

Lemma 12.2. *If a good σ-chain $A \in K_\sigma$, then A satisfies* (G1)

Proof. Clear

Definition 12.7. σ chain A is called *quasi-good* if every non-void half-closed interval $[x, y) = \{z \quad x \leq z < y\}$ in A is good

Lemma 12.3. *Every quasi-good σ-chain is good*

Proof. See the proof of Lemma 14 3

Lemma 12.4. *Every σ-chain in K_σ is good*

Proof. See the proof of Lemma 14 4

Theorem 12 1 is proved

13. Constructing o groups

Theorem 13.1. *For every σ-chain $A \in M_\sigma$ there exists an o-group G such that $\Delta_\sigma G$ is somorphic to A*

Proof. By an induction on A Desired o-groups will be constructed as subgroups of lexicographic sums of countable Archimedean o-groups The operations of gluing and interlacement of Section 5 preserve this property

Let A be one-element Then $(\forall p \in \sigma_1)p'A = 0$ If the only element of A satisfies the predicate E then the naturally ordered group of natural numbers is a desired o-group Let $A \in U_\sigma$ By the Gluing Theorem of Section 5 it can be assumed that $\forall p\ (pA = 0)$ or $pA = 1$ for some p and $qA = 0$ for every $q \neq p$ So Q or Q_p (see Section 3) is a desired o-group

Let $A = B_1 + B_2$ and $B_i \cong \Delta_\sigma H_i$, $i = 1, 2$ Then the lexicographic sum $H_1 + H_2$ is a desired o-group

Let $A = B\ \omega$ and $B \cong \Delta_\sigma H$ Then $H\omega = L\Sigma\{H_i\ \ i \in \omega$ and $H_i = H\}$ is a desired o-group

Let $A = C + B\ \ \omega^*$, $B \cong \Delta_\sigma H$ and $C \in U_\sigma$ If $C \cong 0_\sigma$ then $H\ \ \omega^* = L\Sigma\{H_i\ \ i \in \omega^*$ and $H_i = H\}$ is a desired group Let $p(k, C) \neq 0$ for some p and k

It can be assumed that $p(k, C) = 1$ and $q(l, C) = 0$ if $q \neq p$ or $l \neq k$ Indeed let $p(k, C_{pk}) = 1$ and $q(l, C_{pk}) = 0$ if $q \neq p$ or $l \neq k$ and let $\Delta_\sigma H_{pk} \cong C_{pk} + B\ \ \omega^*$ Then a suitable interlacement of o-groups H_{pk} (see the Interlacement Theorem in Section 5) is a desired o-group

It H is not p-divisible then the o-group G of Section 2 is a desired group Let H be p-divisible

It can be assume that H is a lexicographic multiple of Q Indeed, let $H' = L\Sigma\{H_i\ \ i \in \Delta_\sigma H$ and $H_i = Q\}$ and $\Delta_\sigma G' \cong C + \Delta_\sigma H'\ \ \omega^*$ Then a gluing of $H\ \ \omega^*$ and G' is a desired o-group It can be assumed that $H = Q$ Indeed if $H \not\cong Q$ then H is an interlacement of Q and some H' Let $\Delta_\sigma G' \cong C + Q\omega^*$ Then a suitable interlacement of $H'\omega^*$ and G' is a desired o-group

Now the group G of Section 3 is a desired o-group

Suppose

$$A = D + \tau(F_1 \cup F_2), \qquad D \in U_\sigma,$$

$$F_1 = \{B_i + C_i\ \ 1 \leq i \leq m \text{ and } B_i \cong \Delta_\sigma H_i \text{ and } C_i \in U_\sigma\}$$

and

$$F_2 = \{C_i\ \ m < i \leq n \text{ and } C_i \in U_\sigma\}$$

W l o g $D = 0_\sigma$ Indeed let $\Delta_\sigma G' \cong 0_\sigma + \tau(F_1 \cup F_2)$ Then $\Delta_\sigma(G'/X) \cong B_1 + C_1 + \tau(F_1 \cup F_2)$ for some convex subgroup $X \subset G'$ and $A \cong D + \Delta_\sigma(G'/X)\ \ \omega^*$ See the previous case

Let $A = 0_\sigma + \{D_r\ \ r \in \mathbf{R}\}$ and

$$R_i = \begin{cases} \{r \in \mathbf{R} : D_r = B_i + C_i\}, & \text{if } i \leq m, \\ \{r \in \mathbf{R} : D_r = C_i\}, & \text{if } m < i \end{cases}$$

W.l.o.g. $m = 1$. If $m > 1$ let $A_1 = 0_\sigma + \Sigma\{F_r : r \in \mathbf{R}\}$ where

$$F_r = \begin{cases} B_1 + C_1, & \text{if } r \in R_1, \\ C_j, & \text{if } r \in R_j \text{ and } m < j, \\ 0_\sigma & \text{in other cases} \end{cases}$$

and for $1 < i \leq m$ let $A_i = 0_\sigma + \Sigma\{F_r : r \in \mathbf{R}\}$ where

$$F_r = \begin{cases} B_i + C_i & \text{if } r \in R_i, \\ 0_\sigma & \text{in other cases} \end{cases}$$

Let $\Delta_\sigma G_i \cong A_i$, $i = 1, \ldots, m$. Then the corresponding interlacement of o-groups $G_1, \ldots, G_m$ is a desired o-group.

Below $B = B_1$ and $H = H_1$.

W.l.o.g. $C_i = 0_\sigma$ and $\mathbf{R} - R_i$ is countable for some $i > 1$. Indeed, by the definition of shuffling in Section 12, some $\mathbf{R} - R_i$ is countable and if $i > 1$ then $C_i = 0_\sigma$. Let $\mathbf{R} - R_1$ be countable. There exists a representation $R_1 = \bigcup\{R_{1t} : t \in I\}$ where summands R_{1t} are countable, dense in R and disjoint. Let $u \in I$ and $A_t = 0_\sigma + \Sigma\{F_r^t : r \in \mathbf{R}\}$ where

$$F_r^t = \begin{cases} B + C_1, & \text{if } r \in R_{1t}, \\ C_i, & \text{if } t = u,\ r \in R_i \text{ and } i > 1, \\ 0_\sigma & \text{in other cases} \end{cases}$$

Let $\Delta_\sigma G_t \cong A_t$. The corresponding interlacement of o-groups G_t is a desired o-group.

W.l.o.g. $C_1 = 0_\sigma$. If $C_1 \neq 0_\sigma$ let $R_1 = S_1 \cup S_2$ where summands S_t are dense in $\mathbf{R}$ and disjoint. Let $A_t = 0_\sigma + \Sigma\{F_{tr} : r \in \mathbf{R}\}$ where

$$F_{1r} = \begin{cases} B + 0_\sigma & \text{if } r \in S_1, \\ C_1, & \text{if } r \in S_2, \\ C_i, & \text{if } r \in R_i \text{ and } i > 1, \\ 0_\sigma & \text{in other cases} \end{cases}$$

$$F_{2r} = \begin{cases} C_1, & \text{if } r \in S_1, \\ B + 0_\sigma, & \text{if } r \in S_2, \\ 0_\sigma & \text{in other cases} \end{cases}$$

Let $\Delta_\sigma G_t \cong A_t$. Then the corresponding interlacement of G_1 and G_2 is a desired o-group.

Let $s = \Sigma\{p(k, C_i) : p \in \sigma_1,\ k \in \sigma_3,\ 1 < i \leq n\}$.

W.l.o.g. $s \leq 1$. The statement is proved by induction on s. Let $C_{1i}, C_{2i} \in U_\sigma$ and every $p(k, C_i) = p(k, C_{1i}) + p(k, C_{2i})$. Let $R_1 = S_1 \cup S_2$ and the summands S_t are dense in R and disjunctive. Let $A_t = 0_\sigma + \Sigma\{F_r^t : r \in \mathbf{R}\}$ where

$$F'_r = \begin{cases} B + 0_\sigma, & \text{if } r \in R_1, \\ C_n, & \text{if } r \in R_i \text{ and } 1 < i, \\ 0_\sigma & \text{in other cases} \end{cases}$$

Let $\Delta_\sigma G_i \cong A_i$ Then the corresponding interlacement of G_1 and G_2 is a desired o-group

If $s = 0$ then the lexicographic multiple H R_1 is a desired o-group

Suppose $s = p(k, C_2) = 1$ (and so $F_1 = \{B_1 + 0_\sigma\}$, $F_2 = \{C_2, 0_\sigma\}$)

Case 1 H is not p-divisible There exists a representation $R_1 = \bigcup\{R_{1t}$ $t \in R_2\}$ such that $R_{1t} \cap R_{1u} = \emptyset$ if $t \neq u$ and every chain R_{1t} is isomorphic to ω^* and $\lim R_{1t} = t$ For every $t \in R_2$ let $A_t = C_2 + B$ R_{1t} and G_t be the o-group G of Section 2 Then $\Delta_\sigma G_t \cong A_t$ and the interlacement of o-groups G_t is a desired o-group

Case 2 H is p-divisible W l o g H is a lexicographic multiple of the rational o-group Q Indeed, let subchain $I = \{X \in \Delta^* H$ $X \subset X\}$, $H' = Q$ I and $A_1 = 0_\sigma + \Sigma\{F_r$ $r \in \mathbf{R}\}$ where

$$F_r = \begin{cases} \Delta_\sigma H' + 0_\sigma, & \text{if } r \in R_1, \\ C_2, & \text{if } r \in R_2, \\ 0_\sigma & \text{in other cases} \end{cases}$$

Let $\Delta_\sigma G_1 \cong A_1$ and $G_2 = H$ R_1 Then the corresponding gluing of G_1 and G_2 is desired o-group

W l o g $H \cong Q$ Suppose that H is not isomorphic to Q Then H is isomorphic to lexicographic sum $H_1 + Q + H_2$ where H_1 or H_2 can be zero-group Let $A_1 = 0_\sigma + \{F_r$ $r \in \mathbf{R}\}$ where

$$F_r = \begin{cases} \Delta_\sigma Q + 0_\sigma = 0_\sigma + 0_\sigma, & \text{if } r \in R_1, \\ C_2, & \text{if } r \in R_2, \\ 0_\sigma & \text{in other cases} \end{cases}$$

Let $\Delta_\sigma G_1 \cong A_1$ and $G_2 = (H_1 + H_2)$ R_1 Then the corresponding interlacement of G_1 and G_2 is a desired o-group

Lemma 13.1. *Let X_1, X_2, Y_1, Y_2 be countable dense subsets of the chain* $\mathbf{R}$ *of reals and $X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset$ There exists an automorphism ϕ* $\mathbf{R} \to \mathbf{R}$ *such that $\phi X_i = Y_i$, $i = 1, 2$*

Proof. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ It is enough to construct an isomorphism ϕ $X \to Y$ such that $\phi X_i = Y_i$ Indeed this isomorphism can be extended as follows: $\phi(\lim x_n) = \lim \phi x_n$

Fix a numeration of $X \cup Y$ by naturals A 1–1-function f is called *admissible* if $\operatorname{dom} f$ is finite and $\operatorname{rng}(f \mid X_i) \subseteq Y_i$ A sequence $f_0, f_1,$ of admissible functions is constructed as follows $f_0 = \emptyset$ If $n = 2k$ and x is the element in $X - \operatorname{dom} f_n$ of the minimal number then f_{n+1} is an admissible extension of f_n such that $x \in \operatorname{dom} f_{n+1}$ If $n = 2k + 1$ and y is the element in Y-$\operatorname{rng} f_n$ of the minimal number then f_{n+1} is an admissible extension of f_r such that $y \in \operatorname{rng} f_{n+1}$ Evidently $\lim f_n$ is a desired isomorphism

Now it is clear that the o-group G of Section 6 is a desired o-group Theorem 13 1 is proved

APPENDIX. COMPLETE CHAINS WITH ONE-PLACE PREDICATES

A chain is a linear ordered set A chain A is *complete* if A satisfies the following second order axiom

$$(\forall X \subseteq A)(\forall Y \subseteq A)[X \neq \emptyset \ \& \ Y \neq \emptyset \ \& \ (\forall x \in X)(\forall y \in Y)x < y \supset \exists z \, (\forall x \in X)(\forall y \in Y)x \leqslant z \leqslant y]$$

The decidability of the weak monadic second order theory of complete chains is proved in Section 14 The proof uses [11] and [12] The decidability of the weak monadic second order theory of complete chains with one-place predicates is proved in Section 15, where this theory is reduced to the predecessor theory A similar reduction was used in [5] The decision procedures are primitively recursive

14. Complete chains

L_0 is the weak monadic second order language whose only non-logical constant is "<" K_0 is the class of complete chains, Th K_0 is the L_0-theory of K_0

Definition 14.1. A chain S is the *internal ordinal sum* $\Sigma\{A_i \ \ i \in I\}$ of its convex subchains A_i on a chain I if

(1) $S = \bigcup\{A_i \ \ i \in I\}$ and

(2) $i > j$, $x \in A_i$, $y \in A_j$ imply $x < y$

Definition 14.2. An *external ordinal sum* $S = \Sigma\{A_i \ \ i \in I\}$ of chains A_i on a chain I is defined as follows Elements of S are pairs $(,x)$ where $i \in I$ and $x \in A_i$ $(i, x) < (j, y)$ iff $i < j$ or $i = j$ and $x < y$ In particular $A + B = \Sigma\{A_i \ \ i \in \{0, 1\}$, $0 < 1$, $A_0 = A \ \ B_0 = B\}$ The ordinal product $A \ \ I = \Sigma\{A_i \ \ i \in I$ and $A_i = A\}$

Below ω (respectively ω^*) is the naturally (respectively inversely) ordered set of natural numbers and $\mathbf{Q}$ is the chain of rationals

Definition 14.3. Let F be a finite set of chains An ordinal sum $\Sigma\{A_i \ \ i \in I\}$ is called *F-dense* if

(1) every $A_i \in F$,

(2) for every $B \in F$ the subset $\{i \ \ A_i = B\}$ is dense in A and

(3) I has neither minimal nor maximal elements

Definition 14.4. In the case $I = \mathbf{Q}$ an F-dense sum is called a *shuffling* of F and is denoted by τF

Every two shufflings of F are isomorphic

Lemma 14.1. *Every two F-dense chains are* L_0*-equivalent*

Proof. By the Ehrenfeucht Criterion [2]

Let M be the minimal class of chains such that
(1) M contains all one-element chains,
(2) if A $B \in$ M and either A contains the last element or B contains the first element then $A + B \in$ M,
(3) if $A \in$ M and A contains either the first or the last element then A ω and A ω^* belong to U,
(4) if a finite $F \subset$ M and every member of F contains the first and the last elements then $\tau F \in$ M
Let Th M be the L_0-theory of M

Lemma 14.2. Th $K_0 \subseteq$ Th M

Proof. It is enough to prove that every $A \in$ M is L_0-equivalent to some $A' \in K_0$ An induction on A and the Ehrenfeucht Criterion [2] are used The case of one-element A is trivial $(A+B)' = A' + B'$, $(A \ \ \omega)' = A' \ \ \omega$ and $(A \ \ \omega^*)' = A' \ \ \omega^*$ Let $A = \tau F$ and $F' = \{B' \ \ B \in F\}$ Then A is L_0-equivalent to every F'-dense sum $\Sigma\{A_\iota \ \ \iota \in R\}$ where **R** is the chain of reals

Theorem 14.1. Th M $\subseteq$ Th K_0

Proof. It is enough to prove that for every $n = 1, 2,$ every $A \in K_0$ is n equivalent to some $B \in$ M Fix n Chain A will be called good if it is equivalent to some $B \in$ M Chain A will be called quasi-good if every non-void half-closed interval $[x, y) = \{z \ \ x \leqslant z < y\}$ of A is good

Lemma 14.3. *Every quasi-good chain is good*

Proof. There exists L_0-sentence α such that a chain A is good iff it satisfies α Let $\beta(x, y)$ be obtained from α by the restriction of the quantifiers to the interval $[x \ y)$ Lemma 14 3 states that $\forall xy(x < y \supset \beta(x, y))$ implies α So it is enough to prove Lemma 14 3 only for countable chains Let A be a countable quasi-good chain

Case 1 A has the minimal element a If $A = [a, b] = [a \ b) + \{b\}$ then A is good Suppose A does not contain the maximal element and B be a subset of A such that $B \cong \omega$ and $(\forall x \in A)(\exists y \in B) x < y$ Let $\{x, y\} \sim' \{u, v\}$ iff the intervals

$[x\ y) \cup [y, x)$ and $[u, v) \cup [v, u)$ are non-void and n-equivalent By the Ramsey Theorem [14] there exists an infinite $C \subseteq B$ such that every pair of different elements of C are equivalent Let $b, c \in C$ and $b < c$ By means of the Ehrenfeucht Criterion [2] it is easy to check that A is n-equivalent to $[a, b) + [b, c)\omega$ So A is good

Case 2 A does not contain the minimal element Similarly it is proved that there exists infinite $C \subseteq A$ such that $(\forall x \in A)(\exists y \in C) y < x$ and if $x, y, u, v \in C$ and $x < y$, $u < v$ then $[x, y)$ and $[u, v)$ are n-equivalent Let $b, c \in C$ and $b < c$ Then A is n-equivalent to $[b, c)\omega^* + \{x\ c \leq x\}$ and A is good

Lemma 14.4. *Every complete chain is good*

Proof. Let A be a complete chain For $x, y \in A$ let $x \sim y$ iff $x = y$ or $x \neq y$ and $[x, y) \cup [y, x)$ is quasi-good The introduced relation is an equivalence relation Every $\bar{x} = \{y\ x \sim y\}$ is convex, quasi-good and good Let $\bar{A} = \{\bar{x}\ x \in A\}$ be ordered as follows $\bar{x} < \bar{y} \equiv x < y$ $\bar{A}$ is a dense chain If $\bar{A}$ is one-element Lemma 14 4 is proved Suppose (reductio ad absurdum) $\bar{A}$ is not one-element For $\bar{x} < \bar{y}$ let $F(\bar{x}, \bar{y})$ be a minimal subset of M such that every $\bar{z} \in (\bar{x}, \bar{y})$ is n-equivalent to some $B \in F(\bar{x}, \bar{y})$ Let $F = F(\bar{u}, \bar{v})$ have the minimal possible power Then $\bigcup\{\bar{z}\ \bar{u} < \bar{z} < \bar{v}\}$ is n-equivalent to an F-dense chain and is quasi-good This contradicts to density of $\bar{A}$ Lemma 14 4 is proved

Theorem 14 1 is proved

Theorem 14.2. Th M *is decidable*

Proof. We assume the knowledge of [12] Let $n \geq 2$ We say that n-type t is l-good (r-good) if $t_n(A) = t$ implies $A \models \exists x\ \forall y\ (x \leq y)$ ($A \models \exists x\ \forall y\ (y \leq x)$) The predicates "$l$-good" and "$r$-good" are effective Let S_n be the least set of n-types such that

(1) n-type of one-element chains belongs to S_n,

(2) if $s, t \in S_n$ and either s is r-good or t is l-good then $s +_n t \in S_n$,

(3) if $s \in S_n$ and s is either l-good or r-good then $\omega_n(s)$, $\omega_n^*(s) \in S_n$,

(4) if $X \subseteq S_n$ and every $s \in X$ is l-good and r-good then $\sigma_n(X) \in S_n$

It's easy to see that S_n is the set of n-types of M and S_n effectively depends on n So Th M is decidable

Lemma 14 2 and Theorems 14 1 and 14 2 imply

Theorem 14.3. Th K_c *is decidable*

15. Adding one-place predicates

Let L_m be the weak monadic second order language whose non-logical constants are "<" and the one-place predicate symbols $P_1, \ldots, P_m$. Let K_m be the class of such L_m-models A that L_0-reduction of A is a complete chain. Let K'_m be the class of such models $A \in K_m$ that A satisfies the following axioms

$\bigvee\{P_i(x): 1 \leq i \leq m\}$ and

$P_i(x) \supset \neg P_j(x)$ where $1 \leq i < j \leq m$.

(In other words $A \models \forall x\, \exists! i\, P_i(x)$.)

Let $\mathrm{Th}\, K_m$ (respectively $\mathrm{Th}\, K'_m$) be the L_m-theory of K_m (resp. K'_m).

Lemma 15.1. *$\mathrm{Th}\, K_m$ is uniformly on m reducible to $\mathrm{Th}\, K'_n$ where $n = 2^m$.*

Proof. Clear.

Theorem 15.1. *$\mathrm{Th}\, K'_n$ is uniformly on n reduced to $\mathrm{Th}\, K_0$.*

Proof. The following abbreviations are used:

$$\text{(i)} \qquad y = x^+ \text{ for } x < y \;\&\; \neg\exists z\,(x < z < y) \quad \vee\; x = y \;\&\; (\forall u > x)\exists z\,(x < z < u),$$

$$\text{(ii)} \qquad y = x^- \text{ for } y < x \;\&\; \neg\exists z\,(y < z < x) \quad \vee\; x = y \;\&\; (\forall u < x)\exists z\,(u < z < x),$$

$$\text{(iii)} \quad R_j(x_1) \text{ for } \begin{cases} x_1 = x_1^+, & \text{if } j = 1, \\ \exists x_2 \ldots x_j [\bigwedge\{x_i < x_i^+ = x_{i+1}: 1 \leq i < j\} \;\&\; x_j = x_j^+], & \text{if } 1 < j < n, \\ \exists x_2 \ldots x_n \bigwedge\{x_i < x_i^+ = x_{i+1}: 1 \leq i < n\}, & \text{if } j = n. \end{cases}$$

Let β be an L_n-sentence and β' be obtained from β by

(1) the restriction of quantifiers by $x = x^-$ and

(2) the replacing of every P_i by R_i.

Let $\alpha = (\beta' \;\&\; \exists x\,(x = x^-))$.

Lemma 15.2. *β has a model in K'_n iff α has a model in K_0.*

Proof. Let $A \in K_0$ and $A \models \alpha$. Let $A' = \{x \in A: x = x^-\}$. A' is complete. The definitions $P_i(x) \equiv R_i(x)$ turn A' to an L_n-model satisfying β.

Let $B \in K'_n$ and $B \models \beta$. Let A be the ordinal sum $\Sigma\{C_b: b \in B\}$ of chains C_b which are defined as follows. Let $B \models P_i(b)$. If $x < x^+$ let $C_b = i + \omega^* + \omega$ (where i denotes a chain containing exactly i elements). If $x = x^+$ let $C_b = i + \omega^*$. It is easy to check that A is complete and $A \models \alpha$.

Because β is an arbitrary formula of L_n, Lemma 15 2 implies Theorem 15 1 From Theorem 14 3, Theorem 15 1 and Lemma 15 1 we obtain

Theorem 15.2. Th K_n *is uniformly decidable on m*

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