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Rigid homogeneous chains

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0. MOTIVATION AND PROPAGANDA

Classifying (unordered) sets by the elementary (first order) properties of their automorphism groups was undertaken in (7), (9) and (11). For example, if Ω is a set whose automorphism group, $S(\Omega)$, satisfies

$$\exists x \exists y(x^2 = e \,\&\, y^2 = e \,\&\, \forall z[(zx = xz \,\&\, zy = yz) \to z = e]),$$

then Ω has cardinality at most $\aleph_0$, and conversely (see (7)). We are interested in classifying homogeneous totally ordered sets (homogeneous chains, for short) by the elementary properties of their automorphism groups. (Note that we use 'homogeneous' here to mean that the automorphism group is transitive.) This study was begun in (4) and (5). For any set Ω, $S(\Omega)$ is primitive (i.e. has no congruences). However, the automorphism group of a homogeneous chain need not be o-primitive (i.e. it may have convex congruences). Fortunately, 'o-primitive' is a property that can be captured by a first order sentence for automorphisms of homogeneous chains. Hence our general problem falls naturally into two parts. The first is to classify (first order) the homogeneous chains whose automorphism groups are o-primitive; the second is to determine how the o-primitive components are related for arbitrary homogeneous chains whose automorphism groups are elementarily equivalent.

According to the general theory of automorphism groups of homogeneous chains (3), an o-primitive automorphism group is either o-2 transitive or regular (uniquely transitive). Again, propitiously, these two classes can be distinguished by a first order sentence. The o-2 transitive case was treated in (4) and (5), and bears most resemblance to the unordered case ((7), (9) and (11)) – not surprisingly since $S(\Omega)$ is 2-transitive for any set Ω. In this paper, we completely settle the case of homogeneous chains with uniquely transitive automorphism groups (these are the rigid homogeneous chains of the title).

Specifically, let Ω be a chain and $\mathscr{A}(\Omega) = \mathrm{Aut}(\langle \Omega, \leqslant \rangle)$. $\mathscr{A}(\Omega)$ is a lattice-ordered group under the pointwise ordering. Ω is a *rigid homogeneous chain* if for each $\alpha, \beta \in \Omega$, there is a unique $g \in \mathscr{A}(\Omega)$ such that $\alpha g = \beta$. We will see that Ω is an o-group, and as such is rigid in the model-theoretic sense. T. Ohkuma (8) proved that if Ω is a rigid homogeneous chain, then it is ordermorphic to a subgroup of $\mathbf{R}$, the real numbers, and is ordermorphic to $\mathscr{A}(\Omega)$ which is totally ordered with respect to the pointwise ordering (a self-contained proof is included in the next section). Consequently, we are dealing

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with torsion-free abelian groups. Such groups have an elementary classification determined by a certain set, $\{\gamma_p : p \text{ prime}\}$, of invariants due to Szmielew (12); namely if G is a torsion-free abelian group, its Szmielew p-invariant (p prime), γ_p, is just the dimension of the $\mathbf{Z}_p$ vector space G/pG, where $\mathbf{Z}_p$ is the field of p elements. The main result is:

THEOREM. *If Ω is a rigid homogeneous chain, then $\mathscr{A}(\Omega)$ is a cyclic or dense subgroup of* $\mathbf{R}$; *and in the former case Ω is isomorphic to Z. For each prime p, the Szmielew p-invariant, $\gamma_p(\Omega)$, of $\mathscr{A}(\Omega)$ satisfies $1 \leqslant \gamma_p \leqslant 2^{\aleph_0}$. If $\{\gamma_p : p \text{ prime}\}$ is a set of cardinal numbers with $1 \leqslant \gamma_p \leqslant 2^{\aleph_0}$ for each prime p, then there are 2^c pairwise non-isomorphic rigid homogeneous chains Ω having $\{\gamma_p : p \text{ prime}\}$ as the set of Szmielew invariants for $\mathscr{A}(\Omega)$ (where $c = 2^{\aleph_0}$). If further Ω is dense and Λ is any dense homogeneous chain, $\mathscr{A}(\Lambda) \cong \mathscr{A}(\Omega)$ if and only if Λ is a rigid homogeneous chain and for each prime p, $\gamma_p(\Lambda) = \gamma_p(\Omega)$ or both are infinite.*

Our proof essentially follows that of T. Ohkuma (8) but is somewhat simpler and has necessarily been modified in several ways for our purposes (Ohkuma proved the first sentence of the above theorem and the existence of 2^c pairwise non-isomorphic rigid homogeneous chains – but with no results about Szmielew invariants).

For all unexplained terms, see the next section.

1. BACKGROUND DEFINITIONS AND NOTATION

Let Ω be a totally ordered set. $\mathscr{A}(\Omega)$ is a lattice-ordered group if we define $f \leqslant g$ if and only if $\alpha f \leqslant \alpha g$ for all $\alpha \in \Omega$. Throughout all totally ordered sets will be assumed to be *homogeneous* (i.e., if $\alpha, \beta \in \Omega$, then there exists $g \in \mathscr{A}(\Omega)$ such that $\alpha g = \beta$). If Ω is a rigid homogeneous chain (e.g., $\mathbf{Z}$, the integers), then fix $\alpha_0 \in \Omega$ and define a map from $\mathscr{A}(\Omega)$ onto Ω via: $g \mapsto \alpha_0 g$. This well-defined map preserves order and provides an ordermorphism between $\mathscr{A}(\Omega)$ and Ω; so $\mathscr{A}(\Omega)$ is a totally ordered group. (If $\alpha_0 < \alpha_0 g$ and $\alpha > \alpha g$ for some $\alpha \in \Omega$, let $\bar{g} \in \mathscr{A}(\Omega)$ be defined by

$$\beta\bar{g} = \begin{cases} \beta g & \text{if} \quad \alpha_0 g^n < \beta < \alpha_0 g^m \quad \text{for some} \quad m, n \in \mathbf{Z} \\ \beta & \text{otherwise} \end{cases}$$

Then $\alpha\bar{g} = \alpha$ but $\alpha_0 g \neq \alpha_0$, contradicting the fact that Ω is rigid.) Moreover, $\mathscr{A}(\Omega)$ is Archimedean. (If $e < f^n < g$ for all $n \in \mathbf{Z}^+ = \{1, 2, 3, 4, \ldots\}$, then $\alpha_0 < \alpha_0 f^n < \alpha_0 g$ for all $n \in \mathbf{Z}^+$ since Ω is rigid. Let $\bar{\Omega}$ be the Dedekind completion of the totally ordered set Ω and $\bar{\alpha} = \sup\{\alpha_0 f^n : n \in \mathbf{Z}^+\} \in \bar{\Omega}$. Define $\bar{f} \in \mathscr{A}(\Omega)$ by

$$\alpha\bar{f} = \begin{cases} \alpha f & \text{if} \quad \alpha < \bar{\alpha} \\ \alpha & \text{otherwise.} \end{cases}$$

Then $\bar{f}$ fixes $\alpha_0 g$ but not α_0, contradicting the fact that Ω is rigid.) Hence $\mathscr{A}(\Omega)$ is isomorphic (as a totally ordered group) to a subgroup of $\mathbf{R}$ and if we identify $\mathscr{A}(\Omega)$ with Ω as above, we have that each $g \in \mathscr{A}(\Omega)$ can be realized as a translation by an element of Ω (viz. $\alpha g = \alpha + (\alpha_0 g - \alpha_0)$ for all $\alpha \in \Omega$). Thus each element of $A(\Omega)$ is a translation.

For the rest of this paper, Ω will be a rigid homogeneous chain, and we assume that Ω is a subgroup of $\mathbf{R}$, the real numbers.

Let G be a torsion-free abelian group and p a prime. Let $\gamma_p(G)$ be the dimension of G/pG as a vector space over $\mathbf{Z}_p$, the Galois field of p elements. $\gamma_p(G)$ is called the *pth Szmielew invariant* of G. One way to construct torsion-free abelian groups with given Szmielew invariants $\{\gamma_p: p \text{ prime}\}$ is as follows: Let I_p be an index set for each prime p so that $|I_p| = \gamma_p$ and $I_p \cap I_q = \varnothing$ if p and q are distinct primes. Let $I = \cup\{I_p: p \text{ prime}\}$ and $\{\xi_i: i \in I\}$ be a linearly independent set of elements in a rational vector space of dimension greater than or equal to $|I|$. For each prime p, let $\mathbf{Q}_p$ be the set of rational numbers which, in simplest form, have denominators relatively prime to p. Note that

$$\gamma_q(\mathbf{Q}_p) = \begin{cases} 0 & \text{if} \quad q \neq p \\ 1 & \text{if} \quad q = p. \end{cases}$$

Let $G_p = \Sigma\{\xi_i \mathbf{Q}_p: i \in I_p\}$ and $G = \Sigma\{G_p: p \text{ prime}\}$. Then $\gamma_p(G) = \gamma_p$ for each prime p. If $I_p \neq \varnothing$ for some prime p and $|I| \leqslant 2^{\aleph_0}$, the group constructed above is a dense subgroup of $\mathbf{R}$, if we take $\mathbf{R}$ to be the rational vector space.

THEOREM (*Szmielew* (12)). *If G and H are torsion-free abelian groups, then G and H are elementarily equivalent if and only if, for each prime p, $\gamma_p(G) = \gamma_p(H)$ or both are infinite.*

We will write $\gamma_p(\Omega)$ for $\gamma_p(\mathscr{A}(\Omega))$ for notational convenience – since we are identifying $\mathscr{A}(\Omega)$ and Ω, this is especially permissible!

If Λ is homogeneous and $\mathscr{A}(\Lambda) \equiv \mathscr{A}(\Omega)$ (just as groups), then $\mathscr{A}(\Lambda)$ is also abelian. If $f \in \mathscr{A}(\Lambda)$ and $\lambda f = \lambda$ for some $\lambda \in \Lambda$, let $\sigma \in \Lambda$. There exists $g \in \mathscr{A}(\Lambda)$ such that $\sigma = \lambda g$. Now $\sigma f = \lambda g f = \lambda f g = \sigma$, so $f = e$. Hence Λ is a rigid homogeneous chain and, for each prime p, $\gamma_p(\Lambda) = \gamma_p(\Omega)$ or both are infinite.

Robinson and Zakon (10) have shown that if G and H are dense Archimedean totally ordered groups with $\gamma_p(G) = \gamma_p(H)$ or both infinite (for each prime p), then $G \equiv H$ (as lattice-ordered groups). Hence if Λ is homogeneous and $\mathscr{A}(\Lambda) \equiv \mathscr{A}(\Omega)$ (as groups), then $\mathscr{A}(\Lambda) \equiv \mathscr{A}(\Omega)$ (as lattice-ordered groups) – cf. a similar result for the other o-primitive class of ordered permutation groups (4).

Note that, for each prime p, the map $\alpha \mapsto p\alpha$ fixes $O(\in \mathbf{R})$. Hence not every element of Ω is divisible by p. Therefore $\gamma_p(\Omega) \geqslant 1$ for each prime p.

It remains to prove the existence and cardinality part of the theorem.

For more details of the above and the original proof of Theorem A (without specified Szmielew invariants) of the next section, see (8) or ((3), section 3·2).

2. THE ESSENTIAL PROOF

THEOREM A. *Let Ω_0 be a subgroup of $\mathbf{R}$ having all its Szmielew invariants at least 1. If $|\Omega_0| < 2^{\aleph_0}$, there exists a rigid homogeneous chain $\Omega \supseteq \Omega_0$ having the same Szmielew invariants as Ω_0.*

Note that the hypothesis implies $\gamma_p < 2^{\aleph_0}$ for each p.

Proof. First we may assume that Ω_0 is dense in $\mathbf{R}$ (if it is not, let $\xi \in \mathbf{R}$ be such that $\mathbf{Q}\xi \cap \Omega_0 = \{0\}$. $\Omega_0 \oplus \mathbf{Q}\xi$ has the same cardinality and Szmielew invariants as Ω_0). Let D_0 be the divisible closure of Ω_0 in $\mathbf{R}$ and $E_0 = \{0\}$.

Let $\{f_\lambda : 1 \leqslant \lambda < 2^{\aleph_0}\}$ be an enumeration of the elements of $\mathscr{A}(\mathbf{R})$ which are not translations. Define Ω_λ, D_λ and E_λ by induction on $\lambda (< 2^{\aleph_0})$ so that

(1) Ω_λ is the direct sum of Ω_0 and the divisible subgroup E_λ of $\mathbf{R}$.

(2) D_λ is a divisible subgroup of $\mathbf{R}$ containing Ω_λ.

(3) $|D_\lambda| \leqslant \max\{|\lambda|, |\Omega_0|\}$,

(4) if $\lambda' < \lambda$, then $E_{\lambda'} \subseteq E_\lambda$ and $D_{\lambda'} \backslash \Omega_{\lambda'} \subseteq D_\lambda \backslash \Omega_\lambda$,

(5) $(\Omega_\lambda f_\lambda \cup \Omega_\lambda f_\lambda^{-1}) \cap (D_\lambda \backslash \Omega_\lambda) \neq \varnothing$.

Then $\Omega = \cup \{\Omega_\lambda : \lambda < 2^{\aleph_0}\}$ will be the required rigid homogeneous chain having the same Szmielew invariants as Ω_0. (By (1), each Ω_λ is a subgroup of $\mathbf{R}$ having the same Szmielew invariants as Ω_0, so the same is true of Ω. If $f \in \mathscr{A}(\mathbf{R})$ is not a translation, $f = f_\lambda$ for some λ. By (5), there exists $\alpha \in \Omega_\lambda \subseteq \Omega$ such that αf or $\alpha f^{-1} \in D_\lambda \backslash \Omega_\lambda \subseteq D_\mu \backslash \Omega_\mu$ for all $\mu \geqslant \lambda$. Now αf (or αf^{-1}) $\notin \cup \{\Omega_\mu : \mu \geqslant \lambda\} = \Omega$, so Ω is rigid.)

Suppose D_λ, E_λ and Ω_λ have been defined for all $\lambda < \mu$ so as to satisfy (1)–(5). Let $X_\mu^* = \cup \{X_\lambda : \lambda < \mu\}$ where $X = D$, E or Ω. Then D_μ^*, E_μ^* and Ω_μ^* are subgroups of $\mathbf{R}$ with D_μ^* and E_μ^* divisible, and $\Omega_\mu^* = E_\mu^* \oplus \Omega_0$. Let $f = f_\mu$.

Let $A = \{\alpha \in \mathbf{R} : \alpha f = \alpha + \sigma$ for some $\sigma \in \Omega_\mu^*\}$, $B = \{\alpha \in \mathbf{R} : \alpha f = -\alpha + \sigma$ for some $\sigma \in \Omega_\mu^*\}$ and $C = \{\alpha \in \mathbf{R} : \alpha f = q\alpha + \sigma$ for some $\sigma \in \Omega_\mu^*$ and $q \in \mathbf{Q} \backslash \{0, 1, -1\}\}$.

(i) If $\mathbf{R} \neq A \cup B \cup C \cup \Omega_\mu^* f^{-1} \cup D_\mu^*$, let $\alpha \in \mathbf{R} \backslash (A \cup B \cup C \cup \Omega_\mu^* f^{-1} \cup D_\mu^*)$. Then since $E_\mu^* \subseteq \Omega_\mu^* \subseteq D_\mu^*$, $E_\mu = E_\mu^* \oplus \alpha\mathbf{Q}$ is indeed a direct sum as is $\Omega_\mu = \Omega_0 \oplus E_\mu$. If $\alpha f \in \Omega_\mu$, then $\alpha f = q\alpha + \sigma$ for some $\sigma \in \Omega_\mu^*$ and $q \in \mathbf{Q}$. Hence $\alpha \in A \cup B \cup C \cup \Omega_\mu^* f^{-1}$, a contradiction. Thus (1)–(5) hold for all $\lambda \leqslant \mu$ if we let D_μ be the divisible closure of the subgroup of $\mathbf{R}$ generated by $D_\mu^* \oplus \alpha\mathbf{Q}$ and αf.

(ii) If $\mathbf{R} = A \cup B \cup C \cup \Omega_\mu^* f^{-1} \cup D_\mu^*$, let $h : \alpha \mapsto \alpha f - \alpha$. Then h is continuous and, since f is not a translation, h is not constant. Hence $\mathbf{R}h$ contains an interval. But $(\mathbf{R} \backslash A) h \supseteq \mathbf{R}h \backslash Ah \supseteq \mathbf{R}h \backslash \Omega_\mu^*$ since $Ah \subseteq \Omega_\mu^*$. Since $|\mathbf{R}h| = 2^{\aleph_0}$ and $|\Omega_\mu^*| \leqslant |D_\mu^*| \leqslant \max \{|\mu|, |\Omega_0|\} < 2^{\aleph_0}$, $|\mathbf{R} \backslash A| = 2^{\aleph_0}$.

Now $k : \alpha \mapsto \alpha f + \alpha$ is a strictly increasing function. Hence, for a given $\sigma \in \Omega_\mu^*$, there is at most one $\alpha \in \mathbf{R}$ such that $\alpha k = \sigma$. Thus $|B| \leqslant |\Omega_\mu^*| < 2^{\aleph_0}$.

Consequently, $|C| = 2^{\aleph_0}$, and so $C \backslash D_\mu^* \neq \varnothing$.

If there exists $\delta \in C \backslash D_\mu^*$ with $\delta f = q\delta + \sigma$ for some $\sigma \in \Omega_\mu^*$ and $q \notin \mathbf{Z}$, let $\alpha = \delta$. Otherwise $\alpha = \delta f$ where δ is an arbitrary member of $C \backslash D_\mu^*$. In the latter case, $\alpha = \delta f = m\delta + \sigma$ for some $\sigma \in \Omega_\mu^*$ and $0, \pm 1 \neq m \in \mathbf{Z}$ ($\delta \in C$ so $q \neq 0, \pm 1$). Therefore $\alpha f^{-1} = \delta = \alpha - \sigma/m$. But $\sigma \in D_\mu^*$ and $\delta \notin D_\mu^*$, so $\alpha \notin D_\mu^*$. In either case we have $\alpha \notin D_\mu^*$ with αf (or αf^{-1}) having the form $\frac{k}{n}\alpha + \sigma'$ with $\sigma' \in D_\mu^*$ and k, n relatively prime integers with $n \neq 1$.

If $\sigma' \notin \Omega_\mu^*$, let $E_\mu = E_\mu^* \oplus \alpha\mathbf{Q}$, $\Omega_\mu = \Omega_0 \oplus E_\mu$, and D_μ be the divisible subgroup of R generated by $D_\mu^* \oplus \alpha\mathbf{Q}$ and $\beta (= \alpha f$ or $\alpha f^{-1})$ – the indicated sums being indeed direct. It is straightforward to show that (1)–(5) now hold for all $\lambda \leqslant \mu$ (if $\beta \in \Omega_\mu$, then $\frac{k}{n}\alpha + \sigma' = r\alpha + \tau$ for some $r \in \mathbf{Q}$ and $\tau \in \Omega_\mu^*$. Since $\alpha \notin D_\mu^*$, $\frac{k}{n} = r$ and $\sigma' = \tau \in \Omega_\mu^*$, a contradiction).

If $\sigma' \in \Omega_\mu^*$, let p be a prime such that $p | n$. Let $\xi \in \Omega_0$ be such that $\frac{\xi}{p} \in D_0 \backslash \Omega_0$ ($\gamma_p > 0$ for all primes p, so such a ξ must exist). Let $\zeta = \alpha - \xi$. Then $\zeta \notin D_\mu^*$. Let $E_\mu = E_\mu^* \oplus \zeta\mathbf{Q}$,

$\Omega_\mu = \Omega_0 \oplus E_\mu$ and D_μ be the divisible subgroup of $\mathbf{R}$ generated by D_μ^*, ζ and β ($= \alpha f$ or αf^{-1}) – the indicated sums being indeed direct as is easily shown. Again, a routine verification shows that (1)–(5) now hold for all $\lambda \leqslant \mu$. (Note that $\alpha \in \Omega_\mu$ and $\beta \notin \Omega_\mu$ – if $\beta \in \Omega_\mu$, $\frac{k}{n}\alpha + \sigma' = \beta = \sigma + r\zeta$ for some $r \in \mathbf{Q}$ and $\sigma \in \Omega_\mu^*$. Since $\alpha \notin D_\mu^*$, $\frac{k}{n} = r$. So $\frac{k}{n}\xi \in \Omega_\mu^*$, and hence $\frac{k}{n}\xi \in \Omega_0$, a contradiction.)

This completes the proof of the theorem.

The idea of enumerating the elements of $\mathscr{A}(\mathbf{R})$ (which are not translations) and killing them off one by one is not new. Its origins date back to the isolation of the well-ordering principle around the turn of the century. A specific reference predating Ohkuma is (2).

3. Applications (adaptations of the method)

We now make some minor modifications to the proof of Theorem A to obtain further results concerning the existence and number of certain rigid homogeneous chains. Our first application is to handle the case when some Szmielew invariants actually attain $2^{\aleph_0}$.

THEOREM 1. *Let $\{\gamma_p : p$ prime$\}$ be a set of cardinal numbers with $1 \leqslant \gamma_p \leqslant 2^{\aleph_0}$ for all primes p. There exists a rigid homogeneous chain having $\{\gamma_p : p$ prime$\}$ as its set of Szmielew invariants.*

Proof. Let $I = \{p : p$ prime and $\gamma_p = 2^{\aleph_0}\}$. Let Ω_0 be a dense subgroup of $\mathbf{R}$ having Szmielew invariants γ'_p such that $1 \leqslant \gamma'_p < 2^{\aleph_0}$ and $\gamma'_p = \gamma_p$ for all primes $p \notin I$, with $|\Omega_0| < 2^{\aleph_0}$. Let D_0 be the divisible closure of Ω_0. If $\mu = \nu + p$ for ν a limit ordinal and $p \in I$, let $\tau \in \mathbf{R}\backslash D_{\nu+p-1}$. Let $E_\mu^* = E_{\nu+p-1} \oplus \tau \mathbf{Q}_p$, $\Omega_\mu^* = \Omega_0 \oplus E_\mu^*$, and D_μ^* be the divisible closure of $\Omega_\mu^* \cup D_{\nu+p-1}$. For all μ not of this form, let Ω_μ^*, D_μ^* and E_μ^* be defined as in the proof of Theorem A. Now proceed as before using the new Ω_μ^*, D_μ^*, and E_μ^* in place of the old ones.

THEOREM 2. *Let $\{\gamma_p : p$ prime$\}$ be a set of cardinal numbers with $1 \leqslant \gamma_p \leqslant 2^{\aleph_0}$ for all primes p. There exists a rigid homogeneous chain Ω having $\{\gamma_p : p$ prime$\}$ as its Szmielew invariants such that for each $f \in \mathscr{A}(\mathbf{R})$ not a translation $(\Omega f \cup \Omega f^{-1}) \cap (D\backslash\Omega) \neq \varnothing$, where D is the divisible closure of Ω.*

Proof. We proceed as in the proof of Theorem 1 except that we require D_λ to be the divisible closure of Ω_λ in (2). In order to ensure this at each stage of the induction we proceed exactly as before, constructing first a D_μ which is possibly not the divisible closure of Ω_μ. If $\beta = \alpha f_\mu$ (or αf_μ^{-1}), belongs to the divisible closure of Ω_μ, let Ω_μ and E_μ be as before and D_μ be the divisible closure of Ω_μ. Otherwise, we change Ω_μ, D_μ and E_μ as follows: Let $\xi \in \Omega_0$ be such that $\frac{1}{2}\xi \notin \Omega_0$. Adjoin to E_μ (and hence to Ω_μ) $(\beta - \frac{1}{2}\xi)\,\mathbf{Q}$ and replace D_μ by the divisible closure of the new Ω_μ. Note that $(\beta - \frac{1}{2}\xi)\,\mathbf{Q}$ is adjoined as a direct summand. Further, since $\frac{1}{2}\xi \in D_0\backslash\Omega_0$, β belongs to the new D_μ but not the new Ω_μ. Thus the modified (1)–(5) hold for all $\lambda \leqslant \mu$. Now $\Omega = \cup\{\Omega_\lambda : \lambda < 2^{\aleph_0}\}$ and $D = \cup\{D_\lambda : \lambda < 2^{\aleph_0}\}$ satisfy the conclusion of the theorem.

We now answer a question of Stephen H. McCleary (see (3), page 108).

COROLLARY 3. *Let* $\{\gamma_p: p \text{ prime}\}$ *be a set of cardinal numbers with* $1 \leqslant \gamma_p \leqslant 2^{\aleph_0}$ *for all primes p. There exists a rigid homogeneous chain* Ω *having* $\{\gamma_p: p \text{ prime}\}$ *as its Szmielew invariants such that* $\mathbf{R}$ *is the divisible closure of* Ω.

Proof. Let Ω and D be as in Theorem 2. There exists a divisible subgroup H of $\mathbf{R}$ such that $\mathbf{R} = D \oplus H$. $\Omega \oplus H$ is the desired rigid homogeneous chain.

We now further modify Theorem 2 to help us count the number of isomorphism classes of rigid homogeneous chains of given Szmielew invariants. This proof will not depend on extra set-theoretic assumptions (cf. section 4).

THEOREM 4. *Let* $\{\gamma_p: p \text{ prime}\}$ *be a set of cardinal numbers with* $1 \leqslant \gamma_p \leqslant 2^{\aleph_0}$ *for all primes p. There is a rigid homogeneous chain* Ω *satisfying the conclusion of Theorem 2 such that* $\mathbf{R} = D \oplus \Delta$, *where D is the divisible closure of* Ω *and* $|\Delta| = 2^{\aleph_0}$.

Proof. Let $\{\xi_\nu: \nu < 2^{\aleph_0}\}$ be a basis for $\mathbf{R}$ as a vector space over $\mathbf{Q}$. We construct $\Delta_\lambda (\lambda < 2^{\aleph_0})$ at the same time as we construct Ω_λ, D_λ and E_λ so that Ω_λ, D_λ and E_λ satisfy the modified (1)–(5) of the proof of Theorem 2 and Δ_λ satisfies

(*a*) $\Delta_\lambda \cap D_\lambda = \{0\}$,
(*b*) Δ_λ is a divisible subgroup of $\mathbf{R}$,
(*c*) $\Delta_{\lambda'}$ is a proper direct summand of Δ_λ for each $\lambda' < \lambda$, and
(*d*) $|\Delta_\lambda| = \max\{\aleph_0, |\lambda|\}$.

Let Ω_0, D_0 and E_0 be as in the proof of Theorem 1. Let ν_0 be the least ν such that $\xi_\nu \notin D_0$ and define $\Delta_0 = \mathbf{Q}\xi_{\nu_0}$. Assume that Ω_λ, D_λ, E_λ and Δ_λ have been defined for all $\lambda < \mu$ so that (*a*)–(*d*) hold as well as the modified (1)–(5). Let $X^* = \cup \{X_\lambda: \lambda < \mu\}$ where $X = \Omega, D, E$ or Δ. As in the proof of Theorem 2, we can find $\alpha, \zeta \notin D_\mu^* \oplus \Delta_\mu^*$ such that $\alpha \in \Omega_\mu^* \oplus \zeta\mathbf{Q}$ and $\beta = \alpha f_\mu$ (or $\alpha f_\mu^{-1}) \notin \Omega_\mu^* \oplus \zeta\mathbf{Q} \oplus \Delta_\mu^*$ (all the indicated sums being indeed direct). As before, we can find a divisible E_μ containing E_μ^* as a direct summand such that $\Omega_\mu = \Omega_0 \oplus E_\mu$ has the same cardinality and Szmielew invariants as Ω_μ^* and $\beta \in D_\mu \backslash \Omega_\mu$, where D_μ is the divisible closure of Ω_μ. In either case ($\beta \in (\notin) D_\mu^* \oplus \zeta\mathbf{Q}$), the proof given in Theorem 2 yields D_μ intersecting Δ_μ^* trivially. Let $\bar{\nu}$ be the least ordinal ν such that $\xi_\nu \notin D_\mu \oplus \Delta_\mu^*$, and $\Delta_\mu = \Delta_\mu^* \oplus \mathbf{Q}\xi_{\bar{\nu}}$. Then Ω_λ, D_λ and Δ_λ satisfy (*a*)–(*d*) and the modified (1)–(5) for all $\lambda \leqslant \mu$. Thus $\Omega = \cup \{\Omega_\lambda: \lambda < 2^{\aleph_0}\}$, $D = \cup \{D_\lambda: \lambda < 2^{\aleph_0}\}$ and $\Delta = \cup \{\Delta_\lambda: \lambda < 2^{\aleph_0}\}$ satisfy the conclusion of the theorem.

THEOREM 5. *Let* $\{\gamma_p: p \text{ prime}\}$ *be a set of cardinal numbers with* $1 \leqslant \gamma_p \leqslant 2^{\aleph_0}$ *for all primes p. There are* 2^c *isomorphism classes of rigid homogeneous chains which have* $\{\gamma_p: p \text{ prime}\}$ *as Szmielew invariants, where* $c = 2^{\aleph_0}$.

Proof: Since any order-preserving isomorphism between totally ordered subgroups of $\mathbf{R}$ is obtained by multiplication by a real number and any isomorphism between two rigid homogeneous chains (considered as subgroups of $\mathbf{R}$) is a group isomorphism followed by a translation, there are at most $2^{\aleph_0}$ rigid homogeneous chains in any one isomorphism class. Hence it is enough to show that there are 2^c distinct rigid homogeneous chains having $\{\gamma_p: p \text{ prime}\}$ as Szmielew invariants. Let Ω, D and Δ satisfy the conclusion of Theorem 4, and $\{\xi_\nu: \nu < 2^{\aleph_0}\}$ be a basis for the rational vector space Δ.

For each $X \subseteq 2^{\aleph_0}$, let $\Omega_X = \Omega \oplus \Sigma\{\mathbf{Q}\xi'_\nu : \nu \in X\}$. Then Ω_X is a rigid homogeneous chain having $\{\gamma_p : p \text{ prime}\}$ as its Szmielew invariants. Clearly $\Omega_X \neq \Omega_Y$ if X and Y are distinct subsets of $2^{\aleph_0}$. Thus there are 2^c rigid homogeneous chains having the desired $\{\gamma_p : p \text{ prime}\}$ as Szmielew invariants.

An alternative proof of Theorem 5 can be given by modifying Ohkuma's original proof to include the invariants. This is conceptually easier but messier in its details. In the proof of Theorem 1, at each stage μ we have at least two rationally independent choices α^1_μ and α^2_μ for α. We can either ensure $\alpha^1_\mu \in \Omega_\mu$ and $\alpha^2_\mu \in D_\mu \backslash \Omega_\mu$ or vice versa. In this way we obtain a distinct Ω for each $\phi : 2^{\aleph_0} \to 2 = \{0, 1\}$. This yields 2^c distinct rigid homogeneous chains of the given Szmielew invariants. For details see (8) or (3), 113.

We complete this section with some measure-theoretic results about dense rigid homogeneous chains. The first is easy.

THEOREM 6. *There is a dense rigid homogeneous chain that is not Lebesgue measurable.*

Proof. Since every perfect subset of $\mathbf{R}$ is the closure of a countable set, there are $2^{\aleph_0}$ distinct non-empty perfect subsets of $\mathbf{R}$. Let $\{Y_\mu : \mu < 2^{\aleph_0}\}$ be an enumeration of them. Let D_μ, E_μ and Ω_μ be as constructed in the proof of Theorem 1. Since each Y_μ has cardinality $2^{\aleph_0}$, there is $\eta_\mu \in Y_\mu \backslash D_\mu$. Take $\Omega_\mu \oplus \eta_\mu \mathbf{Q}$, $D_\mu \oplus \eta_\mu \mathbf{Q}$ and $E_\mu \oplus \eta_\mu \mathbf{Q}$ in place of Ω_μ, D_μ and E_μ respectively and build a dense rigid homogeneous chain with them; viz. $\Omega = \cup\{\Omega_\mu : \mu < 2^{\aleph_0}\}$. Note that Ω intersects every non-empty perfect subset of $\mathbf{R}$ and hence is not Lebesgue measurable. (Otherwise, Ω, being a proper subgroup of $\mathbf{R}$ has measure 0. Therefore, its complement contains a non-empty perfect subset of $\mathbf{R}$, a contradiction.)

Note that the proof, combined with that of Theorem 5 actually gives 2^c non-measurable dense rigid homogeneous chains of any prescribed $\{\gamma_p : p \text{ prime}\}$ with $1 \leqslant \gamma_p \leqslant 2^{\aleph_0}$.

The question of whether every dense rigid homogeneous chain is not Lebesgue measurable remains open. The only result we have in this direction is the following.

Let ω be the set of non-negative integers. Then

THEOREM 7. *Let Ω be a dense rigid homogeneous chain. Then Ω cannot be covered by the union of intervals K_n $(n \in \omega)$ where K_n has length less than 9^{-n}.*

Proof. Let $A \subseteq \Omega$ and $B \subseteq \mathbf{R}\backslash\Omega$ be countable dense subsets of $\mathbf{R}$. Let $\mathscr{F}$ be the collection of all strictly increasing functions f whose domain is a finite union of non-adjacent intervals I with end-points in B such that $f|I$ is translation by an element a_I of A. For each $f \in \mathscr{F}$, let $\sigma(f)$ be the set of intervals I whose finite union is the domain of f, and $\tau(f) = \{If : I \in \sigma(f)\}$. Let $\sigma'(f)$ (respectively $\tau'(f)$) be the set of maximal real intervals disjoint from $\operatorname{dom}(f)$ (respectively $\operatorname{rng}(f)$). Let $m_0(f)$ (resp. $m_1(f)$) be the minimal length of intervals in $\sigma'(f)$ (resp. $\tau'(f)$), and

$$m(f) = \min\{m_0(f), m_1(f)\}.$$

We next show:

LEMMA. *If $f \in \mathscr{F}$ and K is a non-empty interval of length $< \frac{1}{3}m(f)$, then there are $g, h \in \mathscr{F}$ extending f such that $K \subseteq \operatorname{dom}(g)$, $K \subseteq \operatorname{rng}(h)$ and $m(g), m(h) > \frac{1}{3}m(f)$.*

Proof of Lemma. By symmetry, it is enough to construct such a g. Since B is dense in $\mathbf{R}$, we may assume that the endpoints of K belong to B, without loss of generality, and, moreover, that there are consecutive intervals $I, J \in \sigma(f)$ with $I < K < J$.

If $\sup I = \inf K$, let g be defined by

$$xg = \begin{cases} xf & \text{if } x \in \operatorname{dom}(f) \\ x + a_I & \text{if } x \in [\inf K, \sup K). \end{cases}$$

Thus I and K together with their point of adjacency have become one interval of $\sigma(g)$ and $g \in \mathscr{F}$. A similar construction works in case $\sup K = \inf J$. In all other cases, we extend f to a g for which $K \in \sigma(g)$ and so that for $x \in K$, $xg = x + a$ with $a \in A$,

$$(\sup I) + a_I + \tfrac{1}{3}m(f) < K + a < (\inf J) + a_J - \tfrac{1}{3}m(f).$$

We can now complete the proof of Theorem 7. Assume $\{K_n : n \in \omega\}$ covers Ω with length $(K_n) < 9^{-n}$. We construct $\{f_n : n \in \omega\} \subseteq \mathscr{F}$ so that

(i) $\sigma(f_0) = \{I, J\}$ for some $I < J$, $a_I \neq a_J$, $\inf J - \sup I > 9$, $(\inf J + a_J) - (\sup I + a_I) > 9$, and $K_0 \subseteq \operatorname{dom}(f_0)$.

(ii) f_n extends f_m if $n \geqslant m$.

(iii) $m(f_n) > 3^{-n+1}$ and

(iv) $K_n \subseteq \operatorname{dom}(f_{2n}) \cap \operatorname{rng}(f_{2n+1})$.

This can be done by the lemma. Let $f = \bigcup \{f_n : n \in \omega\}$; i.e., $\operatorname{dom}(f) = \bigcup \{\operatorname{dom}(f_n) : n \in \omega\}$ and if $x \in \operatorname{dom}(f_n)$, $xf = xf_n$ (f is well-defined by (ii) and strictly increasing). Since $\{K_n : n \in \omega\}$ covers Ω and $\Omega \subseteq \operatorname{dom}(f) \cap \operatorname{rng}(f)$, $\Omega f = \Omega$ by construction. But f is not a translation ($a_I \neq a_J$), so Ω is not rigid, the desired contradiction.

4. A SET-THEORETIC CONSIDERATION

In this section we prove results about rigid homogeneous chains which depend on the particular model of set theory we consider. In particular, we will prove that the existence of dense rigid homogeneous chains of cardinality less than the continuum is independent of the axioms of *ZFC*. For all unexplained set-theoretic terms, see (6).

Do there exist small rigid homogeneous chains (other than $\mathbf{Z}$)? That is, do there exist dense rigid homogeneous chains Ω with $|\Omega| < 2^{\aleph_0}$? If so, for each cardinal $\kappa < 2^{\aleph_0}$, how many of cardinality κ are there? If we assume the continuum hypothesis there are, of course, none. (If $\kappa < 2^{\aleph_0} = \aleph_1$, then $\kappa \leqslant \aleph_0$ and any countable dense subset of $\mathbf{R}$ is isomorphic to $\mathbf{Q}$.) Observe that if Ω is a dense rigid homogeneous chain, then it is $|\Omega|$-dense; i.e., any non-empty open interval of Ω contains $|\Omega|$ points of Ω. In (1), Baumgartner gave a model of set theory in which $2^{\aleph_0} = \aleph_2$ and any two $\aleph_1$-dense sets of reals are isomorphic. In such a model, there can be no dense rigid homogeneous chains of cardinality $\aleph_1$ since there exist non-rigid homogeneous $\aleph_1$-dense chains of cardinality $\aleph_1$ (Take the divisible subgroup of $\mathbf{R}$ generated by $\aleph_1$ rationally independent elements of $\mathbf{R}$). Hence in such a model, there are no small dense rigid homogeneous chains. Actually:

THEOREM 8. *Martin's axiom implies that there are no dense rigid homogeneous chains of cardinality less than* $2^{\aleph_0}$.

Proof. Suppose Ω is a dense subgroup of $\mathbf{R}$ with $|\Omega| < 2^{\aleph_0}$. Then $\mathbf{R}\backslash\Omega$ is dense in $\mathbf{R}$ and we can choose A, B countable dense subsets of $\mathbf{R}$ with $A \subseteq \Omega$ and $B \subseteq \mathbf{R}\backslash\Omega$. Let P be the set of all strictly increasing functions p whose domain is a finite union of non-adjacent intervals I with endpoints in B such that $p|I$ is translation by an element of A. Then P is a partially ordered set if we define $p \leqslant q$ if and only if p extends q. Since P is countable, it satisfies the countable (anti) chain condition. Moreover,

$$D_\alpha = \{p \in P \colon \alpha \in \operatorname{dom}(p)\} \quad \text{and} \quad R_\alpha = \{p \in P \colon (\exists \beta \in \Omega)(\beta p = \alpha)\}$$

are dense subsets of P for each $\alpha \in \Omega$, since B and A (respectively) are dense subsets of $\mathbf{R}$. Also, $D = \{p \in P \colon \exists\alpha\exists\beta(\alpha p - \alpha \neq \beta p - \beta)\}$ is dense in P. By Martin's axiom, there is a filter $G \subseteq P$ such that $G \cap D \neq \varnothing$ and $G \cap D_\alpha \neq \phi \neq G \cap R_\alpha$ for all $\alpha \in \Omega$ ($|\Omega| < 2^{\aleph_0}$). This provides a function g (that is not a translation) such that $\Omega \subseteq \operatorname{dom}(g)$, $\Omega \subseteq \Omega g$, and g extends to an element of $\mathscr{A}(\mathbf{R})$. For all $\alpha \in \Omega$, $\alpha g = \alpha g_\alpha = \alpha + \sigma_\alpha$ for some $\sigma_\alpha \in A$; so $\alpha g \in \Omega$. Thus $\Omega g = \Omega$. Since g is not a translation, Ω is not a rigid homogeneous chain.

The proof given above only uses Martin's axiom for countable p.o. sets. This weak form of Martin's axiom is equivalent to: 'The real line is not a union of less than continuum many nowhere dense subsets.' So the conclusions of Theorem 8 follow from this most natural axiom.

In contrast to Theorem 8.

THEOREM 9. *Consecutively adding ω_1 random reals to a transitive model of ZFC provides a rigid homogeneous chain of cardinality $\aleph_1$. (See below for the definition of random real.)*

Proof. Let M be a transitive model of ZFC. Let P_1 be the p.o. set of Borel subsets of $\mathbf{R}$ (in M) modulo sets of Lebesgue measure zero. We force by P_1 as described in (6), § 20. Let G_1 be a generic subset of P_1 and $\xi_1 = \sup\{r \in \mathbf{Q} \colon (r, \infty)/\mathrm{Null} \in G_1\}$, where Null is the ideal of all subsets of $\mathbf{R}$ (in M) of Lebesgue measure 0. Then $M[G_1] = M[\xi_1]$. ξ_1 is called the *random real* corresponding to G_1. For any $r \in \mathbf{Q}$, $\xi_1 + r$ is also a random real (corresponding to the translated G_1). Hence the set of random reals is dense in the real line of $M[G_1]$.

As P_1 satisfies the countable (anti) chain condition (c.c.c.), the cardinals of $M[G_1]$ are the same as those of M.

Using the Solovay–Tennenbaum iterated forcing procedure (see (6) § 22); we construct a notion of forcing, P_λ, for each $\lambda \leqslant \omega_1$: If $\lambda = \mu + 1 \geqslant 2$, let $P_\lambda = P_\mu * \mathbf{P}_1^\mu$, where $\mathbf{P}_1^\mu$ is a name (with respect to P_μ), and in $M \Vdash_{P_\mu} (\mathbf{P}_1^\mu$ is the p.o. set of Borel sets modulo the null sets). If λ is a limit ordinal, let P_λ be the set of functions f whose domain is a finite subset of λ satisfying: if $\mu \in \operatorname{dom}(f)$, then $\mu f \in P_\mu$. (Forcing with P_λ adds reals $\{\xi_\mu \colon \mu < \lambda\}$). Let $P = P_{\omega_1}$. Each P_λ is c.c.c. ($\lambda < \omega_1$), and hence so is P (see (6), § 22).

Let G be a generic subset of P adding $\xi_\lambda \colon \lambda < \omega_1$. Note that the cardinals of $M[G]$ are precisely those of M. Let C be the additive subgroup of $\mathbf{R}$ (in $M[G]$) generated by $\mathbf{Q}$ and $\{\xi_\lambda \colon \lambda < \omega_1\}$. We prove that C is a rigid homogeneous chain, which establishes the theorem.

For reductio ad absurdum, assume that C is not a rigid homogeneous chain in $M[G]$. Let $f \in M[G]$ be such that $f \in \mathscr{A}(\mathbf{R})^{M[G]}$, f is not a translation and $Cf = C$. Without loss of generality, there exist $x, y \in Q$ such that $x < y$ and $yf - xf < y - x$. $X \subseteq [x, y]$ will be called *good* if X is a closed set of strictly positive Lebesgue measure μX, and there is $n \in \mathbf{Z}^+$ such that $x_1 f - x_2 f = n(x_1 - x_2)$ for all $x_1, x_2 \in X$. Let $\mathscr{F}$ be a maximal family of disjoint good sets. Then $|\mathscr{F}| \leqslant \aleph_0$, so

$$\mu(\cup \mathscr{F}) = \Sigma\{\mu X\colon X \in \mathscr{F}\} \leqslant \Sigma\{\mu(Xf)\colon X \in \mathscr{F}\} \leqslant yf - xf < y - x.$$

Hence there is $D_0 \subseteq [x,y] \backslash \cup \mathscr{F}$ of strictly positive measure. Since any set of strictly positive measure contains a closed subset of strictly positive measure, we may assume that D_0 is closed.

There is $p_0 \in P$ forcing that $\mathbf{f}$ (a P name for f) is a counterexample (to C being a rigid homogeneous chain), and $\mathscr{F}$ is a maximal family of disjoint good subsets of $[x,y]$. For every $r \in \mathbf{Q}$ and $n \in \omega$, there is a condition defining the nth component of rf (Representing $\mathbf{R}$ by elements of ${}^{\omega}2$). The set of such conditions is dense. Therefore there is a maximal antichain of P (which must be countable since P is c.c.c.) contained in this set. Since each $X \in \mathscr{F}$ is closed (and hence Borel), it is coded by a real number. It now follows easily that there is $\lambda < \omega_1$ such that p_0, $\mathscr{F}$, $f \mid \mathbf{Q} \in M_\lambda = M[\{\xi_\mu\colon \mu < \lambda\}]$. $M_\lambda = M[G_\lambda]$ for some generic $G_\lambda \subseteq P_\lambda$. Let $\mathscr{P} \in M_\lambda$ be the quotient notion of forcing P/P_λ; i.e., $P = P_\lambda * \mathscr{P}$. Let $\mathscr{P}_1$ be the p.o. set of Borel sets modulo Null in M_λ. The elements of $\mathscr{P}$ can be identified with finite functions q such that

$$\operatorname{dom}(q) \subseteq \{\mu\colon \lambda \leqslant \mu < \omega_1\}.$$

So for each $D \in \mathscr{P}_1$, $\langle \lambda, D\rangle \in \mathscr{P}$.

Let G' be a generic subset of $\mathscr{P}$ containing $\langle \lambda, D_0\rangle/\mathrm{Null}$. Since $\xi_\lambda \in C$ and $Cf = C$, $\xi_\lambda f = n\xi_\lambda + a$ in $M_\lambda[G']$ where $a = n_1 \xi_{\lambda_1} + \ldots + n_k \xi_{\lambda_k} + a'$ with $a' \in \mathbf{Q}$, $n \in \omega$, $n_1, \ldots, n_k$ non-zero integers, $\lambda_1 < \ldots < \lambda_k$ and $\lambda \notin \{\lambda_1, \ldots, \lambda_k\}$. If $\lambda < \lambda_k$, then ξ_{λ_k} is known in $M_\lambda[\{\xi_\lambda, \xi_{\lambda_1}, \ldots, \xi_{\lambda_{k-1}}\}]$ which is absurd. Hence $a \in M_\lambda$ and there is $q \in \mathscr{P}$ such that $\langle \lambda, D_0\rangle/\mathrm{Null} \in q$ and $q \Vdash_{\mathscr{P}} \boldsymbol{\xi}_\lambda \mathbf{f} = n\boldsymbol{\xi}_\lambda + a$. Let $D_0' = D_0 \backslash D$ where $D = \cup\{I \cap D_0 \colon I$ is a rational open interval with $I \cap D_0$ having Lebesgue measure $0\}$. Then for each rational interval I, $I \cap D_0' = \varnothing$ or $I \cap D_0'$ has strictly positive measure. So, without loss of generality, D_0 enjoys this property. But as

$$D_0/\mathrm{Null} \Vdash_{\mathscr{P}_1} \boldsymbol{\xi}_\lambda \mathbf{f} = n\boldsymbol{\xi}_\lambda + a, \quad (I \cap D_0)/\mathrm{Null} \Vdash_{\mathscr{P}_1} \boldsymbol{\xi}_\lambda \mathbf{f} = n\boldsymbol{\xi}_\lambda + a$$

whenever $I \cap D_0 \neq \varnothing$ (I a rational interval). Indeed D_0/Null forces that the equation $zf = nz + a$ has a solution in each rational interval meeting D_0. Since f is continuous, $zf = nz + a$ for all $z \in D_0$. Thus D_0 is good which contradicts the maximality of $\mathscr{F}$. Consequently, C is a rigid homogeneous chain (of cardinality $\aleph_1$), as desired.

The proof of Theorem 9 can easily be modified to replace ω_1 and $\aleph_1$ by any uncountable cardinal $\kappa \leqslant 2^{\aleph_0}$. Indeed, the ideas of section 3 can be incorporated with the above proof to show that if κ is an uncountable cardinal no bigger than $2^{\aleph_0}$, there is a model of ZFC (dependent on κ) in which there are 2^κ pairwise non-isomorphic rigid homogeneous chains of cardinality κ. The best result in this direction that we are able to obtain is:

THEOREM 10. *Let M be a transitive model of ZFC. There is a notion of forcing, P, in M such that if $G \subseteq P$ is generic, then*

1. *The cardinals of $M[G]$ are those of M,*

2. $(2^{\aleph_0})^{M[G]} = (2^{\aleph_0})^M$,

and

3. *In $M[G]$, for any $\aleph_0 < \kappa \leqslant 2_{\aleph^0}$, there are 2^κ pairwise non-isomorphic rigid homogeneous chains of cardinality κ.*

The proof is a generalization of that of Theorem 9 but involves a more complex iteration of random reals than the Solovay–Tennenbaum one used above. Since the proof is so technical, we have decided to exclude it.

Finally, as a consequence of Theorems 8 and 9 we have:

THEOREM. *The existence of small dense rigid homogeneous chains is independent of the axioms of ZFC. Specifically, Consis (ZFC) $\rightarrow$ Consis(ZFC + $\neg$CH + there are no rigid homogeneous chains of cardinality $\aleph_1$) and Consis(ZFC) $\rightarrow$ Consis(ZFC + $\neg$CH + there exists a rigid homogeneous chain of cardinality $\aleph_1$).*

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