ELEMENTARY THEORY OF AUTOMORPHISM GROUPS OF DOUBLY HOMOGENEOUS CHAINS

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1. INTRODUCTION.

At the Logic Meeting in Storrs, Connecticut (November 1979), the first author presented a survey of the research being done classifying linearly ordered sets by the elementary theory of their automorphism groups. In this paper we wish to present some aspects of our more recent research which may be of interest to logicians. A more up-to-date survey will appear in the Proceedings of the Algebra Conference, Carbondale (1980)--to be published by Springer-Verlag in their lecture note series.

In the early 1970's, (unordered) sets were classified by the elementary (first order) group-theoretic properties of their symmetric groups--i.e., automorphism groups (see [10], [11] and [13]). A natural extension of this work is to try to classify structures of a given signature by the first order properties of their automorphism groups. One such problem is to take the models as $\langle \Omega, \leq \rangle$, linearly ordered sets (or chains, for short). Besides its obvious naturalness, there are two further reasons to study the automorphisms of chains. The first is that if Σ is any set of sentences (of a first order language) having an infinite model and $\langle \Omega, \leq \rangle$ is a chain, there is a model $\mathfrak{A}_\Omega$ of Σ containing Ω as a subset such that each automorphism of $\langle \Omega, \leq \rangle$ extends to an automorphism of the model $\mathfrak{A}_\Omega$; i.e., $\mathrm{Aut}(\langle \Omega, \leq \rangle)$ is a subgroup of $\mathrm{Aut}(\mathfrak{A}_\Omega)$ (See [12] for this and further motivating reasons for model theorists). The second reason is provided by the theorem [3; Appendix I] that every lattice-ordered group can be embedded in the automorphism group of a chain. We will write $\mathcal{A}(\Omega)$ for $\mathrm{Aut}(\langle \Omega, \leq \rangle)$; i.e., $\mathcal{A}(\Omega)$ is the group of all order-preserving permutations of the chain Ω. The classification in this case was begun in [7], [8] and [5]. This paper provides further results.

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The presence of linear ordering complicates matters in two ways. Whereas the symmetric group on a set is always transitive, the same is not necessarily true of $\mathcal{A}(\Omega)$--e.g., if $\Omega = \omega$, then $\mathcal{A}(\Omega) = \{e\}$ (e is the identity element of the group $\mathcal{A}(\Omega)$). So $\mathcal{A}(T) \cong \mathcal{A}(\omega \overleftarrow{\cup} T)$ for any chain T, where $\omega \overleftarrow{\cup} T$ is $\omega \cup T$ ordered by: $n < \tau$ for all $\tau \in T$, $n \in \omega$. In order to obtain any nice classification of chains Ω by the elementary properties of the group $\mathcal{A}(\Omega)$, we will assume that Ω is homogeneous (i.e., for each $\alpha, \beta \in \Omega$, there exists $f \in \mathcal{A}(\Omega)$ such that $f(\alpha) = \beta$; so homogeneous in our sense means 1-homogeneous in the usual model-theoretic sense). If Ω is homogeneous, we will say that $\mathcal{A}(\Omega)$ is transitive. The second complication is that the symmetric group on a set is always primitive (i.e., there is no non-trivial equivalence relation on the set which is respected by the symmetric group). However, even when Ω is homogeneous, there may exist non-trivial equivalence relations on Ω (having convex classes) which are respected by $\mathcal{A}(\Omega)$. For example, let $\Omega = \mathbb{R} \overrightarrow{\times} \mathbb{Z}$, the lexicographic product of the real line, $\mathbb{R}$, and the integers, $\mathbb{Z}$ (i.e., $\mathbb{R} \otimes \mathbb{Z}$ ordered by: $(r,m) > (s,n)$ if $r > s$ or ($r = s$ & $m > n$)). Then $(r,m) \sim (s,n)$ if $r = s$ is an equivalence relation of the desired kind. (Two points of Ω are equivalent only if there are only finitely many points of Ω between them.) Such chains are said to be non-primitive. Fortunately, there is a group-theoretic sentence which is satisfied in a transitive $\mathcal{A}(\Omega)$ if and only if Ω is primitive (see [3; Theorem 4D] or [7; Lemma 4]); so we will confine ourselves to primitive chains in this article. The non-primitive case will be investigated in a later paper.

If Ω is primitive, then [3; Theorem 4.B] either

(i) $\mathcal{A}(\Omega)$ is abelian, or

(ii) Ω is doubly homogeneous (for each $\alpha_i, \beta_i \in \Omega$ $(i = 1,2)$ with $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, there exists $f \in \mathcal{A}(\Omega)$ such that $f(\alpha_i) = \beta_i$ $(i = 1,2)$).

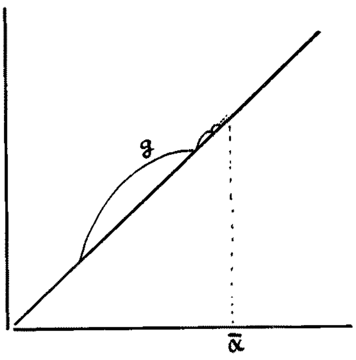
Moreover, (i) and (ii) are disjoint ([3, Lemma 1.6.8]) and so can be distinguished by a group-theoretic sentence about $\mathcal{A}(\Omega)$. Hence we may deal with them separately in attempting to classify homogeneous chains by the elementary properties of their automorphism groups. Case (i), the rigidly homogeneous case, was completely studied in [5]. So we will confine ourselves to doubly homogeneous chains in this article. Our main thrust will be to establish that

the first order language of automorphism groups of doubly homogeneous chains is extremely rich.

Although we are primarily interested in the group $\mathcal{A}(\Omega)$, an auxiliary relation on it will simplify matters. This auxiliary relation is the pointwise ordering on $\mathcal{A}(\Omega)$, with respect to which $\mathcal{A}(\Omega)$ becomes a lattice-ordered group: $f \leq g$ if $f(\alpha) \leq g(\alpha)$ for all $\alpha \in \Omega$; so $g \geq e$ if g moves no points down. In [8] (or [9]) it was shown that there is a formula $\psi(x,y)$ of the group language such that $\mathcal{A}(\Omega) \models \psi(f,g)$ if and only if $e \leq f,g$ or $e \geq f,g$. Hence if $\mathcal{A}(\Omega) \equiv \mathcal{A}(\Lambda)$ as groups, $\langle\mathcal{A}(\Omega),\leq\rangle \equiv \langle\mathcal{A}(\Lambda),\leq\rangle$ or $\langle\mathcal{A}(\Omega),\leq\rangle \equiv \langle\mathcal{A}(\Lambda),\leq^*\rangle$ as lattice-ordered groups where $\leq^*$ is the reverse of the pointwise ordering. Since in most of our results, $\langle\mathcal{A}(\Lambda),\leq\rangle$ satisfies the desired properties if and only if $\langle\mathcal{A}(\Lambda),\leq^*\rangle$ does, we will assume that the pointwise ordering and the inherited lattice operations $\vee$ and $\wedge$ are explicitly in the language. (We use "&" and "or" for the conjunction and disjunction of the language.)

For $g \in \mathcal{A}(\Omega)$, let <u>supp(g)</u> $= \{\alpha \in \Omega: g(\alpha) \neq \alpha\}$, the <u>support</u> of g. If $f,g \in \mathcal{A}(\Omega)$ and $\mathrm{supp}(f) < \mathrm{supp}(g)$ (i.e., $\alpha < \beta$ for all $\alpha \in \mathrm{supp}(f)$ and $\beta \in \mathrm{supp}(g)$), we say that f is to the <u>left</u> of g. Since $\mathrm{supp}(hgh^{-1}) = h(\mathrm{supp}(g))$, it follows that if $f,g > e$, then f is to the left of g if and only if $\mathcal{A}(\Omega) \models (\forall h)(h \geq e \rightarrow f \wedge hgh^{-1} = e)$. We abbreviate this formula to <u>L(f,g)</u>. Hence $g > e$ has bounded support can be expressed in our language by the formula $(\exists f_1 > e)(\exists f_2 > e)(L(f_1,g)\ \&\ L(g,f_2))$.

Let $\overline{\Omega}$ be the Dedekind completion of Ω. Each $g \in \mathcal{A}(\Omega)$ has a unique extension $\overline{g}$ to an element of $\mathcal{A}(\overline{\Omega})$ given by: $\overline{g}(\overline{\alpha}) = \sup\{g(\alpha): \alpha \in \Omega\ \&\ \alpha \leq \overline{\alpha}\}$ $(\overline{\alpha} \in \overline{\Omega})$. We will identify g with $\overline{g}$. If Ω is doubly homogeneous and $\overline{\alpha} \in \overline{\Omega}$, there is $e < g \in \mathcal{A}(\Omega)$ of bounded support such that $\overline{\alpha} = \sup(\mathrm{supp}(g))$. Moreover, if $e < g' \in \mathcal{A}(\Omega)$ and $\overline{\alpha}' = \sup(\mathrm{supp}(g'))$, $\overline{\alpha} = \overline{\alpha}'$ if and only if $\mathcal{A}(\Omega) \models (\forall h > e)(L(g,h) \leftrightarrow L(g',h))$; and $\overline{\alpha} \leq \overline{\alpha}'$ precisely when $\mathcal{A}(\Omega) \models (\forall h > e)(L(g',h) \rightarrow L(g,h))$. We can therefore interpret $\overline{\Omega}$ in $\mathcal{A}(\Omega)$ in this uniform way (the formulae are independent of the particular doubly homogeneous chain Ω). That is, we can interpret (uniformly) $\langle\mathcal{A}(\Omega),\overline{\Omega},\cdot,^{-1},e,\leq_{\mathcal{A}(\Omega)},\leq_{\overline{\Omega}}\rangle$ in $\langle\mathcal{A}(\Omega),\cdot,^{-1},e,\leq_{\mathcal{A}(\Omega)}\rangle$. Moreover, $f(\overline{\alpha}) = \overline{\alpha}'$ if and only if $\mathcal{A}(\Omega) \models (\forall h > e)(L(fgf^{-1},h) \leftrightarrow L(g',h))$.



Consequently, we will assume that our language is equipped explicitly with variables for the points of $\overline{\Omega}$ together with relations for the order on $\overline{\Omega}$ and the action of $\mathcal{A}(\Omega)$ on $\overline{\Omega}$. If $\mathcal{A}(\Omega) \cong \mathcal{A}(\Lambda)$ as groups implies only $\langle \mathcal{A}(\Omega), \leq \rangle \equiv \langle \mathcal{A}(\Lambda), \leq^* \rangle$, we would obtain $\langle \mathcal{A}(\Omega), \overline{\Omega}, \leq_{\mathcal{A}(\Omega)}, \leq_{\overline{\Omega}} \rangle \equiv \langle \mathcal{A}(\Lambda), \overline{\Lambda}, \leq^*_{\mathcal{A}(\Lambda)}, \leq^*_{\overline{\Lambda}} \rangle$. This means that any results of the form: $\mathcal{A}(\Omega) \equiv \mathcal{A}(\Lambda)$ implies $\Omega \cong \Lambda$, really should have the weaker conclusion that the homogeneous chains Ω and Λ ordermorphic or anti-ordermorphic. This makes a difference in Theorem 12 (Cases (a) and (b) become one since $\mathcal{A}(\vec{\mathbb{R}}) \cong \mathcal{A}(\overleftarrow{\mathbb{R}})$ as groups; an isomorphism is furnished by conjugating by an anti-ordermorphism between $\vec{\mathbb{R}}$ and $\overleftarrow{\mathbb{R}}$).

Points $\overline{\alpha}, \overline{\beta} \in \overline{\Omega}$ lie in the same <u>orbit</u> of $\mathcal{A}(\Omega)$ if $f(\overline{\alpha}) = \overline{\beta}$ for some $f \in \mathcal{A}(\Omega)$. As we saw above, this is recognizable in our language. Hence the orbits of $\mathcal{A}(\Omega)$ in $\overline{\Omega}$ are interpretable in our language. Now Ω is an orbit of $\mathcal{A}(\Omega)$. We may not always be able to distinguish it in our language from another orbit T of $\mathcal{A}(\Omega)$ in $\overline{\Omega}$ since it is possible that $\mathcal{A}(\Omega) \cong \mathcal{A}(T)$ as lattice-ordered groups, with the isomorphism being furnished by extending an element of $\mathcal{A}(\Omega)$ to its unique extension in $\mathcal{A}(\overline{\Omega})$ and then restricting the domain to T. However, we will assume that variables for points of Ω are included in our language; thus we can distinguish Ω from any other orbit of $\mathcal{A}(\Omega)$. This means that any results of the form: $\mathcal{A}(\Omega) \equiv \mathcal{A}(\Lambda)$ implies $\Omega \cong \Lambda$, really should have the weaker conclusion that the homogeneous chain Λ is ordermorphic (or anti-ordermorphic) to an orbit of $\mathcal{A}(\Omega)$ in $\overline{\Omega}$; i.e., $\Lambda \cong (\mathcal{A}(\Omega))(\overline{\alpha})$ for some $\overline{\alpha} \in \overline{\Omega}$. For example, in Theorem 5, the conclusion should be: Λ is ordermorphic to the rationals or irrationals.

Throughout this paper, then, our language will be the first order language of lattice-ordered groups, together with variables for points of Ω and for points of $\overline{\Omega}$, and a symbol $f(\overline{\alpha})$ for the action of $f \in \mathcal{A}(\Omega)$ on $\overline{\alpha} \in \overline{\Omega}$. (So if $\overline{\alpha} \in \Omega$, $f(\overline{\alpha}) \in \Omega$.) However, we will use $\mathcal{A}(\Omega)$ as a shorthand for the structures of this language. Our most powerful result is:

THEOREM A: <u>Let</u> Ω <u>be a doubly homogeneous chain. Then countable subsets of</u> Ω <u>**together with** $\subseteq$ **and** membership in them are interpretable in</u> $\mathcal{A}(\Omega)$.

From this we will be able to characterize many chains whose defining properties involve countability; e.g., Suslin, Luzin, Specker. We will also give simpler proofs than those in [7] to show that the language can express that Ω is (isomorphic to) the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, the real long lines, and certain long rational lines.

Also, we will be able to tell (in the language) if Ω can be embedded in $\mathbb{R}$, and whether it can bear the arithmetic structure of an additive subgroup or a subfield of $\mathbb{R}$.

If this background is inadequate, see [3], [4], [7], [8] or [9].

2. RESULTS.

The language $\mathcal{L}$ is the first order language with the usual logical symbols, $=$, variables $f,g,h,\ldots$ for members of $\mathcal{A}(\Omega)$, $\alpha,\beta,\gamma,\ldots$ for members of Ω and $\bar{\alpha},\bar{\beta},\bar{\gamma},\ldots$ for members of $\bar{\Omega}(\supseteq \Omega)$; a constant e for the identity element of $\mathcal{A}(\Omega)$, symbols for multiplication, inverse, least upper bound ($\vee$) and greatest lower bound ($\wedge$) for $\mathcal{A}(\Omega)$ as well as the pointwise order $\leq$ on $\mathcal{A}(\Omega)$ (as a shorthand: $f \leq g$ stands for $f \wedge g = g$); the total order relation ($\leq$) on $\bar{\Omega}$ and the inherited order on Ω; and the action of $\mathcal{A}(\Omega)$ on $\bar{\Omega}$.

An element $e < g \in \mathcal{A}(\Omega)$ is said to have <u>one bump</u> if whenever $\alpha < \bar{\beta} < \gamma$ with $\alpha,\gamma \in \mathrm{supp}(g)$, $g(\bar{\beta}) \neq \bar{\beta}$. This is equivalent to $\mathcal{A}(\Omega) \models (\forall u)(\forall v)(u \wedge v = e \;\&\; u \vee v = g \rightarrow u = e \text{ or } v = e)$. (See [3], [7], [8] or [9].) We will write <u>Bump(g)</u> for this formula of $\mathcal{L}$ (in one free variable g).

Let $e < f \in \mathcal{A}(\Omega)$ and $e < b \in \mathcal{A}(\Omega)$. b is a <u>bump of f</u> if b has just one bump and $f|\mathrm{supp}(b) = b|\mathrm{supp}(b)$. Note that if b_1,b_2 are distinct bumps of f, then $b_1 \wedge b_2 = e$.

LEMMA 0: <u>Let Ω be a homogeneous chain and</u> $e < b,f \in \mathcal{A}(\Omega)$. <u>Then</u> b <u>is a bump of</u> f <u>if and only if</u> $\mathcal{A}(\Omega) \models (b \wedge b^{-1}f = e) \;\&\; \mathrm{Bump}(b)$.

<u>Proof</u>: Let b be a bump of f and $\alpha \in \mathrm{supp}(b)$. Since $f(\alpha) = b(\alpha)$, $b^{-1}f(\alpha) = \alpha$. If $\beta \notin \mathrm{supp}(b)$, $b(\beta) = \beta$. Thus $b \wedge b^{-1}f = e$. Conversely, if $b \wedge b^{-1}f = e$ and $\alpha \in \mathrm{supp}(b)$, $b^{-1}f(\alpha) = \alpha$ so $f(\alpha) = b(\alpha)$. Hence $f|\mathrm{supp}(b) = b|\mathrm{supp}(b)$ and b is a bump of f.

LEMMA 1: <u>Let</u> $\theta_T \equiv (\exists f > e)[\mathrm{Bump}(f) \;\&\; (\forall g)(\neg L(g,f) \;\&\; \neg L(f,g))]$, $\theta_I \equiv (\exists f > e)[\mathrm{Bump}(f) \;\&\; (\forall g)\neg L(g,f)]$ <u>and</u> $\theta_F \equiv (\exists f > e)[\mathrm{Bump}(f) \;\&\; (\forall g)\neg L(f,g)]$. <u>If Ω is a doubly homogeneous, then</u> (i) $\mathcal{A}(\Omega) \models \theta_T$ <u>if and only if Ω has countable coterminality</u>, (ii) $\mathcal{A}(\Omega) \models \theta_I$ <u>if and only if Ω has countable coinitiality</u>, (iii) $\mathcal{A}(\Omega) \models \theta_F$ <u>if and only if Ω has countable cofinality</u>.

<u>Proof</u>: (i) Since $\{f^n(\alpha)\colon n \in \mathbb{Z}\}$ is coterminal in $\mathrm{supp}(f)$ if has one bump, it remains to prove that $\mathcal{A}(\Omega) \models \theta_T$ if Ω is doubly homogeneous and has countable coterminality. Let $\{\alpha_n\colon n \in \mathbb{Z}\}$ be coterminal in Ω. Since Ω is doubly homogeneous, there is an order-morphism $f_n\colon [\alpha_n,\alpha_{n+1}] \cong [\alpha_{n+1},\alpha_{n+2}]$. Let $f = \bigcup\{f_n\colon n \in \mathbb{Z}\}$. Then

$e < f \in \mathcal{A}(\Omega)$ and fixes no point of $\overline{\Omega}$. Moreover, $\text{supp}(f) = \Omega$ so $\mathcal{A}(\Omega) \models \theta_T$.

If $X \subseteq \mathcal{A}(\Omega)$, let <u>$C(X)$</u> $= \{f \in \mathcal{A}(\Omega): (\forall g \in X)(fg = gf)\}$, the <u>centralizer</u> of X in $\mathcal{A}(\Omega)$. If $f \in \mathcal{A}(\Omega)$, write <u>$\langle f\rangle$</u> for the subgroup generated by f: i.e., $\langle f\rangle = \{f^n: n \in \mathbb{Z}\}$.

The following lemma is very similar to Lemma 16 of [6] where the condition of having one bump is removed and $\mathcal{A}(\Omega)$ is replaced by an existentially complete lattice-ordered group.

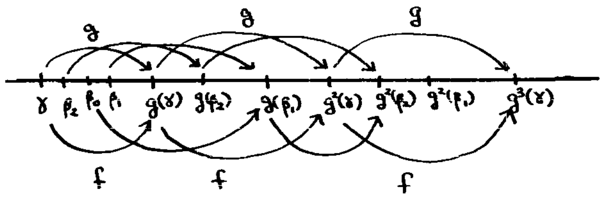
LEMMA 2: <u>If</u> $f \in \mathcal{A}(\Omega)$ <u>and has one bump, then</u> $g \in \langle f\rangle$ <u>if and only if</u> $g \in C(C(f))$.

<u>Proof</u>: Since f has one bump, $f(\alpha) > \alpha$ for all $\alpha \in \text{supp}(f)$. We must show that if $g \in C(C(f))$, then $g \in \langle f\rangle$, the other direction being obvious. If $g \in C(C(f))\setminus\langle f\rangle$, let $\overline{B}$ be the single open interval of support of f in $\overline{\Omega}$; i.e., $\overline{B}$ is the convexification (in $\overline{\Omega}$) of $\text{supp}(f)$. For each $n \in \mathbb{Z}$, the sets $\{\overline{\beta} \in \overline{B}: g(\overline{\beta}) \leq f^n(\overline{\beta})\}$ and $\{\overline{\beta} \in \overline{B}: g(\overline{\beta}) \geq f^{n+1}(\overline{\beta})\}$ are closed (in $\overline{B}$) and disjoint (since $f^n(\overline{\beta}) = f^{n+1}(\overline{\beta})$ implies $\overline{\beta} = f(\overline{\beta})$, contradicting $\overline{\beta} \in \overline{B}$). Since $\overline{B}$ is connected, there exists $\overline{\alpha} \in \overline{B}$ such that $g(\overline{\alpha}) \notin \{f^n(\overline{\alpha}): n \in \mathbb{Z}\}$. Either $g(\overline{\alpha}) \notin \overline{B}$ or, for some $n \in \mathbb{Z}$, $f^n(\overline{\alpha}) < g(\overline{\alpha}) < f^{n+1}(\overline{\alpha})$. In the first case, there exists $h \in \mathcal{A}(\Omega)$ such that $hg(\overline{\alpha}) \neq g(\overline{\alpha})$ and $\text{supp}(h) \cap \overline{B} = \emptyset$. Then $h \in C(f)$ but $g \notin C(h)$ $(hg(\overline{\alpha}) \neq g(\overline{\alpha}) = gh(\overline{\alpha})$, a contradiction. In the second case there is $h \in \mathcal{A}(\Omega)$ such that $g(\overline{\alpha}) \in \text{supp}(h) \subseteq (f^n(\overline{\alpha}), f^{n+1}(\overline{\alpha}))$. Let $h^* \in \mathcal{A}(\Omega)$ be the identity of $\overline{B}$ and agree with $f^m h f^{-m}$ on $(f^{n+m}(\overline{\alpha}), f^{n+m+1}(\overline{\alpha}))$ (for all $m \in \mathbb{Z}$). Since $\alpha = f^0(\alpha)$, $h^*(\alpha) = \alpha$. Then $h^* \in C(f)$ but $g \notin C(h^*)$, the desired contradiction.

We have shown that $g \in \langle f\rangle$ is expressible in our language if f has one bump. We can therefore assume "$g \in \langle f\rangle$" is in $\mathcal{L}$ if f has one bump.

LEMMA 3: <u>The statement</u> "B is a countable bounded subset of Ω" <u>is interpretable in</u> $\mathcal{A}(\Omega)$, <u>as is the formula</u> "$\alpha \in B$". <u>Hence</u> $\subseteq$ <u>between countable bounded subsets of</u> Ω <u>is interpretable in</u> $\mathcal{A}(\Omega)$.

<u>Proof</u>: Let B be a countable bounded non-empty subset of Ω. Let $\beta \in B$ and $\gamma < B$. Since $\mathcal{A}(\Omega)$ is transitive, there exists $e < g \in \mathcal{A}(\Omega)$ such that $B < g(\gamma)$ and g has one bump. Let B be enumerated $\{\beta_n: n \in \omega\}$ with $\beta_0 = \beta$. By double homogeneity, there exists $e < f \in \mathcal{A}(\Omega)$ such that f has one bump and for all $n \in \omega$, $f^n(\beta) = g^n(\beta_n)$ and $f^n(\gamma) = g^n(\gamma)$.

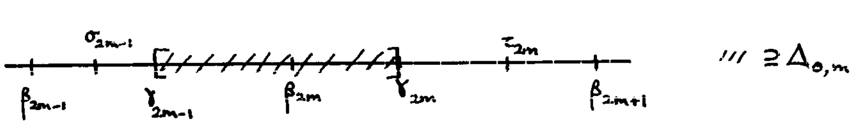


It follows that $B = \{g^{-n}f^{n}(\beta): n \in \omega\} = \{k^{-1}h(\beta): e < h \in \langle f\rangle$ & $e < k \in \langle g\rangle\} \cap (\gamma, g(\gamma))$. By Lemma 2, this last expression is an interpretation of B in $\mathcal{A}(\Omega)$ via the quadruple (β,γ,f,g), where $\beta,\gamma \in \Omega$ with $\gamma < \beta$, and $e < f,g$ have one bump with $\{h(\gamma): h \in \langle f\rangle\} = \{k(\gamma): k \in \langle g\rangle\}$. This last condition can be expressed in our language by $(\forall h \in \langle f\rangle)(\exists k \in \langle g\rangle)(h(\gamma) = k(\gamma))$ & $(\forall k \in \langle g\rangle)(\exists h \in \langle f\rangle)(h(\gamma) = k(\gamma))$. Thus we can determine in $\mathcal{L}$ which quadruples determine the same bounded countable set, etc. Since any quadruple satisfying the above conditions yields a countable bounded set, the lemma is proved.

Note that the interpretation is uniform; i.e., the formulae of $\mathcal{L}$ for interpretation are independent of the particular doubly homogeneous chain Ω. Also, we could choose f,g of the proof of the lemma so that their supports are contained in any open interval (σ,τ) with $\sigma < \inf(B) \leq \sup(B) < \tau$.

We now prove Theorem A.

Let Ω be a doubly homogeneous chain and $\Delta \subseteq \Omega$ be countable. If Δ is bounded, $\Delta = \{b^{-1}a(\beta): e < a \in \langle f\rangle$ & $e < b \in \langle g\rangle\} \cap (\gamma, g(\gamma))$ for some β,γ,f,g where f and g have one bump. If Δ is unbounded in Ω, let $\{\beta_m: m \in \mathbb{Z}\}$, $\{\gamma_m: m \in \mathbb{Z}\}$, $\{\sigma_m: m \in \mathbb{Z}\}$ and $\{\tau_m: m \in \mathbb{Z}\}$ be coterminal in Δ (and hence in Ω) with $\beta_m < \sigma_m < \gamma_m < \tau_m < \beta_{m+1}$ $(m \in \mathbb{Z})$. Let $\Delta_{0,m} = \Delta \cap [\gamma_{2m-1}, \gamma_{2m}]$ and $\Delta_{1,m} = \Delta \cap (\gamma_{2m-1}, \gamma_{2m})$.



By the remark following the proof of Lemma 3, there are $e < f_{i,m}, g_{i,m} \in \mathcal{A}(\Omega)$ $(i = 0,1)$ having one bump with $\mathrm{supp}(f_{0,m}) \cup \mathrm{supp}(g_{0,m}) \subseteq (\beta_{2m-1}, \beta_{2m+1})$, $g_{0,m}(\sigma_{2m-1}) = \tau_{2m}$, $f^n_{0,m}(\sigma_{2m-1}) = g^n_{0,m}(\sigma_{2m-1})$ $(n \in \mathbb{Z})$, $\Delta_{0,m} = \{g^{-n}_{0,m} f^n_{0,m}(\beta_{2m}) : n \in \omega\} = (\sigma_{2m-1}, \tau_{2m}) \cap \{b^{-1}_{0,m} a_{0,m}(\beta_{2m}) : e < a_{0,m} \in \langle f_{0,m} \rangle \; \& \; e < b_{0,m} \in \langle g_{0,m} \rangle\}$, and $\mathrm{supp}(f_{1,m}) \cup \mathrm{supp}(g_{1,m}) \subseteq (\beta_{2m}, \beta_{2m+2})$, $g_{1,m}(\sigma_{2m}) = \tau_{2m+1}$, $f^n_{1,m}(\sigma_{2m}) = g^n_{1,m}(\sigma_{2m})$ $(n \in \mathbb{Z})$, $\Delta_{1,m} = \{g^{-n}_{1,m} f^n_{1,m}(\beta_{2m+1}) : n \in \omega\} = (\sigma_{2m}, \tau_{2m+1}) \cap \{b^{-1}_{1,m} a_{1,m}(\beta_{2m+1}) : e < a_{1,m} \in \langle f_{1,m} \rangle \; \& \; e < b_{1,m} \in \langle g_{1,m} \rangle\}$ $(m \in \mathbb{Z})$. Let f_i and g_i be the supremum of the pairwise disjoint set of elements $\{f_{i,m} : m \in \mathbb{Z}\}$ and $\{g_{i,m} : m \in \mathbb{Z}\}$ respectively $(i = 0,1)$. Then $\Delta = \bigcup_{m \in \mathbb{Z}} \{g_0^{-n} f_0^n(\beta_{2m}) : n \in \omega\} \cup \bigcup_{m \in \mathbb{Z}} \{g_1^{-n} f_1^n(\beta_{2m+1}) : n \in \omega\}$. As in the proof of Lemma 1 (i), there is $e < h \in \mathcal{A}(\Omega)$ having one bump such that $h(\beta_m) = \beta_{m+1}$, $h(\sigma_m) = \sigma_{m+1}$ and $h(\tau_m) = \tau_{m+1}$ $(m \in \mathbb{Z})$. So $\Delta = \{g_0^{-n} f_0^n h^{2m}(\beta_0) : m \in \mathbb{Z}, n \in \omega\} \cup \{g_1^{-n} f_1^n h^{2m} h(\beta_0) : m \in \mathbb{Z}, n \in \omega\}$. We have coded Δ by $(g_0, g_1, f_0, f_1, h, \beta_0, \sigma_0, \tau_0)$ in the following sense: $\delta \in \Delta$ if and only if there are $k \in \langle h^2 \rangle$, $e < a_i \in \langle c_i \rangle$ and $e < b_i \in \langle d_i \rangle$ with $\delta = b_i^{-1} a_i k(\beta_i)$ for $i = 0$ or 1, where $\beta_1 = h(\beta_0)$, $\sigma_1 = h(\sigma_0)$, $\tau_1 = h(\tau_0)$, and c_i and d_i are the unique bumps of f_i and g_i respectively with $c_i k(\sigma_i) = \tau_i = d_i k(\sigma_i)$ and $h^{-1} k(\sigma_i) < b_i^{-1} a_i k(\beta_i) < k(\tau_i)$. By Lemmas 0 and 2, this is expressible in $\mathcal{L}$. Moreover, any octuple $(g_0, g_1, f_0, f_1, h, \beta_0, \sigma_0, \tau_0)$ with $\beta_0 < \sigma_0 < \tau_0 < h(\beta_0)$, $e < h$ has one bump, and $e < g_0, g_1, f_0, f_1$ with $g_i h^{2n}(\sigma_i) = h^{2n+1}(\tau_i) = f_i h^{2n}(\sigma_i)$ $(i = 0,1)$ where $\sigma_1 = h(\sigma_0)$ and $\tau_1 = h(\tau_0)$ gives a countable subset of Ω via $\{g_0^{-n} f_0^n h^{2m}(\beta_0) : m \in \mathbb{Z}, n \in \omega\} \cup \{g_1^{-n} f_1^n h^{2m+1}(\beta_0) : m \in \omega, n \in \mathbb{Z}\}$. Hence Theorem A is proved.

From now on, we will assume that $\mathcal{L}$ includes variables for countable subsets of $\mathcal{L}$ as well as $=$ and $\subseteq$ between them, and membership of elements of Ω in them.

Actually, the following can be proved:

THEOREM B: Let Ω be a doubly homogeneous chain. Then countable subsets of $\overline{\Omega}$ together with membership in them are interpretable in $\mathcal{A}(\Omega)$.

We sketch the proof of Theorem B:

Let $\Delta \subseteq \bar{\Omega}$ be countable. Let $\Delta_1 = \{\delta \in \Delta: (\exists \alpha \in \Omega)[\alpha,\delta) \cap \Delta = \emptyset\}$, $\Delta_2 = \{\delta \in \Delta \setminus \Delta_1: (\exists \beta \in \Omega)(\delta,\beta] \cap \Delta = \emptyset\}$ and $\Delta_3 = \Delta \setminus (\Delta_1 \cup \Delta_2)$. For each $\delta_1 \in \Delta_1$, choose $\alpha = \alpha(\delta_1) \in \Omega$ with $[\alpha_1,\delta_1) \cap \Delta_1 = \emptyset$. Let $e < g_{\delta_1} \in \mathcal{A}(\Omega)$ with the closure of $\text{supp}(g_{\delta_1}) = [\alpha_1,\delta_1]$; so $(\forall f > e)(f \wedge g_{\delta_1} = e \rightarrow L(f,g_{\delta_1})$ or $L(g_{\delta_1},f))$. Let $g_1 \in \mathcal{A}(\Omega)$ be the pointwise supremum of the pairwise disjoint set $\{g_{\delta_1}: \delta_1 \in \Delta_1\}$. Now $A_1 = \{\alpha = \alpha(\delta_1): \delta_1 \in \Delta_1\}$ is a countable subset of Ω. Thus A_1 can be recognized by Theorem A. Hence, using A_1 and g_1 we can recognize Δ_1 and that it is countable. Dually for Δ_2. Now each point of Δ_3 is the supremum and infimum of a countable subset of Δ (and hence of Ω). But, by double--and hence m for all $m \in \omega$--transitivity, there is at most one $\mathcal{A}(\Omega)$ orbit of such points of $\bar{\Omega} \setminus \Omega$. Therefore, Δ_3 comprises at most two countable orbits of $\mathcal{A}(\Omega)$. Hence Δ_3 can be captured in $\mathcal{L}$ by Theorem A. Consequently, so can Δ and Theorem B is proved.

We can now express that Ω is separable (i.e., has a countable subset whose topological closure is Ω) by:
$(\exists$ countable $\Delta)(\forall\alpha)(\forall\beta)(\alpha < \beta \rightarrow (\exists\delta_1,\delta_2,\delta_3 \in \Delta)(\delta_1 < \alpha < \delta_2 < \beta < \delta_3))$.
But Ω can be embedded in $\mathbb{R}$ if and only if it is separable. Hence

COROLLARY 4. *There is a sentence σ such that, for homogeneous Ω, $\mathcal{A}(\Omega) \models \sigma$ if and only if Ω can be embedded in $\mathbb{R}$.*

We next give easy proofs of the main theorems of [7].

THEOREM 5. *There is a sentence ψ of $\mathcal{L}$ such that, for any homogeneous chain Ω, $\mathcal{A}(\Omega) \models \psi$ if and only if $\Omega \cong \mathbb{Q}$.*

Proof: Apply Theorem A. (The only doubly homogeneous countable chain is $\mathbb{Q}$.)

THEOREM 6. *There is a sentence ρ of $\mathcal{L}$ such that, for any homogeneous chain Ω, $\mathcal{A}(\Omega) \models \rho$ if and only if $\Omega \cong \mathbb{R}$.*

Proof: It was shown in [7] that a homogeneous chain Ω is Dedekind complete if and only if $\mathcal{A}(\Omega) \models (\forall f > e)(\text{supp}(f)\text{bounded} \rightarrow (\exists h)[L(f,h^{-1}fh) \,\&\, (\forall g > e)\neg(L(f,g) \,\&\, L(g,h^{-1}fh))])$. The theorem now follows from Corollary 4.

A chain Ω is said to enjoy the *Suslin property* if every pairwise disjoint collection of open intervals of Ω is countable. A homogeneous Dedekind complete chain other than $\mathbb{R}$ that satisfies the Suslin property is called a *Suslin line*. If they exist at all, they exist in profusion (see [1]).

Recall that the bumps of $e < f \in \mathcal{A}(\Omega)$ are pairwise disjoint and recognizable in $\mathcal{L}$--with parameter f (Lemma 0).

THEOREM 7. <u>A doubly homogeneous Ω has the Suslin property if and only if for all $e < f \in \mathcal{A}(\Omega)$, there is a countable $\Delta \subseteq \mathrm{supp}(f)$ such that each bump of f moves exactly one point of Δ. Consequently, there are sentences $\hat{\sigma}_1$, $\hat{\sigma}_2$ of $\mathcal{L}$ such that (1) $\mathcal{A}(\Omega) \models \hat{\sigma}_1$ if and only if Ω enjoys the Suslin property, (2) $\mathcal{A}(\Omega) \models \hat{\sigma}_2$ if and only if Ω is a Suslin line.</u>

<u>Proof</u>: Since the bumps of f have disjoint open intervals of support, the condition is clearly necessary. For sufficiency, by double homogeneity, each open interval A contains the support of some one bump $e < f_A \in \mathcal{A}(\Omega)$. Let $f \in \mathcal{A}(\Omega)$ have $\{f_A : A \in \mathcal{F}\}$ as its set of bumps. Hence $|\mathcal{F}| \leq \aleph_0$, and Ω enjoys the Suslin property.

As noted in the introduction, we may assume that $f \in \mathcal{A}(\Omega)$ is identified with its unique extension to $\mathcal{A}(\bar{\Omega})$. The set of fixed points of f in $\bar{\Omega}$ is always a closed set; the complementary set is a disjoint union of open intervals each being the convexification (in $\bar{\Omega}$) of the support of a bump of f. Hence each connected component of this complementary set has countable coterminality. Conversely, if Ω is doubly homogeneous and $\bar{\Delta}$ is a closed subset of $\bar{\Omega}$ with each connected component of its complement having countable coterminality, then $\bar{\Delta}$ is the fixed point set (in $\bar{\Omega}$) of some $f \in \mathcal{A}(\Omega)$. For, as in the proof of Lemma 1, we may construct a one bump $f_A \in \mathcal{A}(\Omega)$ on each component A of the complement with $\mathrm{supp}(f_A) = A$. The desired function is $f \in \mathcal{A}(\Omega)$ whose set of bumps is just the set of f_A's. If Ω is separable, every interval has countable coterminality. Hence we have proved:

LEMMA 8: <u>If Ω is a separable doubly homogeneous chain, the closed subsets of $\bar{\Omega}$ are precisely the fixed point sets of functions $f \in \mathcal{A}(\Omega)$. Hence the closed subsets of $\bar{\Omega}$ are interpretable in $\mathcal{A}(\Omega)$ for such chains Ω.</u>

A subset $\bar{C}$ of a chain $\bar{\Omega}$ is said to be a <u>Cantor set</u> if $\bar{C}$ is closed, nowhere dense, and has no isolated points.

LEMMA 9: <u>If Ω is a separable doubly homogeneous chain, the Cantor sets of $\bar{\Omega}$ are interpretable in $\mathcal{A}(\Omega)$.</u>

<u>Proof</u>: If Ω is doubly homogeneous and $\bar{C}$ is the fixed point set of $e < f \in \mathcal{A}(\Omega)$, then $\bar{C}$ is nowhere dense if and only if $\bar{C}$ contains no non-empty open interval, which is equivalent to $(\forall g)(f \wedge g = e \rightarrow g = e)$ since every open interval contains the support

of some $e < h \in \mathcal{A}(\Omega)$. $\overline{C}$ has no isolated points precisely when the following formula of $\mathcal{L}$ holds in $\mathcal{A}(\Omega)$:
$(\forall b_1,b_2 \text{ bumps of } f)(L(b_1,b_2) \rightarrow (\exists b_3 \text{ bump of } f)(L(b_1,b_3) \& L(b_3,b_2)))$.
Lemma 9 now follows from Lemma 8.

Again the interpretation given is uniform (for such Ω) in $\mathcal{L}$.

A chain Ω is said to have the Luzin property if every Cantor set of $\overline{\Omega}$ meets Ω in a countable set.

By Theorem A and Lemma 9 we have:

THEOREM 10: There is a sentence θ of $\mathcal{L}$ such that if Ω is a separable doubly homogeneous chain, then $\mathcal{A}(\Omega) \models \theta$ if and only if Ω has the Luzin property.

$\mathbb{Q}$ clearly is separable, doubly homogeneous and has the Luzin property. We now show that there are uncountable chains Ω enjoying these properties, by modifying the standard Luzin construction.

We assume the Continuum Hypothesis. Enumerate the Cantor subsets of $\mathbb{R}$, $\{C_\mu : \mu < \omega_1\}$. For each $\mu < \omega_1$, let $M_\mu = \bigcup\{C_\lambda : \lambda \leq \mu\}$. Then each M_μ is meager; that is, M_μ is a countable union of nowhere dense sets. Let $\langle\langle S\rangle\rangle$ denote the rational subspace generated by $S \subseteq \mathbb{R}$. Now choose, inductively, $x_\mu \in \mathbb{R}$ $(\mu < \omega_1)$ so that $x_0 \notin \mathbb{Q}M_0$, the rational multiples of members of M_0, and for each $\mu < \omega_1$, $x_\mu \notin \mathbb{Q}M_\mu + \langle\langle\{x_\lambda : \lambda < \mu\}\rangle\rangle = N_\mu$. This is possible since each N_μ is meager. We now claim that $\Lambda = \langle\langle\{x_\mu : \mu < \omega_1\}\rangle\rangle$ has the Luzin property. Note that for each $\nu < \omega_1$, if $0 \neq x \in \Lambda \cap M_\nu$, then $x = q_1x_{\nu_1} + \ldots + q_nx_{\nu_n}$ for $0 \neq q_i \in \mathbb{Q}$ and $\nu_1 < \ldots < \nu_n < \omega_1$. This implies that $x_{\nu_n} \in \mathbb{Q}M_\nu + \langle\langle\{x_\lambda : \lambda < \nu_n\}\rangle\rangle$ and hence $\nu_n < \nu$. Therefore $M_\nu \cap \Lambda \subseteq \langle\langle\{x_\lambda : \lambda < \nu\}\rangle\rangle$, a countable set. Since $C_\nu \subseteq M_\nu$, $\Lambda \cap C_\nu$ is countable, as required. Moreover, Λ is uncountable--it contains $\{x_\mu : \mu < \omega_1\}$. Since Λ is a rational vector space, it is homogeneous ($\mathcal{A}(\Lambda)$ contains translations by members of Λ). Multiplication by 2 is also an automorphism of $(\Lambda,\leq)$, so $\mathcal{A}(\Lambda)$ is not abelian. Because of the small translations (say by arbitrary qx_0 $(q \in \mathbb{Q})$), for every interval I of Λ, there is $f \in \mathcal{A}(\Lambda)$ such that $\emptyset \neq I \cap f(I) \neq I$. This is enough to ensure that Λ is primitive and hence doubly homogeneous (see the introduction). Λ is obviously separable as required.

A doubly homogeneous chain Ω is short if it has countable coterminality and for each $\overline{\alpha} \in \overline{\Omega}$, there exist countable bounded $\Gamma,\Delta \subseteq \Omega$ such that $\sup \Gamma = \overline{\alpha} = \inf \Delta$. By Lemma 1 (i) and Lemma 3, shortness is definable in $\mathcal{L}$ and we will assume that it is explicitly in $\mathcal{L}$. An uncountable short chain Ω that contains no uncountable separable subset is said to enjoy the Specker property. "Separable" here means separable in the interval topology of the subset (not necessarily the interval topology of Ω). Thus Ω has the Specker property if and only if it is uncountable, short and doubly homogeneous, and for all countable subsets Γ of Ω, there exists a countable subset Δ of Ω such that for all $\alpha \in \Omega\setminus\Gamma$ there is $\delta \in \Delta$ with no member of Γ between α and δ. Thus

THEOREM 11: There is a sentence χ of $\mathcal{L}$ such that if Ω is homogeneous, $\mathcal{A}(\Omega) \models \chi$ if and only if Ω has the Specker property.

The long (to the right) real line $\vec{\mathbb{R}}$ is constructed by removing the smallest point from the antilexicographically ordered chain $I \overleftarrow{\times} \omega_1$, where I is the half open real interval $[0,1)$. We can define $\overleftarrow{\mathbb{R}}$ and $\overleftrightarrow{\mathbb{R}}$ similarly.

THEOREM 12: Let Ω be a homogeneous set. There are sentences ϕ_i $(i = 1,2,3)$ such that

(a) $\mathcal{A}(\Omega) \models \phi_1$ if and only if $\Omega \cong \vec{\mathbb{R}}$.

(b) $\mathcal{A}(\Omega) \models \phi_2$ if and only if $\Omega \cong \overleftarrow{\mathbb{R}}$.

(c) $\mathcal{A}(\Omega) \models \phi_3$ if and only if $\Omega \cong \overleftrightarrow{\mathbb{R}}$.

Proof: (a) $\vec{\mathbb{R}}$ is completely characterized by the statement that it is Dedekind complete and not separable, yet every subset $\{\alpha: \alpha < \beta\}$ is separable. All of these clauses, together with double homogeneity, are describable in $\mathcal{L}$. ϕ_1 is their conjunction.

(b) and (c) are similar.

The long (to the right) rational lines Ω are constructed as follows. Let I_0 be the set of all rational numbers in the real open interval $(0,1)$. Choose $M \subseteq \omega_1$ with $0 \notin M$. Let $\Omega = \{(q,\nu): q \in I_0, \nu \in \omega_1\} \cup \{(0,\mu): \mu \in M\} \subseteq \vec{\mathbb{R}}$, with the induced order. All the constructions in which M contains a closed unbounded subset of ω_1 (or club for short)--the long rationals with internal club--give rise to ordermorphic chains. Likewise, all constructions in which the complement of M contains a club (the long rationals with external club). The two cases are not ordermorphic and are distinct from all other cases (in which neither M nor its complement contains a club).

THEOREM 13: There are sentences ξ_1, ξ_2, ξ_3 of $\mathcal{L}$ such that if Ω is homogeneous, then

(a) $\mathcal{A}(\Omega) \models \xi_1$ if and only if Ω is ordermorphic to the long rational line with internal club,

(b) $\mathcal{A}(\Omega) \models \xi_2$ if and only if Ω is ordermorphic to the long rational line with external club,

(c) $\mathcal{A}(\Omega) \models \xi_3$ <u>if and only if</u> Ω <u>is ordermorphic to a long rational line with neither internal nor external club</u>.

<u>Proof</u>: Consider Ω a long rational line with internal club. We may assume that $M = \omega_1 \setminus \{0\}$. Then Ω is uncountable but every subset $\{\alpha \in \Omega: \alpha < \beta\}$ is countable. Moreover, $\Gamma = \{(0,\mu): \mu \in M\}$ is a closed unbounded above subset of Ω that is well-ordered; and if $\gamma \in \Gamma$ and $\gamma = \sup_\Gamma\{\delta \in \Gamma: \delta < \gamma\}$, then $\gamma = \sup_\Omega\{\delta \in \Gamma: \delta < \gamma\}$. Now let $e < f \in \mathcal{A}(\Omega)$ have one bump on each interval $\{\alpha: (0,\mu) < \alpha < (0,\mu + 1)\}$ $(\mu \in \omega_1)$. Then the set of bumps of f is well-ordered and $(\forall h)(f \wedge h = e \rightarrow h = e)$. Furthermore, for each bump b of f (except the left-most), there exist $\alpha,\beta \in \Omega$ such that $b(\overline{\delta}) \neq \overline{\delta}$ if and only if $\alpha < \overline{\delta} < \beta$. All of these facts are expressible in $\mathcal{L}$.

Conversely, suppose that Ω is a doubly homogeneous uncountable chain with $\{\alpha \in \Omega: \alpha < \beta\}$ countable for each $\beta \in \Omega$. Assume there exists $e < f \in \mathcal{A}(\Omega)$ such that $(\forall h)(f \wedge h = e \rightarrow h = e)$, the set of bumps of f is well-ordered, and for each bump b of f (except the left-most), there exist $\alpha,\beta \in \Omega$ such that $b(\overline{\delta}) \neq \overline{\delta}$ if and only if $\alpha < \overline{\delta} < \beta$. Let Γ be the set of left endpoints of supports of bumps of f (other than the left-most). Then Γ is a well-ordered uncountable subset of Ω and hence $\Gamma \cong \omega_1$. Also if $\gamma \in \Gamma$ and $\gamma = \sup_\Gamma\{\delta \in \Gamma: \delta < \gamma\}$, then $\gamma = \sup_\Omega\{\delta \in \Gamma: \delta < \gamma\}$. For a bump b of f, we may call the end points of the support of b, μ and $\mu + 1$ $(\mu \in \omega_1)$. Then Ω is the disjoint union of the intervals $[\mu,\mu + 1)$ together with $(-\infty,0)$, each of which is countable and so ordermorphic to the rational interval $[0,1)$ or $(0,1)$. Hence Ω is ordermorphic to $[0,1) \overleftarrow{\times} \omega_1$ with least point $(0,0)$ removed.

(b) is proved similarly, except that we need an f such that all bumps of f have supporting intervals with no endpoints in Ω.

(c) now follows from (a) and (b).

We may also obtain long rational lines inside $\overleftarrow{\mathbb{R}}$ and $\overleftrightarrow{\mathbb{R}}$. Analogous results then hold in these cases.

The characterization of $\mathbb{R}$ and $\mathbb{Q}$ in [7] was achieved with heavy reliance on the arithmetic structure. We have avoided that here by using Theorem A. Still, it is of interest to know whether, for a given chain Ω, it is possible to define an arithmetic on Ω so that Ω becomes an ordered group or an ordered field. We call a chain Ω <u>Archimedean groupable</u> if it is possible to define an operation $+$ on Ω so that $(\Omega,+)$ becomes an Archimedean ordered group. Such groups are abelian and are isomorphic to subgroups of $\mathbb{R}$ (see [2, p. 45]). If, in addition, it is possible to define an operation $\times$ on Ω so that $(\Omega,+,\times)$ becomes an Archimedean ordered field, we say that Ω is <u>Archimedean fieldable</u>. The "field" part of the next theorem is due to Greg Cherlin.

THEOREM 14: <u>There are sentences</u> τ,τ' <u>of</u> $\mathcal{L}$ <u>such that if</u> Ω <u>is a homogeneous chain</u>, $\mathcal{A}(\Omega) \models \tau$ $(\mathcal{A}(\Omega) \models \tau')$ <u>if and only if</u> Ω <u>is Archimedean groupable</u> (<u>Archimedean fieldable</u>).

<u>Proof</u>: Let Ω be Archimedean groupable; so Ω is a subgroup of $\mathbb{R}$ without loss of generality. If $\mathcal{A}(\Omega)$ is abelian, the result follows from [5], and if Ω is countable the result follows from Theorem 4 (or [5] if Ω is discrete). Since Ω is primitive, we are reduced to the doubly homogeneous uncountable case. We may assume $1,\zeta \in \Omega$ for some irrational $\zeta > 0$. Let $t_1,t_\zeta \in \mathcal{A}(\Omega)$ be translations by 1 and ζ, respectively; i.e., $t_1\colon \alpha \mapsto \alpha+1$, $t_\zeta\colon \alpha \mapsto \alpha+\zeta$. As in [7, proof of Theorem], the centralizer $C = C(t_1,t_\zeta)$ in $\mathcal{A}(\Omega)$ consists exactly of the translations $f\colon \alpha \mapsto \alpha+\beta$ $(\beta \in \Omega)$. In particular, C is transitive on Ω, is an abelian totally ordered subgroup of $\mathcal{A}(\Omega)$, and no element of C except e fixes any point of $\bar{\Omega}$ $(=\mathbb{R})$. Now let Λ be any uncountable doubly homogeneous chain such that there exist $e < r,s \in \mathcal{A}(\Lambda)$ with $C(r,s)$, the centralizer of r and s in $\mathcal{A}(\Lambda)$, transitive on Λ, abelian and totally ordered, and no element of $C(r,s)$ other than e fixes any point of $\bar{\Lambda}$. Choose any $\alpha \in \Lambda$. For each $\beta \in \Lambda$, there is a unique $f_\beta \in C(r,s)$ such that $f_\beta(\alpha) = \beta$. The correspondence $\beta \leftrightarrow f_\beta$ provides an ordermorphism between Λ and $C(r,s)$, so it is enough to show that $C(r,s)$ is Archimedean groupable. But $C(r,s)$ is an Archimedean ordered group, for if $e < f < g$ with $f,g \in C(r,s)$, then for any $\beta \in \Lambda$, $\{f^n(\beta)\colon n \in \omega\}$ can have no upper bound (otherwise f would fix the least upper bound in $\bar{\Lambda}$). So for some $n \in \omega$, $f^n(\beta) > g(\beta)$. Since $C(r,s)$ is totally ordered, $f^n > g$.

Now assume that Ω is an Archimedean ordered field. Then Ω is doubly homogeneous and we may assume that it is a subfield of $\mathbb{R}$. If Ω is countable, the result follows from Theorem 4, so assume Ω is uncountable. For any $0 < \beta \in \Omega$, the function $h_\beta\colon \alpha \mapsto \alpha\beta$ belongs to $\mathcal{A}(\Omega)$. Indeed, $h_\beta \in N(C(t_1,t_\zeta))$, the normalizer of $C(t_1,t_\zeta)$, since if $g \in C$ (say $g\colon \alpha \mapsto \alpha+\gamma$), then $(h_\beta g h_\beta^{-1})(\alpha) = (\alpha/\beta+\gamma)\beta = \alpha+\gamma\beta$. So for any $h \in N = N(C)$, the map $g \mapsto hgh^{-1}$ is an order-preserving automorphism of the Archimedean ordered group C, and so must correspond to multiplication by a positive real number by Hion's Lemma [2, p. 46]. In particular, if $e < f \in C$, there is $h \in N$ such that $f = ht_1h^{-1}$ (if

$f: \alpha \mapsto \alpha + \gamma$, $f = h_\gamma t_1 h_\gamma^{-1}$); and if $h_1, h_2 \in N$ and $h_1 t_1 h_1^{-1} = h_2 t_1 h_2^{-1}$, then $h_1 g h_1^{-1} = h_2 g h_2^{-1}$ for all $g \in C$. Now suppose that in $\mathcal{A}(\Lambda)$, for every $e < f \in C(r,s)$ there exists $h \in N'$, the normalizer of $C(r,s)$, such that $hrh^{-1} = f$, and that if $h_1, h_2 \in N'$ with $h_1 r h_1^{-1} = h_2 r h_2^{-1}$, then $h_1 g h_1^{-1} = h_2 g h_2^{-1}$ for all $g \in C(r,s)$. We can define a product $\otimes$ on $C(r,s)$ as follows: Let $f, g \in C(r,s)$ with $e < f$. There exists $h \in N'$ such that $hrh^{-1} = f$. Define $g \otimes f = hgh^{-1}$. This is well-defined and the extension of $\otimes$ to all products (when $f \leq e$) is done in the obvious way. It is straightforward to check that this makes $C(r,s)$ an Archimedean ordered field. Since Λ is ordermorphic to $C(r,s)$, this completes the proof.

So far we have been able to capture every property we want. However, since there are only $2^{\aleph_0}$ complete theories, there exist non-ordermorphic doubly homogeneous chains Ω and Λ with $\mathcal{A}(\Omega) \equiv \mathcal{A}(\Lambda)$; indeed, such Ω and Λ exist with $|\Omega| \neq |\Lambda|$. As yet, we have been unable to explicitly obtain such Ω and Λ. The problem is that the Ehrenfeucht game to be played between $\mathcal{A}(\Omega)$ and $\mathcal{A}(\Lambda)$ (to prove $\mathcal{A}(\Omega) \equiv \mathcal{A}(\Lambda)$) is rather complicated. This is a big gap in our work to date.

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