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MODEST THEORY OF SHORT CHAINS. II

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Abstract. We analyse here the monadic theory of the rational order, the monadic theory of the real line with quantification over "small" subsets and models of these theories. We prove that the results are in some sense the best possible.

§0. Introduction. A *chain* is a linearly ordered set. A chain is *short* iff it embeds neither ω_1 nor ω_1^*. Shortness is expressible in the monadic language of order. For convenience, we shall use the term "chain" to mean "short chain".

Modest chains are defined in §2. A chain is modest iff it is *p-modest* for every positive integer p; p-modesty is expressible in the monadic theory of order.

Let R be the real line. Set $X \subset R$ will be called pseudo-meager iff it is a union of less than c (the cardinality of continuum) nowhere dense subsets. It is well known that the hypothesis "R is not pseudo-meager" can be neither proved nor disproved in ZFC. The Continuum Hypothesis (or Martin's Axiom, see [MS]) implies that each pseudo-meager set is meager. By the Baire Theorem, R is not meager.

Let Q be the chain of rational numbers. The monadic theory of Q is decidable (see [Ra]) but not categorical. By [Sh], there exist nonseparable chains monadically equivalent to Q (i.e. having the same monadic theory as Q) and if R is not pseudo-meager then there are subchains of R of cardinality c monadically equivalent to Q.

THEOREM 1 (SEE §3). *A chain is monadically equivalent to Q iff it is modest and has neither jumps nor endpoints. Moreover there exists an algorithm associating a pair (p, ε) with each sentence φ in the monadic language of order in such a way that either $\varepsilon = 0$ and all p-modest chains without jumps and endpoints satisfy φ or $\varepsilon = 1$ and all these chains do not satisfy φ.*

Each modest chain is embeddable into a modest chain without jumps and endpoints.

COROLLARY 2. *The monadic theory of the class of modest chains is decidable.*

THEOREM 3 (SEE §2). *Assume that R is not pseudo-meager. For each p there exists an everywhere dense subset of R forming a chain which is q-modest for each $q < p$ but is not p-modest. Hence the monadic theory of Q is not finitely axiomatizable in the monadic logic.*

THEOREM 4 (SEE §3). *Assume that each pseudo-meager subset of R is meager. There exists an algorithm associating a sentence φ_p in the monadic language of order with every p and first-order arithmetical sentence φ in such a way that for each chain M which is not p-modest, φ holds in the standard arithmetic iff φ_p holds in M.*

Corollary 2 and Theorem 4 form a dichotomy.

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In §4 we define subsets *modest in* a chain. (A subset X is modest in a chain M iff it is p-modest in M for every positive integer p; p-modesty in M is expressible in the monadic theory of M.) If $X \subset M$ is modest in M then X forms a modest subchain; the converse may be wrong. Subsets modest in a chain M form an ideal closed under countable unions.

Let K_p be the collection of pairs (M, S) where M is a complete chain without jumps and endpoints having an everywhere dense subset p-modest in M and S is a collection of subsets of M p-modest in M and satisfying some natural conditions (conditions (1)—(3) in §5 below). Let T_p be the set of sentences φ in the monadic language of order such that for each $(M, S) \in K_p$, φ belongs to the theory of M with quantification over S.

THEOREM 5 (SEE §5). *There exists an algorithm associating a pair* (p, ε) *with each sentence* φ *in the monadic language of order in such a way that either* $\varepsilon = 0$ *and* φ *belongs to* T_p *or* $\varepsilon = 1$ *and* $\neg\varphi$ *belongs to* T_p.

The *modest theory* of a chain M is the theory of M with quantification over subsets modest in M. Chain N is *absolutely modest* iff it is modest in its completion. Subset X of a complete chain M is modest in M iff it forms an absolutely modest subchain.

COROLLARY 6. *The modest theory of the real line R is decidable. The following theories coincide with the modest theory of R: the modest theory of an arbitrary complete chain without jumps and endpoints having an everywhere dense absolutely modest subset, the theory of R with quantification over countable subsets, the theory of R with quantification over subsets of cardinalities* $< c$.

Note that decidability of the theory of R with quantification over countable subsets follows from [Ra].

The *augmented modest theory* of a chain M is the collection of monadic formulas $F(V)$ such that for each subset X of M the statement $F(X)$ holds in M when the bound set variables of $F(X)$ range over subsets modest in M.

THEOREM 7 (SEE §5). *Assume that each pseudo-meager subset of R is meager. The augmented modest theory of any complete chain without jumps is undecidable.*

Note also that the condition "having an everywhere dense absolutely modest subset" may not be omitted in Corollary 6. There exist complete chains without jumps and endpoints whose modest theories are undecidable.

In §6 we give some phenomena and counterexamples concerning modest subsets of R.

Some results of this paper were announced in [Gu 1] and [GS]. Paper [Gu 2] will be referred to as Part 1. We use here its terminology and results For completeness and reader's convenience we expound here some results of [Gu 1] and [Sh].

§1. Cantor subsets. Let M be a chain. A point $x \in M$ is a *left* (*right*) *limit point* of $Y \subset M$ iff there exists $Z \subset Y$ such that Z is not empty and $x = \inf Z$ ($x = \sup Z$). x is a limit point of Y iff x is either a left or a right limit point of Y. This coincides with the definition of limit point in terms of the *interval topology* on M for which the open intervals form a base. Every chain is regarded to be equipped with the interval topology. Note that the interval topology of a subchain may be weaker than the inherited one.

We say that $X \subset M$ is *dense in* Y iff the closure of $X \cap Y$ includes Y. Thus, X is everywhere dense iff it is dense in M, and X is nowhere dense iff it is not dense in any nonempty open set. The set of limit points of Y is denoted by der(Y). Y is *Cantor* iff it is perfect (i.e. $0 \neq Y = \mathrm{der}(Y)$) and nowhere dense. The collection of Cantor subsets of M will be denoted by Ca(M). The set of two way (left and right) limit points of Y is denoted by Y^0. The cardinality of the continuum is denoted by c. Note that the real line has c perfect subsets and c Cantor subsets and 2^c nowhere dense subsets. It is easy to see that the relations "X is Cantor" and "$X = Y^0$" are expressible in the monadic language of order.

Let us note that each (short) chain is of cardinality $\leq c$ (use the Erdös-Rado generalization (see [ER]) of Ramsey's Theorem).

THEOREM 1.1. *Let*

M be a chain,

$X_0, X_1, \ldots$ be everywhere dense subsets of M,

$X = \bigcup X_n \subset X^0$, and

$A \subset M - X$ be meager in M.

Then there exist a countable $B \subset X$ and a family $S \subset$ Ca(M) of cardinality c such that $B \subset B^0$ and every X_n is dense in B and for each $C \in S$:

(a) *$B \cap C \subset C^0$ and each $B \cap X_n$ is dense in C,*

(b) *$C \cap C_1 \subset B$ for each $C \neq C_1 \in S$, and*

(c) *C is disjoint from A.*

PROOF. Let $A = \bigcup A_n$ where each A_n is nowhere dense in M. Let

$$f : \{0, 1\} \times \{0, 1\} \times \omega \times \omega \to \omega$$

be one-one and onto and $(\alpha n, \beta n, \gamma n, \delta n) = f^{-1}(n)$. Let s and t range over the set of finite sequences of natural numbers, lh(s) be the length of s. s will be regarded as a function from lh(s) to ω. $t = s\hat{\ }n$ means that t extends s by $t(\mathrm{lh}\ s) = n$.

LEMMA 1.2. *There exist intervals $I(s)$ and points $x(s)$ such that*

(i) *$\bar{I}(s) \subset M - \bar{A}_{\mathrm{lh}\, s}$, $\bar{I}(s\hat{\ }n) \subset I(s)$, and $\bar{I}(s\hat{\ }m) \cap \bar{I}(s\hat{\ }n) = 0$ if $m \neq n$;*

(ii) *$x(s) \in I(s)$, $x(s\hat{\ }n) \in X_{\gamma n}$, $\lim x(s\hat{\ }n) = x(s)$, $x(s\hat{\ }n) < x(s)$ if $\alpha n = 0$, otherwise $x(s\hat{\ }n) > x(s)$;*

(iii) *If $x = \lim x(s_n)$ then either $x = x(t)$ for some t or there exists a strictly increasing sequence $t_0 \subset t_1 \subset \cdots$ such that $x \in \bigcap \bar{I}(t_n)$.*

PROOF OF LEMMA 1.2. Take $\bar{I}(0) \subset M - \bar{A}_0$ and $x \in I(0) \cap X_{\gamma 0}$. Suppose that $I(t)$ and $x(t)$ are chosen for every t with $\mathrm{lh}(t) \leq l$ and that the relevant cases of (i) and (ii) hold. Let $\mathrm{lh}(s) = l$ and $G = I(s) - \bar{A}_{l+1}$. Select $y_0 < y_1 < \cdots$ and $z_0 > z_1 > \cdots$ in such a way that $y_n, z_n \in G$ and $\lim y_n = \lim z_n = x(s)$. If $\alpha n = 0$ ($\alpha n = 1$) choose $I(s\hat{\ }n)$ between y_n and y_{n+1} (between z_n and z_{n+1}) in such a way that $\bar{I}(s\hat{\ }n) \subset G$. Pick $x(s\hat{\ }n) \in I(s\hat{\ }n)$.

We have only to check (iii). Let $x = \lim x(s_n)$.

Let $N_i = \{s_n | (i+1) : i < \mathrm{lh}(s_n)\}$. If there exists infinite N_i take the minimal i with infinite N_i. Select a subsequence $s_{n_0}, s_{n_1}, \ldots$ such that $s_{n_0} | i = s_{n_1} | i = \cdots$ and $s_{n_0}(i) < s_{n_1}(i) < \cdots$. Then $\lim x(s_{n_k}) = x(s_{n_0} \mid i) \in X$.

Otherwise by Koenig's Lemma there exists a sequence $t_0 \subset t_1 \subset \cdots$ such that $t_i \in N_i$. Then $x = \lim x(t_i) \in \bigcap \bar{I}(t_i)$. Lemma 1.2 is proved.

We continue the proof of Theorem 1.1. Let $I(S)$, $x(s)$ be as in Lemma 1.2, B be the set of points $x(s)$ and $S = \{\bar{C}_g : g \in {}^{\omega}2\}$ where $C_g = \{x(s) : \text{for each } k < \mathrm{lh}(s), \beta(s(k)) = g(k)\}$. Note that $\lim_{j\to\infty} x(s^\frown f(\varepsilon, g(\mathrm{lh}(s)), i, j) = x(s)$ hence $C_g \subset C_g^0$ and X_i is dense in C_g.

C_g is nowhere dense. For if $x(s)$ belong to an interval I then there exists n such that $I(s^\frown n)$ is disjoint from C_g and intersects I.

Clearly each $\bar{C}_g$ is disjoint from A.

Let us check that $\bar{C}_g \cap \bar{C}_h \subset B$ if $gm \neq hm$. For reduction and absurdum let $x = \lim(s_n) = \lim(t_n) \notin B$ where $x(s_n) \in C_g$ and $x(t_n) \in C_h$. Wlog, sequences s_0, s_1, ... and t_0, t_1, ... are strictly increasing. Then $x \in \bar{I}(s_{m+1}) \cap \bar{I}(t_{m+1}) = 0$.

Theorem 1.1 is proved.

COROLLARY 1.3. *In conditions of Theorem* 1.1 *suppose that M is of cardinality less than c. Then there exists a countable $C = C^0 \in \mathrm{Ca}(M)$ such that $C \subset X$ and each X_n is dense in C.*

PROOF. Let B and S be as in Theorem 1.1. By (b), members of S are pairwise disjoint on $M - B$. Since cardinality of S is bigger than cardinality of $M - B$ there exists $C \in S$ such that $C \subset B$. Now use (a). #

§2. Modest chains.

DEFINITION 2.1. Let M be a chain and $1 \leq \pi \leq \aleph_0$. M is *perfunctorily π-modest* iff $M \subset M^0$ and for each family $\{X_n : n < \pi\}$ of everywhere dense subsets of M there exists $C \in \mathrm{Ca}(M)$ such that $C^0 \subset \bigcup X_n$ and each X_n is dense in C^0. M is *π-modest* iff each subchain of M without jumps and endpoints is perfunctorily π-modest. M is *modest* (*perfunctorily modest*) iff it is π-modest (perfunctorily p-modest) for each finite π.

For each positive integer p, Definition 2.1 actually expresses perfunctory p-modesty and p-modesty in the monadic language of order. By Theorem 2.3 below, every chain of cardinality less than c is $\aleph_0$-modest. One can easily see that the real line is not perfunctorily 1-modest.

LEMMA 2.1. (a) *Chain M is π-modest if all separable subchains of M without jumps and endpoints are perfunctorily π-modest.*

(b) *Suppose that M is separable and has neither jumps nor endpoints. It is perfunctorily π-modest if for each disjoint family $\{X_n : n < \pi\}$ of countable and everywhere dense subsets of M there exists $C \in \mathrm{Ca}(M)$ such that $C^0 \subset \bigcup X_n$ and each X_n is dense in C^0.*

PROOF. (a) It is enough to prove that M is perfunctorily π-modest provided M has neither jumps nor endpoints. Let $\{X_n : n < \pi\}$ be a family of everywhere dense subsets of M. By Theorem 1.1, there exists a countable $Y \subset \bigcup X_n$ such that $Y \subset Y^0$ and each X_n is dense in Y. Let N be the subchain Y^0. If $C \in \mathrm{Ca}(N)$, $C^0 \subset \bigcup X_n$, each X_n is dense in C^0 and D is the closure of C in M then $D \in \mathrm{Ca}(M)$ and $D^0 = C^0$ hence $D^0 \subset \bigcup X_n$ and each X_n is dense in D.

(b) Let $\{I_n : n < \omega\}$ be an open basis of M, and $\{X_q : q < \pi\}$ be a family of everywhere dense subsets of M. Let $f : \omega \times \pi \to \omega$ be one-one and onto, and $(ln, rn) = f^{-1}(n)$. Choose consecutively $y_n \in I_{ln} \cap X_{rn} - \{y_m : m < n\}$. Form $Y_q = \{y_{f(n,q)} : n < \omega\}$. If $C \in \mathrm{Ca}(M)$, $C^0 \subset \bigcup Y_q$ and each Y_q is dense in C^0 then $C^0 \subset \bigcup X_q$ and each X_n is dense in C. #

LEMMA 2.2. *If M is π-modest and $\{X_n : n < \pi\}$ is a family of everywhere dense* *subsets of M then there exists countable $C = C^0 \in \mathrm{Ca}(M)$ such that $C \subset \bigcup X_n$ and each X_n is dense in C.*

PROOF. By Theorem 1.1, there exists a countable $Y = Y^0 \subset \bigcup X_n$ such that every X_n is dense in Y. Since M is π-modest Y is perfunctorily π-modest; hence there exists $C \in \mathrm{Ca}(Y)$ such that $C^0 \subset Y$ and every X_n is dense in C^0. Let D be the closure of C in M. Since C is countable the completion of D is embeddable into the real line. Each point of $D - C^0$ gives a jump in the completion of D hence $D = C^0 \cup (D - C^0)$ is countable. By Corollary 1.3 (with $M = D$ and $X = \bigcup(C^0 \cap X_n)$) there exists $E = E^0 \in \mathrm{Ca}(M)$ such that $E \subset C^0$ and every X_n is dense in E. #

THEOREM 2.3. *Each chain of cardinality less than c is $\aleph_0$-modest.*

PROOF. Use Corollary 1.3. #

THEOREM 2.4. *Suppose that the real line R is not a union of less than c nowhere dense subsets. Then there exists an $\aleph_0$-modest subchain of R of cardinality c.*

PROOF. Let $\{I_\alpha : \alpha < c\}$ be an open basis of R where each element is repeated c times. Let $\mathrm{Ca}(R) = \{C_\alpha : \alpha < c\}$. Pick $x_\alpha \in (I_\alpha - \bigcup_{\beta-\alpha} C_\beta) - \{x_\beta : \beta < \alpha\}$. It is easy to see that the subchain $\{x_\alpha : \alpha < c\}$ is $\aleph_0$-modest. #

There exist chains which are not 1-modest, take for example the real line R. One gets another example setting $p = 0$ in the proof of the following theorem.

THEOREM 2.5. *Assume that R is not a union of less than c nowhere dense subsets. Then for each positive integer p there exists a p-modest subset of R which is not $(p + 1)$-modest.*

PROOF. Let $A_0, \ldots, A_p$ be countable disjoint and everywhere dense subsets of R. Let S_0 (respectively S_1) be the collection of Cantor subsets C of R such that there exists A_i disjoint from C^0 (resp. each A_i is dense in C^0). Let $S_0 = \{B_\alpha : \alpha < c\}$ and $S_1 = \{C_\alpha : \alpha < c\}$. Each B_β is nowhere dense in each C^0_α. Choose consecutively $x_\alpha \in (C^0_\alpha - \bigcup_{\beta<\alpha} B_\beta) - \{x_\beta : \beta < \alpha\}$. Let $W = \{x_\alpha : \alpha < c\}$ and $A = A_0 \cup \cdots \cup A_p \cup W$. Clearly each $|B_\alpha \cap W| < c$ and each $C^0_\alpha \cap W \neq 0$.

A is not $(p + 1)$-modest for there exists no $C \in \mathrm{Ca}(A)$ such that $C^0 \subset \bigcup A_i$ and each A_i is dense in C^0.

We prove that A is p-modest. Let M be a subchain of A without jumps and endpoints, and $\{X_n : n < p\}$ be a family of subsets of A everywhere dense in A.

By induction on $q \leq p$ it is easy to check that there exist an interval I of M and a function $f: q \to (p + 1)$ such that for each $n < q$, $X_n \cap (A_{fn} \cup W)$ is dense in I. Therefore we can assume that each X_n is disjoint from A_0.

By Theorem 1.1, there exists $C \in \mathrm{Ca}(M)$ such that $C^0 \subset \bigcup X_n$ and each X_n is dense in C^0 and C is disjoint from A_0. Let $D = \mathrm{der}(C)$ in R. Clearly $D \in S_0$ hence $|D \cap W| < c$ and $|C| < c$. Now use Theorem 2.3. #

THEOREM 2.6. *Assume that the real line is not a union of less than c nowhere dense subsets. There exists a modest subset of R which is not $\aleph_0$-modest.*

PROOF. Similar to that of Theorem 2.5. Define A as above but with $\aleph_0$ instead of p. We have to check only that A is p-modest for each finite p. Let $\{X_n : n < p\}$ be as above.

Let $A' = \bigcup\{A_m : p < m\}$. There exist I and $f : p \to \omega$ such that for each $n < p$, $X_n \cap (A_{fn} \cup A' \cup W)$ is dense in I. Therefore we can assume that each X_n is disjoint from A_0.

We finish as above. #

Recall that an equivalence relation E on a chain M is called a congruence iff every equivalence class of E is convex.

THEOREM 2.7. *Subchains and quotient chains of π-modest chains are π-modest. If E is a congruence on a chain M, the quotient chain M/E is π-modest and each $X \in M/E$ is π-modest*, then M is π-modest.

PROOF. The first statement is clear. We prove the second one. Let N be a subchain of M without jumps and endpoints. If E does not glue points of N then N is isomorphic to a subchain of M/E and perfunctorily π-modest. Otherwise N has a π-modest interval and therefore is perfunctorily π-modest. #

§3. Monadic theory of the rational line. For each positive integer p let K_p be the class of p-modest chains. Let Rl associate the ring of all subsets of M with each chain $M \in K_1$. It is easy to see that the pair $\langle K_1, \mathrm{Rl}\rangle$ is nice with respect to Definition 3.2 in Part 1. In the following lemma we use terminology of Part 1.

LEMMA 3.1. *There exists an algorithm computing $U_0^1(M, P)$ from* $\mathrm{Th}^0(M, P)$ *whenever there exists p such that $M \in K_p$, and P is a sequence of less than p subsets of M, and the augmented chain $\langle M, P\rangle$ is* 0-*uniform.*

PROOF. Suppose that $M \in K_p$, and $P = \langle P_1, \dots, P_l\rangle$ is a sequence of $l < p$ subsets of M, and the augmented chain $\langle M, P\rangle$ is 0-uniform. Given $\mathrm{Th}^0(M, P)$ we compute $U_0^1(M, P)$.

Set $m = l + 1$, $P_m = M-(P_1 \cup \cdots \cup P_l)$, $P' = \langle P_1, \dots, P_m\rangle$ and $\tilde{M} = \langle M, P'\rangle$. It is sufficient to compute $U_0^1(\tilde{M})$.

Recall that $U_0^1(\tilde{M})$ is the collection of $\mathrm{Th}^0(\tilde{I}/E)$ where $\tilde{I} = \langle I, P' \mid I\rangle$ is an interval of $\tilde{M}$, E is a congruence on I and $\tilde{I}/E$ is 0-uniform. For each convex $X \subset M$ set $\mathrm{th}(X) = [i, j, \varepsilon]$ where

(a) $\min X \in P_i$ or X does not have a minimal point and $i = 0$,

(b) $\max X \in P_j$ or X does not have a maximal point and $j = 0$, and

(c) $\varepsilon = 1$ if X is one-point and $\varepsilon = 0$ otherwise.

If $\tilde{I}$ and E are as above then $\mathrm{Th}^0(\tilde{I}/E)$ is easily computable from $\mathrm{th}(E) = \{\mathrm{th}(X)\colon X \in I/E\}$. Thus it is sufficient to compute $t = \{\mathrm{th}(E)\colon$ there exists an interval $\tilde{I} = \langle I, P' \mid I\rangle$ of $\tilde{M}$ such that E is a congruence on I, I/E is 0-uniform and E is not the identity relation on $I\}$. Let $A = \{0\} \cup \{i\colon P_i \neq 0\}$.

We show that t coincides with the collection t^* of subsets $S \subset A \times A \times \{0,1\}$ such that 0 belongs to the third projection of S. Clearly $t \subset t^*$. Let $S = \{s_1, \dots, s_r\} \in t^*$ and B be the union of the first and the second projections of S. We construct a congruence relation E on M such that $\tilde{M}/E$ is 0-uniform, $\mathrm{th}(E) = S$ and E is not the identity relation on M.

Since M is p-modest and $m \leq p$ there exists a countable $C = C^0 \subset \mathrm{Ca}(M)$ such that $C \subset \{P_k\colon k \in B\}$ and for each $k \in B$, P_k is dense in C. Wlog, C is cofinal in M in both directions.

Let $C = \{a_k\colon k < \omega\}$ and $\{b_k\colon k < \omega\}$ contain an element from each cut of C realized in M. Build a sequence $0 = F_0 \subset F_1 \subset F_2 \subset \cdots$ of finite families of convex subsets of M satisfying the following conditions (a) – (c).

(a) Let $X \in F_k - F_{k-1}$. If $k = q \pmod r$ and $1 \leq q \leq r$ then $\mathrm{th}(X) = s_q$. If X has the minimal (maximal) point then it belongs to C. If X has not the minimal (maximal) point then C is not bounded above (below) in the part of M below (above) X.

(b) F_k is pairwise disjoint and for every $X, Y \in F_k$ there exist elements of F_{k+1} below, between and above X and Y.

(c) $a_k, b_k \in F_0 \cup \cdots \cup F_{k+1}$.

There exists E such that $M/E = \bigcup F_k$. It is the desired congruence. #

THEOREM 3.2. *There exists an algorithm associating a pair (p, ε) with each sentence φ in the monadic language of order in such a way that either $\varepsilon = 0$ and all p-modest chains without jumps and endpoints satisfy φ or $\varepsilon = 1$ and all these chains do not satisfy φ.*

PROOF. Use Theorem 5.3 in Part 1. #

COROLLARY 3.3. *A chain M is monadically equivalent to the rational chain iff it is modest and has neither jumps nor endpoints.*

Each modest chain is easily embeddable into a modest chain without jumps and endpoints.

COROLLARY 3.4. *The monadic theory of modest chains is decidable.*

THEOREM 3.5. *Assume that each pseudo-meager subset of the real line R is meager. The true arithmetic is interpretable in the monadic theory of the class of chains which are not p-modest.*

PROOF. Let M be not p-modest. By Lemma 2.1, there exist a subchain N of M and subsets $X_1, \ldots, X_p$ of N such that N has neither jumps nor endpoints, and $X_1, \ldots, X_p$ are disjoint, countable and dense in N, and there exists no $C \in \mathrm{Ca}(N)$ such that in N: $C^0 \subset \bigcup X_q$ and each X_q is dense in C^0.

Let $D \subset N - \bigcup X_q$ be countable and dense in N and $D = \bigcup\{D_n : n < \omega\}$ be a partition of D into disjoint countable subsets dense in N. Let S be the collection of $C \in \mathrm{Ca}(N)$ such that each X_q is dense in C^0. Let $C \in S$. C will be called *good* iff there exists n such that $C \cap D \subset D_n$. C will be called *bad* iff each good C_1 is nowhere dense in C.

For each $C \in S$, C^0 is not meager in C. Otherwise by Theorem 1.1, there exists $E \in \mathrm{Ca}(N)$ such that $E \subset \bigcup X_q$ and each X_q is dense in E^0 which is impossible.

Let $S = \{C_\alpha : \alpha < c\}$. If C_α is bad pick $x_\alpha \in C_\alpha - \bigcup\{C_\beta : \beta < \alpha$ and C_β is good$\}$. Form $W = \{x_\alpha : C_\alpha$ is bad$\}$. Clearly

(i) $|C \cap W| < c$ if C is good, and

(ii) $C \cap W \neq 0$ if C is bad.

The rest of the proof parallels the proof of Theorem 7.10 in [Sh]. A more detailed proof is in [Gu 3]. #

§4. **Absolute modesty.** Unfortunately the union of two modest subchains may be not modest. Consider for example the case $p = 1$ in the proof of Theorem 2.5. A_0 is $\aleph_0$-modest (for it is countable). W is $\aleph_0$-modest too. (To check that let M be a subset of W without jumps and endpoints and $Z_0, Z_1, \ldots$ be subsets of M dense in N. By Theorem 1.1, there exists $C \in \mathrm{Ca}(R)$ such that each Z_n is dense in C^0 and C is disjoint from A_0. By the construction of W, $|C \cap W| < c$. Now use the $\aleph_0$-modesty of $C \cap W$.) But $A = A_0 \cup W$ is not 1-modest.

DEFINITION 4.1. Let M be a chain and $1 \leq \pi \leq \aleph_0$. A subset X of M is *perfunc-*

torily π-modest in M iff M has neither jumps nor endpoints and for each family $\{Y_n : n < \pi\}$ of everywhere dense subsets of M there exists $C \in \mathrm{Ca}(M)$ such that C^0 is disjoint from $X - \bigcup Y_n$ and each Y_n is dense in C^0. X is *π-modest in M* iff for each subchain N of M without jumps and endpoints, $N \cap X$ is perfunctorily π-modest in N. X is *modest* (*perfunctorily modest*) in M iff X is π-modest (perfunctorily π-modest) in M for each finite π. M is *absolutely modest* (*absolutely π-modest*) iff it is modest (π-modest) in its completion.

For each finite π, the property "X is π-modest in M" is expressible in the monadic theory of chain M.

LEMMA 4.1. *X is π-modest in M iff for each countable $Y \subset M$, the subchain $X \cup Y$ is π-modest.*

PROOF. First suppose that X is π-modest in M. Let $Y \subset M$ be countable and N be a subchain of $X \cup Y$ without jumps and endpoints. We check that N is perfunctorily π-modest. N itself is a union of π-modest and countable subchains. Wlog, $N = M$. Let $\{Z_n : n < \pi\}$ be a family of everywhere dense subsets of M. There exists $C \in \mathrm{Ca}(M)$ such that C^0 is disjoint from $X - \bigcup Z_n$ and each Z_n is dense in C^0. By Theorem 1.1, (with $M = C$ and $X = C^0$) there exists $D \in \mathrm{Ca}(C)$ such that $D^0 \subset \bigcup Z_n$ and each Z_n is dense in D^0.

Now suppose that for each countable Y, the subchain $X \cup Y$ is π-modest. Let N be a subchain of M without jumps and endpoints. We check that $N \cap X$ is perfunctorily π-modest in N. Wlog, $N = M$. Let $\{Z_n : n < \pi\}$ be a family of everywhere dense subsets of M. Take countable A such that $A \subset A^0$ and each Z_n is dense in A^0. Since $X \cup A$ is π-modest there exists $C \in \mathrm{Ca}(M)$ such that C^0 is disjoint from $X - (\bigcup Z_n)$ and each Z_n is dense in C^0. #

Recall that an equivalence relation E on a chain M is called a congruence iff every equivalence class of E is convex.

COROLLARY 4.2. (a) *Let E be a congruence relation on M. If X is π-modest in M then each $Y \subset X$ is π-modest in M and $\{Z \in M/E : X \cap Z \neq 0\}$ is π-modest in M/E. If $\{Z \in M/E : X \cap Z \neq 0\}$ is π-modest in M/E and for each $Z \in M/E$, $X \cap Z$ is π-modest in Z, then X is π-modest in M.* (b) *There exists an $\aleph_0$-modest subset of reals which is not absolutely* 1*-modest provided R is not pseudo-meager.*

PROOF. (a) Use Theorem 2.7. (b) Consider W from the proof of Theorem 2.5. #

THEOREM 4.3. *Let $D_0, D_1, \ldots$ be subsets of a chain M modest in M. Then $\bigcup D_n$ is π-modest in M.*

PROOF. Let N be a subchain of M without jumps and points, and $D = (\bigcup D_n) \cap N$. We have to prove that D is perfunctorily π-modest in N. Wlog, $N = M$. Let $\{X_q : q < \pi\}$ be a family of everywhere dense subsets of M, and $X = \bigcup X_q$. Let S be the collection of $C \in \mathrm{Ca}(M)$ such that each X_q is dense in C^0. Below, C range over S. Let s, t range over the finite sequences of natural numbers, and $f : \{0, 1\} \times \pi \times \omega \to \omega$ be one-one and onto, and $(\alpha n, \beta n, \gamma n) = f^{-1}(n)$. We need the following

LEMMA 4.4. *There exist Cantor subsets $C(s)$ and points $x(s)$ such that*

(i) *$C(s)$ is disjoint from $D_{\mathrm{lh}(s)} - X$, $C(s\hat{\ }n) \subset C(s)$ and $C(s\hat{\ }m) \cap C(s\ n) = 0$ if $m \neq n$;*

(ii) *$x(s) \in C^0(s)$, $x(s\hat{\ }n) \in X_{\beta n}$, $\lim x(s\hat{\ }n) = x(s)$, $x(s\hat{\ }n) < x(s)$ if $\alpha n = 0$, otherwise $x(s\hat{\ }n) > x(s)$;*

(iii) *if* $x = \lim x(s_n)$ *then either* $x = x(t)$ *for some* t *or there exists a strictly increasing sequence* $t_0 \subset t_1 \subset \cdots$ *such that* $x \in \bigcap C(t_n)$.

PROOF OF LEMMA 4.4. Since D_0 is π-modest in M, there exists $C(0)$ disjoint from $D_0 - X$. Pick arbitrary $x(0) \in C^0(0)$. Suppose that $C(t)$ and $x(t)$ are chosen for every t with $\mathrm{lh}(t) \le l$, and that the relevant cases of (i) and (ii) hold. Select $y_0 < y_1 < \cdots$ and $z_0 > z_1 > \cdots$ in such a way that $y_n, z_n \in C^0(s)$ and $\lim y_n = \lim z_n = x(s)$. Since D_l is π-modest in M, for each n there exists $C(s\hat{\ }n) \subset C(s)$ disjoint from $D_l - X$ and located between y_n and y_{n+1} if $\alpha n = 0$, and between z_n and z_{n+1} otherwise. Pick $x(s\hat{\ }n) \in C^0(s\hat{\ }n) \cap X_{\beta n}$. Similar to the corresponding part of the proof of Lemma 1.2 one can check that (iii) holds. Lemma 4.4 is proved. We continue the proof of Theorem 4.3.

Let B be the collection of points $x(s)$, and $C = \bar{B}$. Each X_q is dense in $B \subset C^0$ and C is disjoint from $\bigcup(D_n - X) = D - X$. #

THEOREM 4.5. *Let M be a chain with an everywhere dense subset A π-modest in M. Then for each everywhere dense subset X of M there exists $B \subset X$ everywhere dense and π-modest in M.*

PROOF. By Corollary 4.2(a), it is enough to prove that for each interval I of M there exist a subinterval $J \subset I$ and a set $B \subset J \cap X$ such that B is dense and π-modest in J.

Let I be an interval of M. First suppose that there exist a subinterval $J \subset I$ and a set $Y \subset J$ such that $|Y| < c$ and Y is dense in J. Then there exists an open basis $\{J_\alpha: \alpha < k\}$ of J where $k < c$. Take $B = \{b_\alpha: \alpha < k\}$ where $b_\alpha \in J_\alpha \cap X$ and use Theorem 2.3 and Lemma 4.1.

Now suppose that each subset of I of cardinality $< c$ is nowhere dense in I. Let $\{I_\alpha: \alpha < c\}$ be an open basis of I and $\{Y_\alpha: \alpha < c\}$ be the collection of countable subsets of M. Pick $b_\alpha \in I_\alpha \cap X - \bigcup\{\mathrm{cl}\ Y_\beta: \beta < \alpha\}$ and set $B = \{b_\alpha: \alpha < c\}$.

We check that B is π-dense in I. Let N be a separable subchain of B without jumps and endpoints. It is enough to prove that N is π-modest in I (use Lemmas 2.1(a) and 4.1).

Let $D = \mathrm{der}(N)$ in I. By the construction of B, $|D \cap N| < c$ hence $D \cap N$ is π-modest in I. $N - D$ is discrete in I hence it is isomorphic to a quotient chain of $A \cap I$ hence it is π-modest in I. Now use Theorem 4.3. #

§5. Modest theory of the real line.

Let K be a class of complete chains and Rl associate a ring $\mathrm{Rl}(M)$ of subsets of M with each $M \in K$. Suppose that members of $\mathrm{Rl}(M)$ are modest in M and the pair $\langle K, \mathrm{Rl}\rangle$ is nice with respect to Definition 3.2 in Part 1. In the following lemma we use terminology of Part 1.

LEMMA 5.1. *There exists an algorithm computing $U_0^1(\tilde{M})$ from* $\mathrm{Th}^0(\tilde{M})$ *whenever $\tilde{M}$ is a* 0-*uniform augmented chain.*

PROOF. Let $M \in K$ and $P = \langle P_1, \ldots, P_m\rangle$ be a sequence of members of $\mathrm{Rl}(M)$. Suppose that the augmented chain $\tilde{M} = \langle M, P\rangle$ is 0-uniform. Given $\mathrm{Th}^0(\tilde{M})$ we compute $U_0^1(\tilde{M})$.

By the definition, $U_0^1(\tilde{M})$ in the collection of $\mathrm{Th}^0(\tilde{I}/E)$ where $\tilde{I} = \langle I, P|I\rangle$ is an interval of $\tilde{M}$, E is a congruence on I and $\tilde{I}/E$ is 0-uniform. Let $N = P_1 \cup \cdots \cup P_m$ and N^* be the augmented chain $\langle N, P\rangle$. If $\tilde{I}$ and E are as above then $I^* =$

$\langle I \cap N, P|I \rangle$ is an interval of N^*, $E^* = E \cap (N \times N)$ is a congruence on $I \cap N$ and I^*/E^* is 0-uniform. It is easy to construct an algorithm f computing $\mathrm{Th}^0(\tilde{I}/E)$ from $\mathrm{Th}^0(I^*/E^*)$ and to check that $U_0^1(\tilde{M}) = \{f(t) : t \in U_0^1(N^*)\}$. Since the chain N is modest we can use the algorithm of Lemma 3.1 to compute $U_0^1(N^*)$ from the 0-theory $\mathrm{Th}^0(N^*)$ of N^* which is easily computable from $\mathrm{Th}^0(\tilde{M})$. Thus $U_0^1(\tilde{M})$ is computable from $\mathrm{Th}^0(M)$. #

THEOREM 5.2. *There exists an algorithm associating a sentence φ' with each sentence φ in the monadic language of order in such a way that φ' is either φ or the negation of φ and for every chain M in K without jumps and endpoints, φ' is a theorem in the theory of M with quantification over* Rl(M).

PROOF. Use Corollary 5.4 in Part 1. #

The *modest theory* of chain M is the theory of M with quantification over subsets modest in M.

COROLLARY 5.3. *Let M be a complete chain without jumps and endpoints having an everywhere dense subset modest in M. Then the modest theory of M coincides with the modest theory of the real line R. The modest theory of R is decidable.*

COROLLARY 5.4. *The following theories coincide: the modest theory of R, the theory of R with quantification over countable subsets, the theory of R with quantification over subsets of cardinalities $< c$.*

Note. Let W be a subset of R which is p-modest in R but not $(p + 1)$-modest in R. Let M be a chain obtained from R by replacing of points of W by copies of $[0, 1]$. M has an everywhere dense subset p-modest in M but it does not have an everywhere dense subset $(p + 1)$-modest in M.

Let L_1 be the monadic language of order enriched by a set constant. If X is a subset of a chain M then (M, X) is a model of L_1. The theory of (M, X) in L_1 when set variables range over subsets modest in M will be called the modest theory of (M, X).

THEOREM 5.5. *Assume that each pseudo-meager subset of R is meager. Then for each complete chain M without jumps and endpoints and for each $X \subset M$ which is not modest in M the modest theory of (M, X) is undecidable.*

See the detailed proof in [Gu 3]. #

Now we turn our attention to the condition. "M has an everywhere dense subset modest in M" occurring in Corollary 5.3. This condition is essential.

THEOREM 5.6. *Assume that R is not pseudo-meager. There exists a complete modest chain without jumps and endpoints whose modest theory is undecidable.*

PROOF. Let Q be the set of rationals. According to Theorem 7.11 in [Sh], there exists $W \subset R - Q$ such that a finitely axiomatizable undecidable first-order theory T is interpretable in the theory of (R, W) with quantification over subsets of Q. W is not 1-modest in any interval of R.

Let M be a chain obtained from R by replacing each point of W by a copy of $[0, 1]$. Clearly T is interpretable in the modest theory of the completion of M. #

§6. **Addition.** Here we prove some phenomena concerning modest subsets of the real line R. The first phenomenon contradicts somewhat the feeling that modest subsets are small.

Note. The sequences built in the proofs of the theorems below are continuous.

THEOREM 6.1. *Assume that each pseudo-meager subset of R is meager. Then there exists a partition $R = A \cup B$ such that both A and B are perfunctorily $\aleph_0$-modest in R.*

PROOF. Let $\{\langle X_{\alpha 0}, X_{\alpha 1}, \ldots\rangle : 0 < \alpha < c$ and α is even$\}$ be the collection of sequences $\langle X_0, X_1, \ldots\rangle$ such that each X_n is a countable and everywhere dense subset of R. Let $R = \{y_\alpha : \alpha < c$ and α is odd$\}$. By induction on α we build meager approximations A_α and B_α. $A_0 = B_0 = 0$. Suppose that α is odd. If y_α does not belong to $A_\alpha \cup B_\alpha$ set $A_{\alpha+1} = A_\alpha \cup \{y_\alpha\}$ and $B_{\alpha+1} = B_\alpha$; otherwise set $A_{\alpha+1} = A_\alpha$ and $B_{\alpha+1} = B_\alpha$. Let α be even. By Theorem 1.1, there exist $C_{\alpha 0}, C_{\alpha 1} \in \mathrm{Ca}(R)$ such that each $X_{\alpha n}$ is dense in each $C^0_{\alpha\varepsilon}$, and $C_{\alpha 0} \cap C_{\alpha 1} \subset X_\alpha = \bigcup\{X_{\alpha n} : n < \omega\}$ and each $C_{\alpha\varepsilon}$ is disjoint from $A_\alpha \cup B_\alpha - X_\alpha$. Set $A_{\alpha+1} = A_\alpha \cup (C_{\alpha 0} - B_\alpha)$ and $B_{\alpha+1} = B_\alpha \cup (C_{\alpha 1} - A_{\alpha+1})$. Set $A = \bigcup A_\alpha$ and $B = \bigcup B_\alpha$. Given $\langle X_{\alpha 0}, X_{\alpha 1}, \ldots\rangle$ take $C_{\alpha 0}$ (respectively $C_{\alpha 1}$) to check perfunctorily $\aleph_0$-modesty of A (respectively B). #

LEMMA 6.2. *Theorem 1.1 remains true if ($X \subset X^0$) and ($B \cap C \subset C^0$) are replaced by ($X \subset$ der X) and ($B \cap C^0 = 0$) respectively.*

PROOF. Replace ($\alpha n = 0$) by ($x(s)$ is a left limit point of X) in Lemma 1.2. #

COROLLARY 6.3. *Let X be a countable subset of R without isolated points (i.e. $X \subset$ der X), and $X_0, X_1, \ldots$ be subsets of X dense in X, and $A \subset R - X$ be of cardinality $< c$. Then there exists $C \in \mathrm{Ca}(R)$ such that each X_n is dense in C and C^0 is disjoint from $A \cup X$.*

PROOF. By Lemma 6.2 (with M equal to the closure of X), there exist $B \subset X$ and $S \subset \mathrm{Ca}(R)$ such that $|S| = c$, and each X_n is dense in each $C \in S$, and B is disjoint from C^0 for each $C \in S$, and S is disjoint on $R - B$. There exists $C \in S$ disjoint from A and $(X - B)$. Clearly C^0 is disjoint from $A \cup X$. #

Let M be a chain and $1 \leq \pi \leq \aleph_0$. Subset $A \subset M$ is *topologically π-modest* iff for each $X \subset M$ with $X \subset \mathrm{der}(X)$ and each family $\{X_n : n < \pi\}$ of subsets of X dense in X there exists $C \in \mathrm{Ca}(M)$ such that $C \subset X$ and each X_n is dense in C.

THEOREM 6.4. *Assume that R is not pseudo-meager. Then there exists an everywhere dense $A \subset R$ such that A is topologically $\aleph_0$-modest in R but A is not even perfunctorily 1-modest.*

PROOF. Let $\{\langle X_{\alpha 0}, X_{\alpha 1}, \ldots\rangle : \alpha < c$ and α is odd$\}$ be the collection of sequences $\langle X_0, X_1, \ldots\rangle$ such that $\bigcup X_n$ is countable and has no isolated points in R and each X_n is dense in $\bigcup X_n$. Let $\{C_\alpha : 0 < \alpha < c$ and α is even$\}$ be the collection of subsets $C \in \mathrm{Ca}(R)$ such that the set Q of rational numbers is dense in C^0. By induction on α we build disjoint and increasing A_α and B_α such that $|A_\alpha| \leq \alpha + \omega$ and B_α is meager.

$A_0 = Q$ and $B_0 = 0$. Suppose α is odd. By Corollary 6.3, there exists $D_\alpha \in \mathrm{Ca}(R)$ such that each $X_{\alpha n}$ is dense in D_α and D^0_α is disjoint from $\bigcup\{X_{\alpha n} : n < \omega\}$ and from A_α. Set $A_{\alpha+1} = A_\alpha$ and $B_{\alpha+1} = B_\alpha \cup (D_\alpha - A_\alpha)$. Note that D^0_α is disjoint from Q hence D_α is nowhere dense in any C_β. Now suppose that $\alpha > 0$ is even. Pick $x_\alpha \in C^0_\alpha - (B_\alpha \cup Q)$, set $A_{\alpha+1} = A_\alpha \cup \{x_\alpha\}$ and $B_{\alpha+1} = B_\alpha$. Set $A = \bigcup A_\alpha$. A is topologically $\aleph_0$-modest in R but there is no $C \in \mathrm{Ca}(A)$ such that Q is dense in C^0 and C^0 is disjoint from $Q - A$. #

LEMMA 6.5. *Let X be an everywhere dense subset of R and $D_0, D_1, \ldots$ be nowhere dense subsets of R. There exists an everywhere dense $Y \subset X$ such that each $\bar{D}_n \cap Y$ is finite.*

PROOF. Let $\{I_n : n < \omega\}$ be an open basis of R. Pick $y_n \in I_n - (\bar{D}_0 \cup \cdots \cup \bar{D}_n)$. #

THEOREM 6.6. *Assume the Continuum Hypothesis. Then there exists an $\aleph_0$-modest $A \subset R$ such that for each everywhere dense $E \subset R - A$, $A \cup E$ is not even perfunctorily* 1*-modest.*

PROOF. Let $\{\langle X_{\alpha 0}, X_{\alpha 1}, \ldots \rangle : \alpha < c \text{ is odd}\}$ be the collection of sequences $\langle X_0, X_1, \ldots \rangle$ such that $\bigcup X_n$ is countable and forms a subchain without jumps and endpoints and each X_n is everywhere dense in that subchain. Let $X_\alpha = \bigcup\{X_{\alpha n}: n < \omega\}$.

Let $\mathrm{Ca}(R) = \{C_\alpha : 0 < \alpha < c \text{ and } \alpha \text{ is even}\}$ where each Cantor subset of R is repeated c times.

By induction on α we build A_α, A_α^* and B_α such that $A_\alpha \subset A_\alpha^*$, and $A_\alpha \cap B_\alpha = 0$, and A_α^*, B_α are meager in R. $A_0 = A_0^* = Q$ and $B_0 = 0$.

If α is odd and $X_\alpha \subset A_\alpha$ let M_α be the subchain of R formed by the closure of X_α. By Theorem 1.1, there exists $D_\alpha \in \mathrm{Ca}(M_\alpha)$ such that D_α is disjoint from $A_\alpha^* - X_\alpha$ and in M_α: each $X_{\alpha n}$ is dense in D_α^0. Set $A_{\alpha+1} = A_\alpha$, $A_{\alpha+1}^* = A_\alpha^*$ and $B_{\alpha+1} = B_\alpha \cup (D_\alpha - X_\alpha)$.

If α is odd and X_α is disjoint from A_α select an everywhere dense $Y_\alpha \subset X_\alpha$ such that each $D_\beta \cap Y_\alpha$ is finite where $\beta < \alpha$. Set $A_{\alpha+1} = A_\alpha$, $A_{\alpha+1}^* = A_\alpha^* \cup Y_\alpha$ and $B_{\alpha+1} = B_\alpha$.

If $\alpha > 0$ is even and there exists $\beta < \alpha$ such that Y_β is dense in C_α^0 then each D_γ with $\gamma < \alpha$ is nowhere dense in C_α^0. Pick $a_\alpha \in C_\alpha^0 - (A_\alpha^* \cup B_\alpha)$. Set $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$, $A_{\alpha+1}^* = A_\alpha^* \cup \{a_\alpha\}$ and $B_{\alpha+1} = B_\alpha$.

In the other cases set $A_{\alpha+1} = A_\alpha$, $A_{\alpha+1}^* = A_\alpha^*$ and $B_{\alpha+1} = B_\alpha$. Set $A = \bigcup A_\alpha$. Clearly A is $\aleph_0$-modest. Let $E \subset R - A$ be everywhere dense. We check that $A \cup E$ is not perfunctorily 1-modest. There exists α such that $Y_\alpha \subset X_\alpha \subset E$. If $Z \in \mathrm{Ca}(A)$ and Y_α is dense in Z then there exists β such that $\bar{Z} = C_\beta$ and $a_\beta \in A$ hence $(Z^0 \subset Y_\alpha)$ does not hold. #

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