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MONADIC THEORY OF ORDER AND TOPOLOGY, 1

BY

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ABSTRACT

We deal with the monadic theory of linearly ordered sets and topological spaces, disprove two of Shelah's conjectures and prove some more results. In particular, if the Continuum Hypothesis holds, then there exist monadic formulae expressing the predicates "X is countable" and "X is meager" in the real line and in Cantor's Discontinuum.

Introduction

The *pure monadic language* has two sorts of variables: for points and for sets of points. Its atomic formulae have the form $x_i \in X_j$. The rest of its formulae are built up from the atomic ones by means of the ordinary propositional connectives and quantifiers, both for point and set variables. The *monadic language of order* is obtained from the pure monadic language by adding a new point predicate "$<$" (so the new atomic formulae have the form $x_i < x_j$). The *monadic topological language* is obtained from the pure monadic language by adding the closure operation (so the new atomic formulae have the form $X_i = \bar{X}_j$). Formulae in the latter language will be called *topological.*

For the sake of brevity, linearly ordered sets will be called *chains.* The *monadic theory of a chain M* is the theory of M in the monadic language of order when the set variables range over all subsets of M. The *monadic theory of a topological space U* is the theory of U in the monadic topological language when the set variables range over all subsets of U. A chain M can be regarded as a top. space. The monadic theory of this space is easily interpretable in the monadic theory of the chain M.

A short survey of results concerning the monadic theory of chains can be found in [12]. The monadic theory of topology was studied in [3]. The present

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paper and its forthcoming continuation [5] are closely connected with [12]. We generalize Shelah's results, prove some of his conjectures, disprove several of the others and prove some more results. Some of the results are announced in [4] and [7].

We will now summarize the contents of the present paper. In the course of this summary we will have to mention some results which will be proved in [5] and [6].

Recall that a subset X of a top. space U is called meager (in U) iff it is a union of $\leqq \aleph_0$ nowhere dense sets. Otherwise X is of the second category (in U). Below, c is the cardinality of the continuum. We say that X is *pseudo-meager* in U iff it is a union of $< c$ nwd sets. Otherwise X is *of the third category*. X is *hereditarily of the third category* iff each perfect subset of X is of the third category.

Assuming

(*) The real line is of the third category

Shelah interpreted the true arithmetic (i.e. the first order theory of the standard model of arithmetic) in the monadic theory of the real line (see theorem 7.10 in [12]). Clearly (*) follows from the Continuum Hypothesis (CH). (*) follows also from Martin's Axiom (see [10]). It is well known that (*) cannot be proved in ZFC.

Theorem 7.10 in [12] can be generalized as follows.

THEOREM 1. *The true arithmetic is interpretable in the monadic theory of the class of top. spaces which are dense, T_3, first countable, of cardinality c, and hereditarily of the third category.*

In particular, the true arithmetic is interpretable in the monadic (topological) theory of Cantor's Discontinuum if (*) holds. Assuming $V = L$ we can replace the true arithmetic by the full second order theory of c in Theorem 1. For a detailed proof of Theorem 1 in both its variants see [5], though most of the proof is given here in Sections 1–5.

The main results of the present paper concern expressibility. Let K be the class of top. spaces described in Theorem 1. Let $SK = \{U: U \in K$ and U is separable$\}$. We say that a top. space U is *quasi-separable* iff for each non-empty open $G \subset U$ there exists a non-empty open separable $G' \subset G$. Let $QSK = \{U: U \in K$ and U is quasi-separable$\}$. If (*) holds then the real line and Cantor's Discontinuum belong to SK.

THEOREM 2 (see §6). *The predicate "$2^{|X|} \leqq c$" is expressible by a top. formula in SK.*

THEOREM 3 (see §6). *Assume $2^{\kappa} < 2^{c}$ for each $\kappa < c$. Then the predicate "$|X| = c$" is expressible by a top. formula in SK.*

COROLLARY 4 (Martin's Axiom). *The predicate "$|X| = c$" is expressible in SK.*

COROLLARY 5 (CH). *The predicate "X is at most countable" is expressible in SK.*

Corollary 5 disproves Conjecture 7G in [12].

THEOREM 6 (see §7). *Assume that c is regular, and $2^{\kappa} < 2^{c}$ for each $\kappa < c$. Then the predicate "X is of the third category" is expressible by a top. formula in SK.*

THEOREM 7 (see §7). *Assume Martin's Axiom. Then the predicate "X is meager" is expressible by a top. formula in QSK.*

COROLLARY 8 (CH). *The predicate "X is meager" is expressible in QSK.*

Corollary 5 gives the strongest result in a certain sense. According to §8, the predicate "X is at most countable" is not expressible in QSK even in the compact case. Neither "X is finite" nor "The space is compact" is expressible in SK even in the locally compact case. "X is finite" is easily expressible in the compact case. Finiteness and compactness are easily expressible in the monadic theory of chains. Under some set-theoretic assumptions "X is countable" is expressible in countably complete chains (see [5]) which implies categoricity and finite axiomatizability of the real line in monadic logic.

Modest spaces are defined in §2. A chain M is *perfunctorily n-modest* iff it has no jumps and for every everywhere dense subsets $X_1, \cdots, X_n$ of M there exists a perfect $Y \subset M$ without jumps such that $Y \subset X_1 \cup \cdots \cup X_n$ and each $X_i \cap Y$ is dense in Y. Cf. Remark 2 on page 409 in [12]. M is *n-modest* iff each subchain of M without jumps is perfunctorily n-modest. M is *modest* iff it is n-modest for every n. This paper was almost finished when I noticed that modesty plays a fundamental role in the monadic theory of rational chain. In discussions with Shelah it was cleared up that modesty plays a fundamental role in the monadic theory of all *short* (embedding neither ω_1 nor ω_1^*) chains.

THEOREM 9. *A chain is monadically equivalent to (i.e. has the same monadic theory as) the rational chain Q iff it is dense, without endpoints, short, and modest.*

Theorem 9 generalizes Theorem 6.2.B and 6.3 in [12] and has essentially the same proof. By [11], the monadic theory of Q is decidable.

COROLLARY 10. *The monadic theory of short modest chains is decidable.*

The proof of Theorem 9 really gives a kind of an elimination of quantifiers. In particular, for each sentence F in the monadic theory of Q there exists an n such that n-modesty decides F in the theory of dense short chains without endpoints. Together with a chain variant of Theorem 2.7 below this fact gives

THEOREM 11. *Assume* $(*)$. *The monadic theory of Q is not finitely axiomatizable in monadic logic (i.e. there exists no sentence F in the monadic theory of Q such that each chain satisfying F is monadically equivalent to Q).*

Theorem 11 disproves Conjecture 03 in [12].

Let the *modest theory* of a chain M be the theory of M in the monadic language of order when the set variables range over the modest subchains of M which remain modest after adding an arbitrary countable subset of M.

THEOREM 12 (Gurevich and Shelah). *The modest theory of real line is decidable and coincides with that of any complete short chain without jumps and endpoints satisfying the following modest axiom* $\exists X \forall y\,(y \in \bar{X})$.

Theorem 12 generalizes Theorem 6.5 in [12] and can be proved by methods of [12]. We give a detailed proof of Theorems 9, 11 and 12 in [6].

Shelah noticed that assuming CH it is possible to interpret the true arithmetic not only in the monadic theory of the real line R but also in the monadic theory of each non-modest subchain of R. This gives rise to

THEOREM 13 (see §9). *Assume CH. For each $n \geqq 1$, true arithmetic is interpretable in the monadic theory of the class of first countable T_3 space of cardinality c which are not n-modest.*

COROLLARY 14 (CH). *The true arithmetic is interpretable in the monadic theory of each short chain which is not modest.*

Corollaries 10 and 14 form a dichotomy.

Assuming $V = L$ we can replace the true arithmetic by the full second order theory of c in Theorem 13 and Corollary 14 (see [5]). For each $n \geqq 1$ let K_n be the class of top. spaces described in Theorem 13. Let SK_n (respectively QSK) be the class of separable (resp. quasi-separable) spaces belonging to K_n. Using §9 it is easy to check

THEOREM 15 (CH). *Corollaries 5 and 8 remain true if SK and QSK are replaced by SK_n and QSK_n respectively.*

By Theorem 6.2 and 6.3 in [12] the predicates "X is countable" and "X is meager" are not expressible in the monadic theory of modest chains.

At an earlier stage of the present work I conjectured that, in the case of the real line R, no top. formula discriminates between subsets of cardinality κ and subsets of cardinality λ if $2^\kappa = 2^\lambda$. This conjecture fails (Magidor and Shelah) in the standard Cohen model for blowing c to ω_3 (see the end of §6).

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§0. Terminology and notation

[8] is used as the source of set-theoretic terminology and notation, but $X \subset Y$ is consistent with $X = Y$ here and the power set of X is denoted by $PS(X)$. Additionally $\Sigma\{X_i : i \in I\}$ denotes $\cup\{X_i : i \in I\}$ in the case that $\{X_i : i \in I\}$ is disjoint, c is the cardinality of the continuum, and $\exp \kappa = 2^\kappa$.

[9] is used as the source of topological terminology and notation. U denotes the top. space in which we work. $A, B, C, D, E, W, X, Y, Z$ are arbitrary point sets i.e. subsets of U. G and H are non-empty open point sets. $\mathrm{cl}\, X = \bar{X}$, $\mathrm{der}\, X$ is the set of limit points of X.

We deviate from [9] in using the notion of density. Here X is dense iff $X \subset \mathrm{der}\, X$, X is dense in Y iff $Y \subset \mathrm{cl}(X \cap Y)$, X is nowhere dense in Y iff $Y - \mathrm{cl}(X \cap Y)$ is dense in Y, X is everywhere dense iff X is dense in U, X is nowhere dense iff X is nowhere dense in U. "Everywhere dense" and "nowhere dense" are abbreviated as "ewd" and "nwd" respectively.

We define:

$X \subsetsim Y$ iff $X - Y$ is nwd,

$X \approx Y$ iff $X \subsetsim Y$ and $Y \subsetsim X$, and

$X < Y$ iff $X \subsetsim Y$ and $Y - X$ is ewd.

We say that a top. formula $F(X_1, \cdots, X_n)$ holds in G iff $G \models F(G \cap X_1, \cdots, G \cap X_n)$. For example "$X \subset Y$ in G" means that $G \cap X \subset G \cap Y$. We define the *domain* of $F(X_1, \cdots, X_n)$ as $\mathrm{do}\, F(X_1, \cdots, X_n) = \cup\{G : F(X_1, \cdots, X_n)$ holds in $G\}$. For example $\mathrm{do}\,(X \subset Y) = \cup\{G : G \cap X \subset G \cap Y\}$.

We use the term "regular" as a synonym of "T_3". Below U is dense, regular, and first countable. The last property means that for each point $p \in U$ there exists a countable basis of neighbourhoods of p. In Sections 3–7 we also assume that U is of cardinality c and hereditarily of the third category.

"WLOG" and "nbd" abbreviate "without loss of generality" and "neighbourhood" respectively.

§1. Almost disjoint division

We work in a top. space U which is dense, regular, and first countable.

DEFINITION 1.1. A point set $X \subset U$ is called *Cantor* (in U) iff X is perfect (i.e. $X = \operatorname{der} X \neq 0$) and nowhere dense. $\mathrm{Ca}(X) = \{Y\colon Y \subset X$ and Y is Cantor in the subspace $X\}$, $\mathrm{SCa}(X) = \{Y\colon Y \in \mathrm{Ca}(X)$ and Y is separable$\}$.

The following theorem generalizes the statement $(*)$ on page 413 in [12]. We deal with a meager subset and use first countability instead of second countability.

THEOREM 1.1. *Let $\bigcup_{n<\omega} X_n \subset X$ where each X_n is ewd. Let $A \subset U - X$ be meager. Then there exists a family $S \subset \mathrm{SCa}(U)$ of cardinality c such that:*
(1.1) *each X_n is dense in each $Y \in S$,*
(1.2) *$Y \cap Z \subset X$ for every different $Y, Z \in S$, and*
(1.3) *each $Y \in S$ is disjoint from A.*

PROOF. Let $A = \bigcup A_n$ where each A_n is nwd. Let $f\colon \{0,1\} \times \omega \times \omega \to \omega$ be one-one and onto, and $(gn, hn, h'n) = f^{-1}(n)$. Let s and t range over the finite sequences of natural numbers, $\mathrm{lh}\, s$ be the length of s. s is regarded as a function from $\mathrm{lh}\, s$ to ω. $t = s\,\hat{}\,n$ means that t extends s by $t(\mathrm{lh}\, s) = n$.

LEMMA 1.2. *There exist open sets $G(s)$ and points $p(s)$ such that:*
(1.4) *$\bar G(s) \subset U - \bar A_{\mathrm{lh}\, s}$, $\bar G(s\,\hat{}\,n) \subset G(s)$ and $\bar G(s\,\hat{}\,m) \cap \bar G(s\,\hat{}\,n) = 0$ if $m \neq n$;*
(1.5) *$p(s) \in G(s)$, $p(s\,\hat{}\,n) \in X_{hn}$ and $\lim p(s\,\hat{}\,n) = p(s)$; and*
(1.6) *If $p = \lim p(s_n) \notin X$ then there exists a strictly increasing sequence $t_0 \subset t_1 \subset \cdots$ such that $p \in \bigcap \bar G(t_n)$.*

PROOF OF LEMMA 1.2. Form $G(0) = U - \bar A_0$ and pick $p(0) \in G(0)$. Suppose that $G(t)$ and $p(t)$ are chosen for every t with $\mathrm{lh}\, t \leqq l$ and that the relevant cases of (1.4) and (1.5) hold. Let $\mathrm{lh}\, s = l$ and $B_0 \supset B_1 \supset \cdots$ be a nbd basis for $p(s)$. Pick consecutively

$$p(s\,\hat{}\,n) \in G(s) \cap B_n \cap X_{hn} - (\bar A_{l+1} \cup \{p(s)\} \cup \{p(s\,\hat{}\,m)\colon m < n\}).$$

Using the regularity of U, choose consecutively disjoint $G(s\,\hat{}\,n)$ containing $p(s\,\hat{}\,n)$, and $H(s\,\hat{}\,n)$ including $U - G(s)$, $\bar A_{l+1}$, $p(s)$, $\{p(s\,\hat{}\,m)\colon m \neq n\}$ and $\bigcup \{\bar G(s\,\hat{}\,m)\colon m < n\}$. Therefore, (1.4) and (1.5) are proved.

Now we prove (1.6). Suppose that $p = \lim p(s_n) \notin X$. Then every $N_i = \{s_n \mid (i+1): i < \mathrm{lh}\, s_n\}$ is finite.

In the other case take the minimal i with infinite N_i. Then there exists a subsequence $s_{n_0}, s_{n_1}, \cdots$ such that $s_{n_0} \mid i = s_{n_1} \mid i = \cdots$ and $s_{n_0}(i) < s_{n_1}(i) < \cdots$. Clearly $\lim p(s_{n_k}) = p(s_{n_0} \mid i) \in X$.

By König's Lemma there exists a sequence $t_0 \subset t_1 \subset \cdots$ such that $t_i \in N_i$. Clearly $\lim p(t_n) = p$. Lemma 1.2 is proved.

We continue the proof of Theorem 1.1. Let $S = \{\bar{Y}_\xi : \xi \in {}^\omega 2\}$ where

$$Y_\xi = \{p(s): \forall k (k < \mathrm{lh}\, s \rightarrow g(s(k)) = \xi(k))\}.$$

Note that $\lim_{j \to \infty} p(s^\wedge f(\xi(\mathrm{lh}\, s), i, j) = p(s)$. Hence Y_ξ is dense and X_i is dense in Y_ξ.

Y_ξ is nwd. For, if $p(s) \in G$ then there exists n such that $G(s^\wedge n)$ is disjoint from Y_ξ and intersects G.

So $S \subset \mathrm{SCa}(U)$, $|S| = c$ and (1.1) holds.

We now prove (1.3). Let $p = \lim p(s_n) \in Y_\xi - X$. WLOG, the sequence $s_0, s_1, \cdots$ is strictly increasing and $p \in \bigcap \bar{G}(s_n)$. Now use (1.4).

We prove (1.2) by reduction to absurdity. Let $p \in \bar{Y}_\xi \cap \bar{Y}_\eta - X$ where $\xi m \neq \eta m$. Then $p = \lim p(s_n) = \lim p(t_n)$ where $p(s_n) \in Y_\xi$ and $p(t_n) \in Y_\eta$. WLOG, the sequences $s_0, s_1, \cdots$ and $t_0, t_1, \cdots$ are strictly increasing and $p \in \bar{G}(s_{m+1}) \cap \bar{G}(t_{m+1}) = 0$.

Theorem 1.1 is proved. □

COROLLARY 1.3. *If $|Y| < c$ and $Y_0, Y_1, \cdots$ are dense in Y, then there exists a countable $C \in \mathrm{Ca}(Y)$ such that $C \subset \bigcup Y_n$ and each Y_n is dense in C.*

PROOF. There exists a countable $X \subset Y$ such that $X \subset \bigcup Y_n$ and each Y_n is dense in X (use first countability). By Theorem 1.1 there exists $S \subset \mathrm{Ca}(\bar{X})$ such that $|S| = c$, each Y_n is dense in each $Z \in S$ and S is disjoint on $Y - X$. Hence there exists $Z \in S$ disjoint from $Y - X$. Build $C = Y \cap Z$. □

In the following lemma we allow $H_i = 0$.

LEMMA 1.4. *For each family $\{G_i : i \in I\}$ there exists a disjoint family $\{H_i : i \in I\}$ such that $H_i \subset G_i$ and $\Sigma H_i \approx \bigcup G_i$.*

PROOF. Consider functions f such that do $f = I$, fi is an open subset of G_i and the range of f is disjoint. Take f with maximal Σfi. □

LEMMA 1.5. *If $\{H_\alpha : \alpha < \kappa\}$ is an open basis of U, and for each $\alpha < \kappa$, $|H_\alpha \cap X| \geqq \lambda \geqq \kappa$, then X can be partitioned into λ disjoint ewd parts.*

PROOF. WLOG, $|X| = \lambda$. Let $f: \kappa \times \lambda \to \lambda$ be one-one and onto, and $(l\alpha, r\alpha) = f^{-1}(\alpha)$. Pick $x_\alpha \in H_{l\alpha} \cap X - \{x_\beta: \beta < \alpha\}$. Build $X_\alpha = \{x_{f(\beta,\alpha)}: \beta < \kappa\}$. □

§2. Modesty and Shelah's wonder sets

DEFINITION 2.1. A subset D of U is *κ-modest* iff $\kappa = 0$ or $\kappa > 0$ and for every X and $\{X_\alpha: \alpha < \kappa\}$, if X is a dense subset of D and each X_α is dense in X, then there exists $C \in \mathrm{Ca}(D)$ such that $C \subset X$ and each X_α is dense in C.

For each n, the predicate "D is n-modest" is expressible by a top. formula.

LEMMA 2.1. *If $1 \leqq \kappa \leqq \aleph_0$, D is κ-modest, X and $\{X_\alpha: \alpha < \kappa\}$ are as above, then there exists a countable $C \subset \mathrm{Ca}(D)$ such that $C \subset X$ and each X_α is dense in C.*

PROOF. Using first countability choose a countable dense $Y \subset X$ such that each X_α is dense in Y. □

LEMMA 2.2. *If each $C \in \mathrm{SCa}(D)$ is either of cardinality $< c$ or meager in $\bar{C}$, then D is $\aleph_0$-modest.*

PROOF. Let $X_0, X_1, \cdots \subset X \subset D$, X be dense, and each X_n be dense in X. By Theorem 1.1 there exists $C_0 \in \mathrm{SCa}(\bar{X})$ such that each X_n is dense in C_0. $C_0 \cap D$ is Cantor in D. If it is of cardinality $< c$ then, by Lemma 1.3, there exists $C \in \mathrm{Ca}(C_0 \cap D) \subset \mathrm{Ca}(D)$ such that $C \subset X$ and each X_n is dense in C. If it is meager in C_0, then $C_0 \cap D - X$ is meager in C_0 and, by Theorem 1.1, there exists $C_1 \in \mathrm{Ca}(C_0)$ such that C_1 is disjoint from $C_0 \cap D - X$ and each X_n is dense in C_1. Let $C = C_1 \cap D$. C is Cantor in D, $C \subset X$, and each X_n is dense in C. □

Suppose that $|U| = c$. (According to [1] each first countable compact top. space is of cardinality c.) Then U has an open base of cardinality $\leqq c$ (use first countability) and $|\mathrm{SCa}(U)| = c$.

LEMMA 2.3. *Suppose that for each G, $G \cap X$ is of the third category. Then there exists $E \subset X$ such that*

(i) *for each G, $|E \cap G| = c$, and*

(ii) *each separable Cantor subset of E is of cardinality $< c$.*

PROOF. Let $\{H_\alpha: \alpha < c\}$ be an open basis of U where each H_α occurs c times. Let $\{Y_\alpha: \alpha < c\} = \mathrm{SCa}(U)$. Pick $p_\alpha \in H_\alpha \cap X - \bigcup_{\beta<\alpha} Y_\beta - \{p_\beta: \beta < \alpha\}$. Then $E = \{p_\alpha: \alpha < c\}$ is the desired set. □

Let us fix $D \subset U$. Let C range over $\mathrm{Ca}(D)$ and $P \subset \mathrm{PS}(D)$.

DEFINITION 2.2. C is *P-good* iff there exists $A \in P$ such that $C \subset A$. C is *P-bad* iff each P-good C' is nwd in C. C is *very P-bad* iff there exists no $C' \subset C$ which is P-good.

Note that if C is not P-bad then there exists G such that $C \cap G \neq 0$ and $C \cap \mathrm{cl}(C \cap G)$ is P-good.

DEFINITION 2.3. W is a *wonder set* for P iff $W \subset U - D$ and for each separable C,

(i) $|\bar{C} \cap W| < c$ if C is P-good, and

(ii) $\bar{C} \cap W \neq 0$ if C is P-bad.

LEMMA 2.4 (cf. lemma 7.4 in [12]). *Suppose that for every separable C, $\bar{C} - D$ is of the third category in $\bar{C}$. Then there exists a wonder set for P.*

PROOF. First, note that if C_1 is P-good and C_2 is P-bad, then $\bar{C}_1$ is nwd in $\bar{C}_2$. Because, if $\bar{C}_1$ is dense in $\bar{C}_2 \cap G$, then $\bar{C}_2 \cap G \subset \bar{C}_1$ and $C_2 \cap G = D \cap \bar{C}_2 \cap G \subset D \cap \bar{C}_1 = C_1$.

Let $\mathrm{SCa}(D) = \{C_\alpha : \alpha < c\}$. If C_α is P-bad pick $p_\alpha \in (\bar{C}_\alpha - D) - \cup \{\bar{C}_\beta : \beta < \alpha$ and C_β is P-good$\}$. Then $W = \{p_\alpha : C_\alpha$ is P-bad$\}$ is the desired set. □

Suppose that U is hereditarily of the third category.

LEMMA 2.5. *If D is 1-modest and E is perfect, then $E - D$ is of the third category in E.*

PROOF. Suppose the contrary: $E - D = \cup \{X_\alpha : \alpha < \kappa\}$ where $\kappa < c$ and each X_α is nwd in E. Let $Y = E - \cup \bar{X}_\alpha$. U is hereditarily of the third category hence Y is dense. As D is 1-modest there exists a countable $C \subset Y$. $\bar{C} = C + \cup (\bar{C} \cap \bar{X}_\alpha)$ and each $\bar{C} \cap \bar{X}_\alpha$ is nwd in C. So $\bar{C}$ is pseudo-meager, which is impossible. □

THEOREM 2.6. *If D is 1-modest there exists a wonder set for P.*

PROOF. See Lemmas 2.4 and 2.5. □

Clearly R is not 1-modest.

THEOREM 2.7. *Assume that the real line R is of the third category. Then for each $n \geqq 1$ there exists an ewd $X \subset R$ which is n-modest but not $(n+1)$-modest.*

PROOF. Let $D \subset R$ be countable and ewd. Partition D into $n+1$ disjoint and ewd parts: $D = D_0 + \cdots + D_n$. Let $S_0 = \{E : E \in \mathrm{Ca}(R)$ and there exists an i such that E is disjoint from $D_i\}$, and $S_1 = \{E : E \in \mathrm{Ca}(R)$ and each D_i is dense in $E\}$.

Imitating the construction of Shelah's wonder set one can build W such that:

(i) $|E \cap W| < c$ if $E \in S_0$, and

(ii) $E \cap W \neq 0$ if $E \in S_1$.

Build $X = D + W$. X is not $(n+1)$-modest. For, there exists no $E \in \mathrm{Ca}(X)$ such that $E \subset D$ and each D_i is dense in E.

We prove now that X is n-modest. Let Y and $Y_0, \cdots, Y_{n-1}$ be subsets of X such that Y is dense and each Y_i is dense in Y. We look for $E \in \mathrm{Ca}(X)$ such that $E \subset Y$ and each D_i is dense in E.

WLOG, there exists $f: n \to (n+1)$ such that for each $i < n$, $Y_i \subset D_{fi} + W$.

We prove this as follows. There exist G_0 and j such that $G_0 \cap Y \neq 0$ and $Y_0 \cap (D_j + W)$ is dense in $G_0 \cap Y$. There exist $G_1 \subset G_0$ and k such that $G_1 \cap Y \neq 0$ and $Y_1 \cap (D_k + W)$ is dense in $G_1 \cap Y$. And so on. Now replace Y and $Y_0, Y_1, \cdots$ by $G_n \cap Y$ and $Y_0 \cap (D_j + W)$, $Y_1 \cap (D_k + W), \cdots$ respectively.

Hence there exists an i such that for each $j < n$, $Y_j \subset X - D_i$. WLOG, $i = 0$ and $Y \subset X - D_0$. By Theorem 1.1 there exists $S \subset \mathrm{Ca}(X)$ such that $|S| = c$, each Y_i is dense in each $E \in S$, and S is disjoint on $X - Y$. There exists $E_0 \in S$ which is disjoint from D_0 (since D_0 is countable). By (i), $|E_0| < c$. By Lemma 1.3 there exists $E \subset \mathrm{Ca}(E_0)$ such that $E \subset Y$ and each Y_i is dense in E. $\square$

§3. Representation of a family of point sets

In this section a rather general family P of point sets is described by a top. formula (with a wonder set for P among the parameters) in such a way that the elements of P satisfy this formula and every X satisfying this formula coincides locally with elements of P.

As before we work in a top. space U which is dense, regular, and first countable. Until the end of §7 we also suppose that U is of cardinality c and hereditarily of the third category.

Select a 2-modest D. Let C range over $\mathrm{Ca}(D)$ and $X \subset D$.

Definition 3.1. Let $W \subset U - D$. C is *W-good* iff $\bar{C}$ is disjoint from W. X is *W-bad with witnesses Y and Z* iff X is dense, Y and Z are dense in X, and there exists no W-good $C \subset X$ such that Y and Z are dense in C. X is *very W-bad* iff there exists no $C \subset X$ which is W-good.

Let $P \subset \mathrm{PS}(D)$, A range over P, and W be a wonder set for P.

Lemma 3.1. *C is very P-bad iff it is very W-bad.*

Proof. If C is not very W-bad take a W-good $C_0 \subset C$. Select a separable

$C_1 \subset C_0$. According to Definition 2.3, C_1 is not P-bad. Hence there exists a P-good $C_2 \subset C_1$. So C is not very P-bad.

If C is not very P-bad take a P-good $C_0 \subset C$. Select a separable $C_1 \subset C_0$. According to Definition 2.3 $|\bar{C}_1 \cap W| < c$. Use Theorem 1.1 to choose a W-good $C_2 \subset C_1$. C is not very W-bad. □

LEMMA 3.2. *If C is W-bad (with some witnesses), then it is P-bad.*

PROOF. Let Y, Z witness that C is W-bad. For reduction to absurdity suppose that C is not P-bad. Then there exists G such that $C \cap G \neq 0$ and $C_0 = C \cap \mathrm{cl}(C \cap G)$ is P-good. C_0 is W-bad with witnesses Y and Z. Using the 2-modesty of D take a countable $C_1 \subset C_0$ such that Y, Z are dense in C_1. According to Definition 2.3, $|\bar{C}_1 \cap W| < c$. Using Theorem 1.1 one can choose a W-good $C_2 \subset C_1$ with Y, Z dense in C_2 which contradicts the fact that C is W-bad with witnesses Y, Z. □

LEMMA 3.3. *Suppose that P is disjoint, and X is dense and has no very P-bad subsets. Then for each G with $G \cap X \neq 0$ there exist $G' \subset G$ and $A \in P$ such that $G' \cap X \neq 0$ and A is dense in $G' \cap X$.*

PROOF. Suppose the contrary. Then there exists G such that $G \cap X \neq 0$ and each A is nwd in $G \cap X$. Pick $x_0, x_1, \cdots \in G \cap X$ such that

(i) $x_m \in A$ and $m < n$ imply $x_n \notin A$, and

(ii) each $x_m \in \mathrm{cl}\{x_n : m < n\}$.

Using the 1-modesty of D take $C \subset \{x_n : n < \omega\}$. C is very P-bad which contradicts the conditions of the lemma. □

Select $D^0 \subset D$ and suppose that $\{A \cap D^0 : A \in P\}$ is disjoint. For each Y let $Y^0 = D^0 \cap Y$.

LEMMA 3.4. *Suppose that C^0 has no very P-bad subsets and is dense in C. If C is P-bad, then it is W-bad with one of the witnesses in D^0.*

PROOF. Let g, h range over the non-empty open subsets of C. By virtue of Lemma 1.4 it is enough to prove that for each g there exists $h \subset g$ which is W-bad with one of the witnesses in D^0. Select g. By Lemma 3.3 there exist $h \subset g$ and A which is dense in h^0. Let $Y = h^0 \cap A$ and $Z = h - A$. Y and Z witness that h is W-bad. Because, let $C_1 \subset Y \cup Z$ and Y, Z be dense in C_1. Using the 2-modesty of D take a countable $C_2 \subset C_1$ with Y, Z dense in C_2. If C_1 is W-good, then C_2 is W-good. According to Definition 2.3, C_2 is not P-bad. But C_2 is P-bad. □

THEOREM 3.5. *The following statements are equivalent*:

(i) *X has no W-bad subset with one of the witnesses in D^0*,

(ii) *For every countable $C \subset X$, if C^0 is dense in C, then $\bigcup_A \mathrm{do}(C \subset A)$ is dense in C, and*

(iii) *For each perfect $E \subset U$, if X^0 is dense in $E \cap X$ then $E \models (\bigcup_A \mathrm{do}(E \cap X \subset A \cap E)$ is ewd).*

PROOF. (i)$\rightarrow$(iii). Without loss of generality $E = U$. For reduction to absurdity let $G \subset U - \bigcup_A \mathrm{do}(X \subset A)$. Then X^0 is dense in G. By Lemma 3.3 there exist $G' \subset G$ and A such that A is dense in $G' \cap X^0 \neq 0$. Let $Y = G' \cap X^0 \cap A$ and $Z = G' \cap X - A$. Using the 2-modesty of D take $C \subset Y \cup Z$ such that Y and Z are dense in C. Now use Lemma 3.4.

That (iii)$\rightarrow$(ii) is clear.

(ii)$\rightarrow$(i). Let $X_1 \subset X$ be W-bad with witnesses $Y \subset D^0$ and Z. Take a countable $C \subset X_1$ such that Y and Z are dense in C. By Lemma 3.2 C is P-bad, hence $C \cap \mathrm{do}(C \subset A) = 0$ for every A. $\square$

Let $\mathrm{St}'(X, D, D^0, W)$ be a topological formula saying that $X \subset D$ and

(3.1) X has no W-bad subset, and

(3.2) If $Y \subset X^0$, $Z \subset D - X$, and Y, Z are dense in $Y + Z$, then Y and Z witness that $Y + Z$ is W-bad.

LEMMA 3.6. *Each $A \in P$ satisfies* St$'$.

PROOF. By virtue of Lemma 3.2 A satisfies (3.1). If $Y \subset A^0$, $Z \subset D - A$, and Y, Z are dense in $Y + Z$, take $C \subset Y + Z$ with Y and Z dense in C. By Lemma 3.4, C is W-bad. $\square$

THEOREM 3.7. (cf. lemma 7.7 in [12]). *Suppose that X satisfies* St$'$, *$E \subset U$ is perfect, and X^0 is dense in E. Then $E \models (\Sigma_A \mathrm{do}(E \cap X = A \cap E)$ is ewd).*

PROOF. Without loss of generality $E = U$. For reduction to absurdity let $G \subset U - \bigcup_A \mathrm{do}(X = A)$. By Theorem 3.5, there exist A and $G' \subset G \cap \mathrm{do}(X^0 \subset A)$. Let $Y = G' \cap X^0$ and $Z = G' \cap A - X$. By (3.2), $Y + Z$ is W-bad. On the other hand, $Y + Z$ is included in A which has no W-bad subset. $\square$

Let $\mathrm{St}(X, D, D^0, W)$ say that $X \subset D$, and for each G there exists H such that $\mathrm{St}'(H \cap X, D \cap H, D^0 \cap H, H \cap W)$ holds in H, and $D \subset \mathrm{cl}\, X^0$. Let D be ewd.

THEOREM 3.8. *X satisfies* St *iff $X \subset D$ and $\Sigma \mathrm{do}(X = A)$ is ewd.*

PROOF is clear. $\square$

§4. Towers

Here we present a top. formula defining *towers* of point sets.

Recall that $X < Y$ iff $X - Y$ is nwd and $Y - X$ is ewd. Correspondingly $X < Y$ in G iff $X - Y$ is nwd in G and $Y - X$ is dense in G.

Definition 4.1. A quadruple $T = (D, D^0, D^1, W)$ is a *tower* iff it satisfies conditions (4.1)–(4.3) below where A, B range over $[T] = \{X\colon \mathrm{St}(X, D, D^0, W)\}$, $A^\varepsilon = A \cap D^\varepsilon$ and $B^\varepsilon = B \cap D^\varepsilon$.

(4.1) D is 2-modest, $D^0 + D^1 \subset D$, D^0 and D^1 are ewd, and $W \subset U - D$.

(4.2) $\mathrm{do}(A^0 = B^0) + \mathrm{do}(A^0 \cap B^0 = 0)$ is ewd, $\mathrm{do}(A^1 \subset B^1) \cup \mathrm{do}(B^1 \subset A^1)$ is ewd, and $\mathrm{do}(A^0 = B^0) \approx \mathrm{do}(A^1 = B^1) \approx \mathrm{do}(A = B)$.

(4.3) There exist no G and $P \subset [T]$ such that $P \neq 0$ and for each $A \in P$ there exists $B \in P$ such that $B^1 < A^1$ in G.

Lemma 4.1. *Suppose that* (4.1) *and* (4.2) *hold. Then* (4.3) *is equivalent to*

(4.3′) *There exist no G and W' such that*

$$P' = \{X\colon G \models \mathrm{St}(X, D \cap G, D^0 \cap G, G \cap W')\} \neq 0$$

and for each $X \in P'$ there exist $A \in [T]$ and $Y \in P'$ such that $A \cap G \approx X$ and $Y^1 < X^1$ in G.

Proof. Suppose $\neg(4.3)$. Pick $A_0, A_1, \cdots \in P$ such that $A_0^1 > A_1^1 > \cdots$ in G. By (4.2), $m < n$ implies $A_m^0 \cap A_n^0 \cap G \approx 0$. Let $X_n = (A_n^0 - \bigcup_{m<n} A_m^0) + A_n^1$ and W' be a wonder set for $\{X_n \cap G\colon n \in \omega\}$. $\neg(4.3')$ is proved.

Suppose $\neg(4.3')$. Pick $X_0, X_1, \cdots \in P'$ and $A_0, A_1, \cdots \in [T]$ such that for each n, $A_n^1 > A_{n+1}^1$ in G and $A_n \cap G \approx X_n$. Let $P = \{A_n\colon n \in \omega\}$. $\neg(4.3)$ is proved. □

Corollary 4.2. *The predicate "(X_1, X_2, X_3, X_4) is a tower" is expressible by a top. formula.*

If (X_1, X_2, X_3, X_4) is a tower, then the elements of $\{X\colon \mathrm{St}(X, X_1, X_2, X_4)\}$ will be called *storeys* of this tower. In what follows $T = (D, D^0, D^1, W)$ is a tower, A and B range over the storeys of T, and $X^\varepsilon = D^\varepsilon \cap X$ if $X \subset D$ and $\varepsilon = 0, 1$.

Lemma 4.3. *Suppose that* $\cup\{G_\iota\colon \iota \in I\}$ *is ewd. Given a family* $\{A_\iota\colon \iota \in I\}$ *of storeys one can find a disjoint family* $\{H_\iota\colon \iota \in I\}$ *and a storey B such that* $H_\iota \subset G_\iota$, ΣH_ι *is ewd and* $B = A_\iota$ *in* H_ι.

Proof. Use Lemma 1.4 to find appropriate H_ι's. Build $B = \Sigma(A_\iota \cap H_\iota)$. □

LEMMA 4.4. *If P_0 is a family of storeys and $P_1 = \{A : B \in P_0 \rightarrow B^1 < A^1\} \neq 0$, then P_1 has a minimal (according to $\subsetneq$) element.*

PROOF. Suppose the contrary. By virtue of Lemma 4.3 there exists G such that for any $A \in P_1$ and $G' \subset G$, A is not minimal in G'. By virtue of (4.3) it is enough to prove that for each $A \in P_1$ there exists $B \in P_1$ such that $B^1 < A^1$ in G. Let $A \in P_1$. A is not minimal in any $G' \subset G$. Hence for each $G' \subset G$ there exists B and $G'' \subset G'$ such that $B^1 < A^1$ in G''. Now use Lemma 4.3. □

THEOREM 4.5. *There exist an ordinal $h(T)$ (the* height *of T), a family $\{D_\alpha : \alpha < h(T)\}$ of T-storeys (a* skeleton *of T), and a non-empty open set* do(T) *(the* domain *of T) such that*

(4.4) *$\alpha < \beta < h(T)$ implies $D^1_\alpha < D^1_\beta$,*

(4.5) *For each A, $\Sigma\{\mathrm{do}(A = D_\alpha) : \alpha < h(T)\}$ is dense in* do(T), *and*

(4.6) *If $\Sigma\,\mathrm{do}(A = D_\alpha)$ is dense in G for each A, then $G \subset \mathrm{do}(T)$.*

NOTE. (4.5) is equivalent to

(4.5′) For every A and $G \subset \mathrm{do}(T)$ there exist $G' \subset G$ and α such that $A \approx D_\alpha$ in G'.

PROOF OF THEOREM 4.5. If $\{D_\beta : \beta < \alpha\}$ is already built let $P_\alpha = \{A : \text{for each } \beta < \alpha,\ D^1_\beta < A^1 \text{ in } U\}$. If $P_\alpha \neq 0$ let D_α be a minimal element in P_α. If $P_\alpha = 0$ end the process and let $h(T) = \alpha$. Let $P(G) = \{A : \text{for each } \alpha,\ D^1_\alpha < A^1 \text{ on } G\}$, $G_0 = \cup\{G : P(G) \neq 0\}$, and $\mathrm{do}\,T = U - \mathrm{cl}\,G_0$. By Lemma 4.3, $P(G_0) \neq 0$, hence $\mathrm{do}\,T \neq 0$. (4.4) and (4.6) are clear. We check (4.5′). Choose A and $G \subset \mathrm{do}\,T$. $P(G) = 0$ and (4.2) imply that there exist α and $G' \subset G$ such that $A^1 - D^1_\alpha$ is nwd in G'. If α is the least possible such that this condition holds, then $D^1_\alpha - A^1$ is nwd in G' by the construction of D_α. Hence $A^1 \cap G' \approx D^1_\alpha \cap G'$ and by (4.2), $A \cap G' \approx D_\alpha \cap G'$. □

§5. More about towers

Here we prove some auxiliary results which will be used later on.

If U is separable, then there exists a countable family $\{G_n : n \in \omega\}$ such that each G includes some G_n. In this case U is Suslin (i.e. each family of disjoint open subsets of U is at most countable) and for each ewd $X \subset U$ there exists a countable ewd $Y \subset X$. A separable chain is embeddable in the real line, hence it is second countable.

LEMMA 5.1. *If U is separable, then each tower is of height $< c^+$.*

PROOF. Let $\{D_\alpha: \alpha < h(T)\}$ be a skeleton of tower $T = (D, D^0, D^1, W)$. For each $\alpha < h(T)$ choose an ewd countable $X_\alpha \subset D^0 \cap D_\alpha$. By (4.2), $X_\alpha \neq X_\beta$ if $\alpha \neq \beta$. So $|h(T)| \leqq c^{\aleph_0} = c$. □

DEFINITION 5.1. A tower T is *exponential on* E iff for each $X \subset E$ there exists a T-storey A such that $A \cap E \approx X$.

LEMMA 5.2. *Let $T = (D, D^0, D^1, W)$ be a tower. Suppose that T is exponential on D^2 and for each G, $|D^2 \cap G| \geqq \kappa \geqq \aleph_0$. Then $h(T) \geqq \exp \kappa$.*

PROOF. WLOG $\mathrm{do}(T) = U$ and $|D^2| = \kappa$. Because, there exist λ and $G \subset \mathrm{do}\, T$ such that for each $G' \subset G$, $|D^2 \cap G'| = \lambda$, and we can work in G.

Let $\{D_\alpha: \alpha < h(T)\}$ be a skeleton of T. By Lemma 1.5, $D^2 = \Sigma\{X_\beta: \beta < \kappa\}$ for some ewd X_β's. Let $S = \{(\alpha, I): \alpha < h(T)$ and $I \subset \kappa$ and there exists G such that $D_\alpha = \Sigma\{X_\beta: \beta \in I\}$ in $G\}$. For each I there exists α such that $(\alpha, I) \in S$. Hence $|S| \geqq \exp \kappa$. Each disjoint family of open subsets of U is of cardinality $\leqq \kappa$. Therefore, for each α there are at most κ different I's such that $(\alpha, I) \in S$. Hence $\kappa \cdot |h(T)| \geqq |S| \geqq \exp \kappa$ and $h(T) \geqq \exp \kappa$. □

LEMMA 5.3. *Suppose that ΣH_ι is ewd. For each i let $T_\iota = (D_\iota, D^0_\iota, D^1_\iota, W_\iota)$ be a tower in the subspace H_ι. Then $T = (\Sigma D_\iota, \Sigma D^0_\iota, \Sigma D^1_\iota, \Sigma W_\iota)$ is a tower in U. If, for each i, T_ι is exponential on $E_\iota \subset D_\iota$, then T is exponential on ΣE_ι.*

PROOF. Lemma 5.3 is obvious. □

LEMMA 5.4. *Suppose that D is 2-modest and for each G, $|D \cap G| \geqq \lambda \geqq \aleph_0$. Then there exists a tower $T = (D, D^0, D^1, W)$ with skeleton $\{D_\alpha: \alpha < h(T)\}$ such that $h(T) \geqq \lambda$ and $\{D^0 \cap D_\alpha: \alpha < h(T)\}$ is disjoint. Moreover, if $E \subset D$, $\lambda \geqq 2^\kappa$ and $\forall G\ \exists G'$ ($G' \subset G$ and $|E \cap G'| \leqq \kappa$), then T can be chosen in such a way that $D^0 + D^1$ is disjoint from E and T is exponential on E.*

PROOF. By virtue of Lemmas 1.4 and 5.3 we can suppose that for each G, $|D \cap G| = \lambda$ and $|E \cap G| = \kappa$. By Lemma 1.5, $D - E$ can be partitioned into λ disjoint ewd parts. Let $\{X_\alpha: \alpha < \lambda\} + \{Y_\alpha: \alpha < \lambda\}$ be the family of these parts. Let $\mathrm{PS}(E) = \{Z_\alpha: \alpha < \lambda\}$. Build $D_\alpha = X_\alpha + \Sigma_{\beta < \alpha} Y_\beta + Z_\alpha$. By Theorem 2.6 there exists a wonder set W for $\{D_\alpha: \alpha < \lambda\}$. $T = (D, \Sigma X_\alpha, \Sigma Y_\alpha, W)$ is the desired tower. □

Recall that $X \subsetneq Y$ iff $X - Y$ is nwd. Correspondingly, $X \subsetneq Y$ in G iff $X - Y$ is nwd in G. Let $T = (D, D^0, D^1, W)$ be a tower with skeleton $\{D_\alpha: \alpha < h(T)\}$, and $X^\epsilon = D^\epsilon \cap X$.

DEFINITION 5.2. T is *stable* iff for each tower $T' = (E, E^*, E', W')$ with

$D^0 \subset E$ there exists $X \subset D^0$ such that for each T-storey A and each G, $A^0 \subsetneq X$ in G iff there exists T'-storey B with $A^0 \approx B^0$ in G.

LEMMA 5.5. *If* $\mathrm{do}(T) = U$ *and* $\{D^0_\alpha : \alpha < h(T)\}$ *is disjoint, then* T *is stable.*

PROOF. Take $X = \cup\{D^0_\alpha \cap G$: there exists a T'-storey B with $D^0_\alpha \cap G = B^0 \cap G\}$. $\square$

LEMMA 5.6. *Suppose that* T *is stable and* $|D| < c$. *Let* $I \subset h(T)$ *be of cardinality* $\leqq c$. *Then there exists* $X \subset D^0$ *such that for every* G *and* $\alpha < h(T)$, (i) $\alpha \in I$ *implies* $D^0_\alpha \subsetneq X$, *and* (ii) $D^0_\alpha \subsetneq X$ *in* G *implies* $\alpha \in I$.

PROOF. By Lemma 2.5, $U - D$ is hereditary of the third category. By Lemmas 2.2 and 2.3 there exists a 2-modest $E^* \subset U - D$ such that for each G, $|G \cap E^*| = c$. By Lemma 1.5, $E^* = \Sigma_{\alpha<c} E^*_\alpha$ for some ewd E^*_α's. Let $E = D + E^*$, $E_\alpha = D_\alpha + E^*_\alpha$, and W' be a wonder set for $\{E_\alpha : \alpha \in I\}$. Clearly, $T' = (E, E^*, D^1, W')$ is a tower. Let X be as in Definition 5.2. If $\alpha \in I$, then E_α is a storey of T'. Hence $D^0_\alpha \subsetneq X$. If $D^0_\alpha \subsetneq X$ in G, then $D^0_\alpha \approx B^0$ in G for some storey B of T'. By Theorem 4.5 there exists $\beta \in I$ such that $G \cap \mathrm{do}(B = E_\beta) \neq 0$. Clearly, $\alpha = \beta \in I$. $\square$

COROLLARY 5.7. *If* T *is stable on* D^2, *then* $\exp|D| \geqq |h(T)|^c$.

LEMMA 5.8. *Suppose that* U *is separable. Then for each* n, *the predicate* "$h(T) = \omega_n$" *is expressible by a top. formula.*

PROOF is easy.

It is not difficult to express "$h(T) = \omega_n$", "$h(T) = c$" and many other properties.

§6. Countable sets and sets of cardinality c

Suppose that U is separable. Recall that $\exp \kappa = 2^\kappa$.

LEMMA 6.1. *The following statements are equivalent*:

(6.1) $\forall G\ \exists G'$ $(G' \subset G$ *and* $\exp|G' \cap X| \leqq c)$, *and*

(6.2) *There exist a tower* $T = (D, D^0, D^1, W)$ *and* $D^2 \subset D - (D^0 + D^1)$ *such that* $D^2 \approx X$ *and* T *is exponential on* D^2.

PROOF. Suppose that (6.1) holds. WLOG, there exists κ such that for each G, $|G \cap X| = \kappa$. Because, let $G_\kappa = \cup\{G$: for each $G' \subset G$, $|G' \cap X| = \kappa\}$. Then ΣG_κ is ewd. Now use Lemma 5.3.

By Lemma 2.3 there exists $E \subset U - X$ such that for each G, $|E \cap G| = c$, and each separable Cantor subset of E is countable. Let $D = E + X$. By Lemma 2.2, D is 2-modest. Now use Lemma 5.4.

Suppose that (6.2) holds. For reduction to absurdity suppose that there exists G such that for each $G' \subset G$, $\exp|G' \cap X| > c$. Then for each $G' \subset G$, $\exp|G' \cap D^2| > c$. By Lemma 5.2 it follows that $|h(T)| > c$ which contradicts Lemma 5.1. $\square$

THEOREM 6.2. *The predicate* "$\exp|X| \leqq c$" *is expressible by a top. formula.*

PROOF. Say X is small iff it satisfies (6.1). According to Lemma 6.1, the predicate "X is small" is expressible by a top. formula. Therefore, it is enough to prove that $\exp|X| \leqq c$ is equivalent to:

(6.3) For each perfect set E, $E \cap X$ is small in E.

Clearly $\exp|X| \leqq c$ implies (6.3). Suppose that $\exp|X| > c$. Let $G = \cup\{G'$: $\exp|G' \cap X| \leqq c\}$ and $E = U - G$. Clearly $E \neq 0$ and $E \cap X$ is not small in E. $\square$

COROLLARY 6.3 (CH). *The predicate "X is at most countable" is expressible by a top. formula.*

LEMMA 6.4. *Assume that* $\forall \kappa(\kappa < c \rightarrow \exp \kappa < \exp c)$. *The following statements are equivalent*:

(6.4) *For each G, $|G \cap X| = c$, and*

(6.5) *If $G \cap X$ is 2-modest then there exists a tower $T = (D, D^0, D^1, W)$ in G such that $D \subset X$, and T is stable and exponential on some Y which is dense in G.*

PROOF. (6.5) implies (6.4) by Lemma 2.2, Lemma 5.2 and Corollary 5.7. (6.4) implies (6.5) by Lemmas 5.4 and 5.5. $\square$

THEOREM 6.5. *Assume that* $\forall \kappa(\kappa < c \rightarrow \exp \kappa < \exp c)$. *Then the predicate* $|X| = c$ *is expressible by a top. formula.*

PROOF. Let us say that X is *everywhere big* iff it satisfies (6.4). By Lemma 6.4, the predicate "X is everywhere big" is expressible by a top. formula. But $|X| = c$ iff there exists a perfect E such that $E \cap X$ is everywhere big in E. $\square$

Let M be the standard Cohen model for blowing c to ω_3. In M, $\exp \aleph_0 = \exp \aleph_2 = \aleph_3$. At our request Menachem Magidor verified the following phenomena in M:

(i) The real line R is of the third category; and

(ii) If $\langle X_\alpha \colon \alpha < \alpha^* \rangle$ is a sequence of sets of rational numbers and $\alpha < \beta \rightarrow X_\alpha < X_\beta$, then $\alpha^* < \aleph_2$.

The following note is due to Saharon Shelah. Let $\varphi(X)$ state that there exist a perfect E and a tower $T = (D, D^0, D^1, W)$ in E such that $D \subset X$ and $h(T) = \omega_2$. This formula discriminates between countable subsets and subsets of cardinality $\aleph_2$ of R.

§7. Meager sets

LEMMA 7.1. *Assume that c is regular. Then the following statements are equivalent:*

(7.1) *For each G, $G \cap X$ is of the third category (in U), and*

(7.2) *There exists $D \subset X$ such that for each G, $|D \cap G| = c$, and each Cantor subset of D is of cardinality $< c$.*

PROOF. By Lemma 2.3, (7.1) implies (7.2). The other implication is obvious. □

Let us say that X is *everywhere of the third category* iff it satisfies (7.1).

LEMMA 7.2. *Suppose that U is separable. Then X is of the third category iff there exists G such that $G \cap X$ is everywhere of the third category in G.*

PROOF. Suppose that for each G there exists $G' \subset G$ such that $G' \cap X$ is pseudo-meager. By Lemma 1.4, there exists a disjoint family $\{H_i : i \in I\}$ such that ΣH_i is ewd and for each i there exists $\kappa_i < c$ such that $H_i \cap X$ is a union of κ_i nwd sets, WLOG, $I = \omega$, since U is separable. Let $\kappa = \kappa_0 + \kappa_1 + \cdots$. $\kappa < c$ since $\mathrm{cf}(c) > \omega$. X is a union of κ nwd sets, hence X is pseudo-meager.

The other implication is clear. □

THEOREM 7.3. *Assume*

(7.3) *c is regular, and $\exp\kappa < \exp c$ for each $\kappa < c$.*

If U is separable then the predicate "X is of the third category" is expressible by a top. formula in U.

PROOF. See Lemmas 6.5, 7.1 and 7.2. □

LEMMA 7.4. *Lemma 7.2 remains true if "U is separable" is replaced by*

(7.4) *There exists $\kappa < c$ such that each pseudo-meager subset of U is a union of $\leqq \kappa$ nwd sets.*

PROOF is clear. □

(7.4) holds if c is a successor.

Suppose that U is quasi-separable; i.e. for each G there exists a separable $G' \subset G$.

THEOREM 7.5. *Suppose* (7.3) *and* (7.4). *Then the predicate "X is of the third category" is expressible by a top. formula in U.*

PROOF. By Lemma 6.5, there exists a top. formula $F(G, X)$ stating $|G \cap X| = c$ in case G is separable. By Lemma 7.1, there exists a top. formula $F'(G, X)$ stating that $G \cap X$ is everywhere of the third category in G in case G is separable. By Lemma 7.4, X is of the third category iff $\exists G \forall G'(G' \subset G \rightarrow F'(G', X))$. $\square$

COROLLARY 7.6 (CH). *The predicate "X is meager" is expressible by a top. formula in U.*

THEOREM 7.7. *Assume Martin's Axiom. Then the predicate "X is meager" is expressible by a top. formula in U.*

PROOF. By virtue of Martin's Axiom, if $\kappa < c$ then $\exp \kappa \leqq c$ (see [10]). Hence the assumption in Theorem 6.5 holds.

Further, each pseudo-meager subset of U is meager. By virtue of Lemma 1.4 it is enough to check this statement in the case where U is separable. In this case U has a countable quasi-basis; i.e. there exists $\{G_n : n \in \omega\}$ such that each G includes some G_n. Martin and Solovay proved in [10] that each pseudo-meager subset of the real line is meager. Their proof is valid for each top. space with a countable quasi-basis.

Hence Lemma 7.1 remains true if the supposition "c is regular" is omitted. Now use the proof of Theorem 7.5. $\square$

§8. Inexpressible properties

Let CD be Cantor's Discontinuum (as a top. space). For each $\kappa > 0$ let CD $\times \kappa$ be a top. space which splits into κ disjoint sets which are closed, open, and homeomorphic with CD. If κ is infinite let CD $\times \kappa + 1$ be a one point compactification of CD $\times \kappa$.

THEOREM 8.1. *For each $\kappa > 0$,* CD $\times \kappa$ *has the same monadic theory as* CD. *For each infinite κ,* CD $\times \kappa + 1$ *has the same monadic theory as* CD.

PROOF. Note that CD $\times n$ and CD $\times \aleph_0 + 1$ are homeomorphic with CD. Now use Ehrenfeucht's Game Criterion. $\square$

COROLLARY 8.2. *Let K be the class of top. spaces which are dense, regular, locally compact, and second countable. The proposition "The space is compact" and the predicate "X is finite" are not expressible by top. formulae in K.*

COROLLARY 8.3. *Let K be the class of dense and regular top. spaces which are either*

(i) *locally compact and locally second countable (i.e. for every G and $p \in G$ there exists a second countable G' such that $p \in G' \subset G$), or*

(ii) *compact and quasi-second countable (i.e. for each G there exists a second countable $G' \subset G$).*

Then for any infinite κ neither "$|X| = \kappa$" nor "$|X| \leqq \kappa$" is expressible by a top. formula in K.

§9. Immodesty

THEOREM 9.1 (CH). *For each n, the true arithmetic is interpretable in the monadic theory of the class of regular first countable top. spaces of cardinality c which are not n-modest.*

PROOF. Let U be a regular first countable top. space of cardinality c which is not n-modest. Then there exists $D \subset U$ and $X_1, \cdots, X_n \subset D$ such that D is dense, and each X_i is dense in D and there exists no $C \in \mathrm{Ca}(D)$ such that each X_i is dense in C. WLOG, D is countable (use first countability) and ewd (we will work in $\bar{D}$ only).

For each perfect E, if $X_1, \cdots, X_n$ are dense in E then $E - D$ is of the second category in E.

For, suppose that $E - D$ is meager in E. Then, by Theorem 1.1, there exists $C \in \mathrm{Ca}(E)$ which is disjoint from $E - X$ (i.e. $C \subset D$) and each X_i is dense in C.

Let $S = \{C\colon C \in \mathrm{SCa}(D)$ and $X_1, \cdots, X_n$ are dense in $C\}$. Following the proof of Lemma 2.4 it is easy to check that for each $P \subset \mathrm{PS}(D)$ there exists $W \subset U - D$ such that for each $C \in S$

(9.1) $|\bar{C} \cap W| < c$ if $\exists A (C \subset A \in P)$, and

(9.2) $\bar{C} \cap W \neq 0$ if there exist no G and $A \in P$ such that $0 \neq C \cap G \subset A$.

The rest of the proof of Theorem 9.1 parallels the proof of theorem 7.10 in [12]. A more detailed proof will be presented in [5].

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