Two notes on formalized topology

by

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Abstract. The first order topology and the full second order logic are interpretable each in the other. The monadic topological theory of the Euclidean plane and the full second order theory of $2^{\aleph_0}$ are interpretable each in the other.

§ 0. Introduction and the results. The full (pure) second order language has variables for elements, monadic predicates, dyadic predicates and so on; all these variables are quantifiable. It has no non-logical constants. Without loss of generality it has no functional variables. The full second order theory of a non-zero cardinal $\varkappa$ is the theory of a set S of cardinality $\varkappa$ in the full second order language (n-adic predicate variables range over all n-adic predicates on S). The full second order logic is the theory of all non-empty sets in the full second order language.

The pure monadic language is the part of the full pure second order language having variables for elements and monadic predicates only. In this case it is convenient for us to interpret the predicate variables as set variables and to write $x \in Y$ instead of $Y(x)$. The (monadic) *topological language* is obtained from the pure monadic language by adding the symbol of the closure operation. Let U be a topological space. The *monadic theory* of U is the theory of U in the monadic topological language (when the set variables range over all subsets of U).

According to [4], the first order theory of U is the first order theory of the lattice of closed subsets of U. It is a little more convenient for us to define the *first order theory* of U as the theory of U in the topological language when the set variables range over the closed subsets of U. It is essentially the same theory in the case of T_1 spaces (only T_1 spaces are regarded in [4]).

We say here that theory T_1 is interpretable in theory T_2 iff there exists an algorithm f associating a sentence $f(\varphi)$ in the language of T_2 with each sentence φ in the language of T_1 in such a way that φ is a theorem of T_1 iff $f(\varphi)$ is a theorem of T_2.

Grzegorczyk interpreted the first order arithmetic (i.e., the first order theory of the standard model of arithmetic) in the first order topological theory of the Euclidean plane (see [1]). Rabin proved decidability of the first order theory of the real line (see [5]). Assuming the Continuum Hypothesis Shelah interpreted the first order arithmetic in the monadic topological theory of the real line (see [6]).

According to [4], the first order topological theory of the Euclidean plane and the second order arithmetic are interpretable each in the other. The second order arithmetic and the full second order theory of $\aleph_0$ are interpretable each in the other. The last theory is an object of the set theory and in that sense the authors did the maximum. The authors interpreted also the second order arithmetic in the first order theory of the class of all T_1 spaces.

Note that the full third order logic is easily interpretable in the full second order one. Hence the first order theory of the class of T_1 spaces (respectively normal spaces, compact Hausdorff spaces and so on) is interpretable in the full second order logic. We prove here the following two theorems.

THEOREM 1. *The full second order logic is interpretable in the first order theory of any class of topological spaces containing all compact Hausdorff spaces.*

THEOREM 2. *The monadic topological theory of the Euclidean plane and the full second order theory of $2^{\aleph_0}$ are interpretable each in the other.*

These two theorems were announced in [2]. According to [3], the monadic topological theory of the real line and the full second order theory of $2^{\aleph_0}$ are mutually interpretable in the constructive universe (in ZF+V = L).

§ 1. Proof of Theorem 1. The pure dyadic language is the part of the full second order language having variables for elements and dyadic predicates only. Let DL be the theory of non-empty sets in the pure dyadic language. The full second order logic is easily interpretable in DL. (Treat $\{x\colon E(x,x)\}$ as the set of points, $\{a\colon P(a,a)\}$ as the set of pairs of points, $L(a,x)$ as "x is the left component of a", $R(a,x)$ as "x is the right component of a". Then for example,

$$\exists a\big(L(a,x)\wedge R(a,y)\wedge Q(a,z)\big)$$

is an arbitrary 3-adic predicate on the set of points.)

Let SDL be the theory of non-empty sets in the pure dyadic language when the predicate variables range over dyadic symmetric irreflexive predicates.

DL is interpretable in SDL. (Let $E_i = \{x\colon E(c_i,x)\}$ where $i = 0,1,2$. Treat E_0 as the main set of points and E_1, E_2 as auxiliary sets. If E_0, E_1, E_2 are mutually disjoint and $L(R)$ is a one-one correspondence between E_0 and E_1 (between E_0 and E_2) then $\exists ab\big(L(x,a)\wedge R(y,b)\wedge Q(a,b)\big)$ is an arbitrary dyadic predicate on E_0. The trick is well-known.)

Let K be a class of topological spaces containing all compact Hausdorff spaces. In the rest of this section we interpret SDL in the first order theory of K.

Any discrete subset of a topological space is the difference of two closed sets: its closure and its derived set (this note is due to [4]). Hence without loss of generality we can assume that the set variables range over closed and discrete subsets.

It is easy to write a formula $\varphi(x,y,z,X,Z)$ in the topological language stating in the first order topology that X and Z are disjoint, x and y are different points tn X, $z\in Z$ and there exists a closed Y such that $X\cap Y = \{x,y\}$, $Y\cap Z = \{z\}$ and $Y-\{x,y\}$ is connected. In order to interpret SDL in the first order theory of K it is enough, given a non-zero cardinal $\varkappa$, to find $U\in K$ and discrete disjoint $X,Z\subset U$ such that $|X| = \varkappa$ and φ defines a one-one correspondence between Z and $\{\{x,y\}\colon x$ and y are different elements of $X\}$.

Fix a non-zero cardinal $\varkappa$. Firstly, we build an auxiliary topological space V. The set of points of V is equal to $\varkappa\cup\{\langle\alpha,\beta,t\rangle\colon \alpha<\beta<\varkappa$, t is a real number and $0<t<1\}$. The open basis of V consists of the following sets:

(i) $\{\langle\alpha,\beta,t\rangle\colon t\in I\}$ where $\alpha<\beta$ and I is an open subinterval of $(0,1)$; and

(ii) $\{\beta\}\cup\{\langle\alpha,\beta,t\rangle\colon \alpha<\beta$ and $t>t_\alpha\}\cup\{\langle\beta,\gamma,t\rangle\colon \beta<\gamma$ and $t<t_\gamma\}$ where t_α, t_γ are positive real numbers less than 1.

In other words we regard $\varkappa$ as a discrete space and for every $\alpha<\beta$ we add a set $(\alpha,\beta) = \{\langle\alpha,\beta,t\rangle\colon 0<t<1\}$ in such a way that $[\alpha,\beta] = (\alpha,\beta)\cup\{\alpha,\beta\}$ is a closed arc connecting α and β.

V is normal. To prove that, let A_1 and A_2 be closed and disjoint subsets of V. Each arc $[\alpha,\beta]$ is normal, hence there exist $G_1(\alpha,\beta)$ and $G_2(\alpha,\beta)$ such that $A_i\cap[\alpha,\beta]\subset G_i(\alpha,\beta)\subset[\alpha,\beta]$ and $G_1(\alpha,\beta)$, $G_2(\alpha,\beta)$ are disjoint and open in $[\alpha,\beta]$. Without loss of generality, $A_i\cap\{\alpha,\beta\} = G_i(\alpha,\beta)\cap\{\alpha,\beta\}$. Let $G_i = \bigcup\{G_i(\alpha,\beta)\colon \alpha<\beta\}$. G_1 and G_2 are open and disjoint neighborhoods of A_1 and A_2, respectively.

Let U be a compactification of V, we work in U. Let $X = \varkappa$ and

$$Z = \{\langle\alpha,\beta,\tfrac{1}{2}\rangle\colon \alpha<\beta\}.$$

X and Z are discrete, and $|X| = \varkappa$. We prove that φ defines the function $f(\alpha,\beta) = \langle\alpha,\beta,\frac{1}{2}\rangle$. For every $\alpha<\beta$, $Y = [\alpha,\beta]$ is closed, $X\cap[\alpha,\beta] = \{\alpha,\beta\}$, $Y\cap Z = \{\langle\alpha,\beta,\frac{1}{2}\rangle\}$ and (α,β) is connected. Suppose that Y is closed, $X\cap Y = \{\alpha,\beta\}$, $X\cap Y = \{\langle\gamma,\delta,\frac{1}{2}\rangle\}$ and $Y-\{\alpha,\beta\}$ is connected. For reduction to absurdity suppose that $\{\alpha,\beta\}\neq\{\gamma,\delta\}$.

Let us check that (γ,δ) is open. Clearly, (γ,δ) is open in V. Hence $(\gamma,\delta) = G\cap V$ for some open G. In particular, $\gamma,\delta\notin G$. V is dense in U hence (γ,δ) is dense in G and $G\subset[\gamma,\delta]$. So $(\gamma,\delta)\subset G\subset[\gamma,\delta]-\{\gamma,\delta\}$.

Let B be the boundary of (γ,δ). We check that $B = \{\gamma,\delta\}$. Clearly, $\gamma,\delta\in B$. Let $u\in B$. Then $u\in$ closure $(\gamma,\delta) = [\gamma,\delta]$ and $u\in$ closure $\big(U-(\gamma,\delta)\big) = U-(\gamma,\delta)$. Hence $u\in\{\gamma,\delta\}$.

Since $Y-\{\alpha,\beta\}$ is connected and intersects both (γ,δ) and $U-(\gamma,\delta)$ then $Y-\{\alpha,\beta\}$ intersects the boundary of (γ,δ). But $B\cap(Y-\{\alpha,\beta\})\subset\varkappa\cap(U-\varkappa) = 0$ so we have a contradiction.

§ 2. Proof of Theorem 2. Let R be the set of real numbers. By R^2 we denote both the Euclidean plane and the set of ordered pairs of real numbers. Let T be the monadic topological theory of R^2. It is routine to interpret T in the full second order logic. According to § 1 it is enough to interpret the dyadic logic DL in T.

For each $Z\subset R^2$ define:

$Z^* = \{Y\colon Y$ is a component of Z or Y is a component of $R^2-Z\}$; and $x\sim y(\mathrm{mod}\,Z)$ iff there exists $Y\in Z^*$ such that $x\in Y$ and $y\in Y$.

It is easy to write formula $\psi(x, y, z, X, Y, Z_1, Z_2, Z_3)$ in the topological language stating (in T) that $x, y \in X$ and $X \in Z_1^*$, $Y \in Z_2^*$, $x \sim z (\mathrm{mod} Z_2)$ and there exists $y^1 \in Y$ such that $y \sim y^1 (\mathrm{mod} Z_3)$ and $y^1 \sim z (\mathrm{mod} Z_1)$.

In order to interpret DL in T it is enough to find $X, Y, Z_1, Z_2, Z_3 \subset R^2$ such that $|X| = 2^{\aleph_0}$ and ψ defines a one-one correspondence between $\{\langle x, y\rangle: x, y \in X\}$ and $\{z: z \in R^2\}$. Let Q be the set of rational numbers. Choose $X = R \times \{0\}$, $Y = \{0\} \times R$, $Z_1 = R \times Q$, $Z_2 = Q \times R$ and $Z_3 = \{\langle a, b\rangle: a - b \in Q\}$. Then $\psi(x, y, z, X, Y, Z_1, Z_2, Z_3)$ holds iff there exist $a, b \in R$ such that $x = \langle a, 0\rangle$, $y = \langle b, 0\rangle$ and $z = \langle a, b\rangle$.

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Borel sets with $F_{\sigma\delta}$-sections

by

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Abstract. Let E, F be compact metric spaces. We characterize Borel sets A in $E \times F$ with $F_{\sigma\delta}$-sections.

Introduction. We consider two fixed compact metric spaces E and F. The class $\mathscr{C}$ will consist of the Borel subsets A of $E \times F$ such that for each $x \in E$ the section $A(x) = \{y \in F;\ (x, y) \in A\}$ is closed in F. We will prove the following:

THEOREM 1. *If A is a Borel subset of $E \times F$ such that each section $A(x)$ is $F_{\sigma\delta}$ in F, then A belongs to the class $\mathscr{C}_{\sigma\delta}$.*

This is an extension of the work of J. Saint-Raymond (see [13]), who established:

THEOREM 2. *If a is a Borel subset of $E \times F$ such that each section $A(x)$ is F_σ in F, then A belongs to the class $\mathscr{C}_\sigma$.*

Theorem 1 is also related to my earlier paper [2].

Preliminaries. N will denote the set of all positive integers. Let $\mathscr{R} = \bigcup_k N^k$, taking $N^0 = \{\emptyset\}$. Thus $\mathscr{R}$ consists of the finite complexes of integers. If $c \in \mathscr{R}$, let $|c|$ be the length of c. If $c, d \in \mathscr{R}$, we write $c < d$ if c is an initial section of d. Let $(p_k)_k$ be an enumeration of all prime numbers. If we associate 0 to $\emptyset$ and the integer $p_1^{n_1} \ldots p_k^{n_k}$ to the complex $c = (n_1, \ldots, n_k)$, a one-one map of $\mathscr{R}$ into N is established. The induced ordering of $\mathscr{R}$ will be called the *standard ordering*. Let $\mathscr{N} = N^N$. if $\nu \in \mathscr{N}$ and $c \in \mathscr{R}$, we write $c < \nu$ if c is an initial section of ν.

If L is a compact metric space, then $\underline{\underline{K}}(L)$ consists of all closed subsets of L and is equipped with the exponential or Vietoris topology. This topology is compact metrizable. I recall the following result (see [7]).

LEMMA 1. *Let P be a Polish subspace of the compact metric space L. Then the subspace $\underline{\underline{F}}(P)$ of $\underline{\underline{K}}(L)$ consisting of those compact sets K in L such that $K = \overline{K \cap P}$, is Polish.*