# CRUMBLY SPACES

YURI GUREVICH

*Ben-Gurion University, Beer-Sheva, Israel, and Simon Fraser University, Canada*

## 1. Introduction

HENSON, JOCKUSCH, JR., RUBEL, TAKEUTI (1977) define the first-order theory of a top. space $U$ as the first-order theory of the lattice of closed subsets of $U$. We define and study crumbly spaces in order to answer the following question of HENSON *et al.* (1977).

Q3. *Are any two* 0*-dimensional separable metric spaces without isolated points elementary equivalent? In particular, are the rationals, the irrationals and the Cantor set elementary equivalent as top. spaces?*

The answer is yes. All crumbly spaces without isolated points have the same first-order theory, and any 0-dimensional separable metric space is crumbly. The first-order theory of crumbly spaces is decidable. The method used is that of GUREVICH (1979), essentially Shelah's.

Speaking about a space or a top. space, we always mean a non-empty $T_1$ space. A space and its universe are denoted in the same way. The derivative (the set of limit points) of a point set $X$ is denoted by $X'$. As usual, $X$ is a deleted neighbourhood of a point $y$ if $y \notin X$ and $X \cup \{y\}$ is a nbd (neighbourhood) of $y$.

Considering a chain (linearly ordered set) as a top. (= topological) space, we always have in mind the interval topology.

"W.l.o.g." and "nwd" abbreviate "without loss of generality" and "nowhere dense", respectively.

## 2. Congruences

Let $U$ be a top. space and let $E$ be an equivalence relation on $U$ with closed fibers (= equivalence classes). The fibers of $E$ form the quotient space $U/E$. By definition a set $A$ of fibers is open in $U/E$ if $\bigcup A$ is open in $U$.

$E$ will be called a *congruence* if

(Co 1) *All non-singleton fibers of $E$ are open,* and
(Co 2) *The quotient mapping $x \to x/E$ is closed.*

The quotient mapping is open if (Co 1) holds. However, (Co 1) does not imply (Co 2). Consider a chain

$$\dots -2, -1, 0, 1, 2, \dots, \omega$$

and the equivalence relation with fibers $\{0\}, \{-1, +1\}, \{-2, +2\}, \dots, \{\omega\}$.
For $X \subseteq U$ let $X_E = \bigcup \{x/E: x \in X\}$.

CLAIM 1. *In presence of* (Co 1) *condition* (Co 2) *is equivalent to*
(Co 3) *If $X \subseteq U$ and $y \in (X_E)' - X_E$, then each nbd of $y$ includes some $x/E$ with $x \in X$.*

PROOF: (Co 2) means that $X_E$ is closed for any closed $X$. For a closed $X$ (Co 3) implies $(X_E)' \subseteq X_E \cup X' = X_E$. If $X$, $y$ and a nbd $H$ of $y$ give a counterexample for (Co 3) select a point $fx$ from each $(x/E) - H$ with $x \in X$. Let $F$ be the closure of $\{fx: x \in X\}$. Then $X_E \subseteq F_E$ but $y \notin F_E$; hence $F_E$ is not closed and (Co 2) fails.

CLAIM 2. *Suppose that $E$ is a congruence on $U$. The reduction of $E$ on an arbitrary subspace $V$ of $U$ is a congruence on $V$.*
PROOF: Check (Co 1) and (Co 3).

CLAIM 3. *Let $U$ be a chain.*
(i) *If all non-singleton fibers of $E$ are clopen and convex, then $E$ is a congruence.*
(ii) *Suppose that $E$ is a congruence on $U$. Define $xey$ if $xEy$ and $xEz$ for every $z$ between $x$ and $y$. Then $e$ is a congruence on $U$.*
PROOF: (i) Check condition (Co 3). (ii) Use statement (i).

Given a top. space $I$ and a disjoint family $\{V_i: i \in I\}$ of top. spaces such that $V_i$ is singleton for $i \in I'$, we define $V = \sum \{V_i: i \in I\}$ as follows. $x \in V$ if $x$ belongs to some $V_i$ and an arbitrary $X \subseteq V$ is closed in $V$ if $V_i \cap X$ is closed in $V_i$, for each $i$ and $\{i: V_i \text{ meets } X\}$ is closed in $I$ (so that $X$ is open in $V$ iff $V_i \cap X$ is open in $V_i$ for each $i$ and $\{i: V_i \subseteq X\}$ is open in $I$). Check that $V$ is really a top. space, every $V_i$ is a closed subspace of $V$ and the relation "$x$, $y$ are in the same $V_i$" is a congruence on $V$. We say that $V$ is the *discrete sum* of spaces $V_i$ if $I$ is discrete.

CLAIM 4. *Suppose that $E$ is a congruence on $U$. Form $I \subseteq U$ by selecting a point from each fiber of $E$. Then $I$ is closed, $U = \sum \{i/E\colon i \in I\}$ and the mapping $i/E \to i$ is a homeomorphism of $U/E$ onto $I$.*

PROOF: If $x \in I'$, then $x/E$ cannot be open; hence $x/E$ is singleton and $x \in I$.

Given $X \subseteq U$ with $(i/E) \cap X$ closed for each $i$, we check that $X$ is closed iff $J = \{i\colon i/E \text{ meets } X\}$ is closed. If $X$ is closed and $y \in J'$, then $y \in (X_E)' \subseteq X_E$ and $y/E$ is singleton; hence $y \in X$. If $J$ is closed and $y \in X'$, then $y \in J'_E \subseteq J_E$ and either $y \in ((y/E) \cap X)' \subseteq X$ or $y/E = \{y\} \subseteq X$.

If $A$ is a closed set in $U/E$, then $\bigcup A$ is closed in $U$ and $(\bigcup A) \cap I$ is closed in $I$. If $J$ is closed in $I$, it is closed in $U$; hence $\{i/E\colon i \in J\}$ is closed in $U/E$.

## 3. Crumbly spaces

Given a family $A$ of open sets in a top. space $U$ and a congruence $E$ on $U$, we say that $E$ *crumbles* $A$ if $E$ is the identity on $U - \bigcup A$ and for each $x \in \bigcup A$ the fiber $x/E$ is an open subset of some $a \in A$. If $E$ crumbles $A = \{G\}$, we say that $E$ crumbles $G$.

DEFINITION. A space $U$ is *crumbly* if

(Cr 1) *For each family $A$ of open sets in $U$ there is a congruence on $U$ crumbling $A$*, and

(Cr 2) *Each discrete $X \subseteq U$ can be split into $Y$ and $X - Y$ such that $Y' = (X - Y)'$.*

(Cr 1) does not imply (Cr 2): consider $U = \omega \cup \{F\}$ where $F$ is a non-principal ultrafilter on $\omega$ and $X \subseteq U$ is open iff $X \subseteq \omega$ or $X - \{F\} \in F$.

CLAIM 1. *Let $U$ be a crumbly space.*

(i) *Any subspace of $U$ is crumbly.*

(ii) *For any congruence $E$ the quotient space $U/E$ is crumbly.*

PROOF: (i) is straightforward. (ii) follows from (i) and Claim 4 in § 2.

LEMMA 2. *Let $f$ be a continuous mapping from a space $U$ onto a space $I$ and $U_i = f^{-1}(i)$ for $i \in I$. Suppose that if $y \in U_i \cap (U - U_i)'$ and $H$ is a nbd of $y$, then there is a deleted nbd $J$ of $i$ in $I$ such that $\bigcup \{U_j\colon j \in J\} \subseteq H$. If $I$ and all subspaces $U_i$ satisfy* (Cr 2), *then $U$ satisfies* (Cr 2).

PROOF: Given a discrete $X \subseteq U$ form $K = \{i\colon U_i \cap X \text{ is singleton}\}$ and split it into $K_1, K_2$ such that $K'_1 = K'_2$. Split each $U_i \cap X$ into $X_{1i}, X_{2i}$ such that $X'_{1i} = X'_{2i}$, and both $X_{si}$ are not empty if $|U_i \cap X| \geqslant 2$, and $X_{si}$ is not empty if $i \in K_s$. Form $X_s = \bigcup \{X_{si}\colon i \in I\}$.

Given $y \in U_i \cap X'$ and a nbd $H$ of $y$, we show that $H$ meets $X_s$. That is clear if $y \in (U_i \cap X)'$. Suppose $y \in (X - U_i)'$. W.l.o.g. $H - U_i = \bigcup \{U_j : j \in J\}$ for some deleted nbd $J$ of $i$. If $|U_j \cap X| \geqslant 2$ for some $j \in J$, then $H$ meets $X_s$. Otherwise $J$ meets $K$; hence $J$ meets $K_s$, hence $H$ meets $X_s$. □

THEOREM 3. *$U = \sum \{U_i: i \in I\}$ is crumbly if $I$ and all $U_i$ are crumbly spaces.*

PROOF: In virtue of Lemma 2 it suffices to check (Cr 1). Given a family $A$ of open sets in $U$, form the family $B$ of open sets $\{i: U_i \subseteq a\}$ in $I$ where $a \in A$ and crumble $B$ by a congruence $e$ on $I$. If $i \in I - \bigcup B$, crumble $A|U_i$ by a congruence $E_i$ on $U_i$. For $x \in U_i$ and $y \in U_j$ define $xEy$ if $iej$ or $i = j$ and $xE_iy$. $E$ is an equivalence relation on $U$. It crumbles $A$ if it is a congruence. In the remaining part of the proof we show that $E$ satisfies conditions (Co 1) and (Co 3).

If $F$ is a fiber of $E$, then either $F = \bigcup \{U_i: i \in J\}$ where $J$ is an open fiber of $e$ or else $F$ is a fiber of some $E_i$. In the first case $F$ is open in $U$. In the second case, if $F$ is not singleton, then it is open in $U_i$ and $U_i$ is open in $U$, hence $F$ is open in $U$.

Let $X \subseteq U$, $y \in (X_E)' - X_E$, $y \in U_i$ and let $H$ be a nbd of $y$. W.l.o.g. $i \in I'$ and there is a nbd $J$ of $i$ with $\bigcup \{U_j: j \in J\} \subseteq H$. If $x/E \subseteq U_i$ for some $x \in X$ and $j \in J$, then $x/E \subseteq H$. Otherwise, by (Co 3) applied to $e$, $J$ includes a fiber $K$ of $e$ with $\bigcup \{U_j: j \in K\} = x/E$ for some $x \in X$ which implies $x/E \subseteq H$.

CLAIM 4. *Any crumbly space without isolated points has a perfect nwd set.*

PROOF: Build a sequence $E_0, E_1, \ldots$ of congruences and a sequence $X_0, X_1, \ldots$ of point sets in such a way that $X_n$ comprises a point from each non-singleton fiber of $E_n$, and $E_{n+1}$ refines $E_n$ and crumbles the compliment of $X_n$, and all points are $E_0$-equivalent.

For each $n > 0$ choose $Y_n \subset X_n$ such that $Y'_n = (X_n - Y_n)'$. The difference between $\bigcup \{X_n: n \geqslant 0\}$ and $\bigcup \{(Y_n)_{E_n}: n > 0\}$ is perfect and nwd.

## 4. Crumbly chains

Let $U$ be a chain.

CLAIM 1. *$U$ satisfies* (Cr 2).

PROOF: Given a discrete $X \subseteq U$ define $xey$ if $x, y \in X$ and there is no points of $X'$ between $x$, $y$. Let $X_1$ be the union of non-singleton fibers of $e$. Splitting the non-singleton fibers one gets $Y_1 \subseteq X_1$ with $Y'_1 = (X_1 - Y_1)'$. It remains to split $X - X_1$. W.l.o.g. $X_1 = 0$.

Let $\varkappa$ be an infinite cardinal. An interval $I$ of the subchain $X$ will be called $\varkappa$-*good* if $\varkappa = |I| = |(x, y)|$ for any $x<y$ in $I$. Any non-empty non-singleton interval of the subchain $X$ has a subinterval $\varkappa$-good for some $\varkappa$. If $I$ is $\varkappa$-good arrange a list $\langle I_\alpha: \alpha<\varkappa\rangle$ of all subintervals $(x, y)$ with $x<y$ in $I$ and build disjoint $Y, Z \subseteq I$ meeting each $I_\alpha$, then $Y' = (I-Y)'$. Now consider the equivalence relation $xEy$ if $x, y \in X$ and either $x = y$ or there is a subset $Y$ of $I = \{z \in X\colon z \text{ between } x, y\}$ such that $Y' = (I-Y)'$. □

$U$ is 0-dimensional iff no interval of $U$ with at least two points is connected iff between any two different points of $U$ there is a jump or an empty Dedekind cut. (A jump between $x<y$ means a pair $u<v$ such that $x \leqslant u < v \leqslant y$ and the interval $(u, v)$ is empty.) Suppose that $U$ is 0-dimensional.

THEOREM 2. *The following statements are equivalent*:

(1) *$U$ is crumbly*,

(2) *Each non-empty open interval of $U$ without a minimum* (*respectively maximum*) *point can be partitioned into a chain of open subintervals without a first* (*respectively last*) *member*.

(3) *There are no $\varkappa$, $S$ and $f\colon S \to U$ such that $\varkappa$ is a regular uncountable cardinal, $S$ is a stationary subset of $\varkappa$, $f$ is a continuous mapping from the subspace $S$ of the chain $\varkappa$ into $U$ preserving or reversing the order.*

PROOF: (1)→(2). Given a non-empty open interval $I$ without a maximum point crumble the family of open initial proper subintervals of $I$ and use Claim 3 in § 2.

(2)→(3). Let $\varkappa$, $S$ and $f\colon S \to U$ give a counterexample for (3) where $f$ preserves the order. Partition the least initial interval of $U$ including $fS$ into a chain $J$ of open subintervals without a last member. The set $C = \{\alpha\colon$ there is an initial proper subinterval $K$ of $J$ such that for each $\beta \in S$, $f\beta \in \bigcup K$ iff $\beta<\alpha\}$ is closed and unbounded in $\varkappa$. Consider $\alpha \in C \cap S$ to get a contradiction.

(3)→(2). Let $I$ be a non-empty open interval of $U$ without a maximum point and $\varkappa$ be the cofinality of $I$. Choose an ascending sequence $\langle I_\alpha: \alpha<\varkappa\rangle$ of initial subintervals of $I$ covering $I$ and such that no $I_\alpha$ has a supremum in $I-I_\alpha$. If $\varkappa = \omega$ partition $I$ in the obvious way. If $\varkappa>\omega$ consider the function $f\alpha = \sup \bigcup \{I_\beta\colon \beta<\alpha\}$. By (3) the domain $S$ of $f$ is not stationary. Use a closed unbounded $C \subseteq \varkappa - S$ to partition $I$.

(2)→(1). By Claim 1 it suffices to check (Cr1). Let $A$ be a family of open sets in $U$. A clopen interval $I$ will be called a *crumb* if there is $a \in A$ including $I$. A set $X \subseteq U$ will be called *good* if there is a disjoint family $B$ of crumbs such that $\bigcup B$ is convex and includes $X$. It suffices to prove that each maximal interval in $\bigcup A$ is good. W.l.o.g. $\bigcup A = U$.

Define $xEy$ if $\{x,y\}$ is good. If $x<y<z$, $xEy$ and $yEz$, take disjoint families $B$, $C$ of crumbs such that $\bigcup B$, $\bigcup C$ are convex and include $\{x,y\}$, $\{y,z\}$, respectively. If $y \in b \in B$ and $y \in c \in C$ then $\{x,z\}$ is covered by $b \cap c$, the part of $b$ below $c$, the part of $c$ above $b$, the crumbs of $B$ below $b$ and the crumbs of $C$ above $c$. Hence $E$ is an equivalence relation.

Any fiber of $E$ is an open interval. It suffices to prove that each fiber of $E$ is good. Now use (2).

COROLLARY 3. *Any separable* 0*-dimensional metric space is crumbly.*

PROOF: By classical theorems any separable metric space is second-countable, and any 0-dimensional second-countable metric space is embeddable into the Cantor set. The Cantor set is crumbly by Theorem 2. Now use Claim 1 of § 3.

## 5. The adjusted theory

A point set $X$ is *sandy* if each point of $X$ is isolated in the space.

CLAIM 1. *Let $U$ be a crumbly space with nwd $U'$. Each closed $X \subseteq U'$ is the derivative of some sandy set.*

PROOF: Crumble $U-X$ by a congruence and pick an isolated point in each open fiber.

CLAIM 2. *Let $U$ be a crumbly space with nwd $U'$. For each first-order sentence $\varphi$ in the language $\{\leqslant\}$ the following statements are equivalent:*

(1) *The collection of closed subsets of $U'$ ordered by inclusion satisfies $\varphi$.*

(2) *The collection of sandy subsets of $U$ with $X \leqslant Y$ meaning $X' \subseteq Y'$ satisfies $\varphi$.*

PROOF: Identification of indistinguishable members in (2) gives a model isomorphic to the model of (1).

THEOREM 3. *For each crumbly space $I$ there is a crumbly superspace $U$ such that $I = U'$ and $I$ is nwd in $U$.*

PROOF: The idea is to sew on tails to enough points of $I$. Choose an everywhere dense $I_0 \subseteq I$ and a disjoint family $\{U_i \colon i \in I\}$ of crumbly spaces such that $U_i = \{i\}$ if $i \in I - I_0$ and $U_i' = \{i\}$ if $i \in I_0$. Define a space $U$ as follows: $x \in U$ if $x$ belongs to some $U_i$, and an arbitrary $X \subseteq U$ is closed in $U$ if $U_i \cap X$ is closed in $U_i$ for each $i$ and $X$ includes the derivative in $I$ of $\{i\colon\ U_i \text{ meets } X\}$ (so that $X$ is open in $U$ iff $U_i \cap X$ is open in $U_i$ for each $i$ and $\{i\colon\ U_i \subseteq X\}$ includes a deleted nbd in $I$ of any $j \in I \cap X$). Check that $U$ is really a top. space, all spaces $U_i$ are closed subspaces of $U$, $I = U'$, $I$ is nwd in $U$, and associating $i$ with each point in $U_i$ gives a continuous mapping from $U$ onto $I$.

We prove that $U$ is crumbly. In virtue of Lemma 2 in § 3 it suffices to check (Cr1) only. Given a family $A$ of open sets in $U$ crumble $A|I$ by a congruence $e$ on $I$. If $J \subseteq (\bigcup A) \cap I$ is a fiber of $e$, choose $gJ \in A$ with $J \subseteq gJ$ and form $hJ = gJ \cap \bigcup \{U_i\colon\ i \in J\}$. The sets $hJ$ are clopen and disjoint. Consider the equivalence relation $E$ on $U$ whose only non-singleton fibers are the sets $hJ$. Clearly, $E$ crumbles $A$ if it is a congruence. Clearly, $E$ satisfies (Co1). It remains to check (Co3). Let $X \subseteq U$, $y \in (X_E)' - X_E$ and $H$ be a nbd of $y$ in $U$. Then $y \in I$ and there is a deleted nbd $K$ of $y$ in $I$ such that $\bigcup \{U_i\colon\ i \in K\} \subseteq H$. If $H$ does not include any singleton $x/E$ with $x \in X$, then, by (Co3) applied to $e$, $K$ includes some $J$ with $hJ = x/E$ for some $x \in X$ which implies $x/E \subseteq H$.

Definition. The *adjusted first-order theory* of a crumbly space $U$ with a nwd derivative is the first-order theory of the Boolean algebra of sandy subsets of $U$ with an additional relation $X \leqslant Y$ meaning $X' \subseteq Y'$. $T$ is the adjusted first-order theory of all crumbly spaces with nwd derivatives.

Corollary 4. *The first-order theory of crumbly spaces is interpretable in $T$. All crumbly spaces without isolated points have the same first-order theory if all crumbly spaces $U$ such that $U'$ is nwd and $0 \neq U' = U''$ have the same adjusted first-order theory.*

Claim 5. *Every two crumbly spaces $U$, $V$ with singleton derivative have the same adjusted first-order theory.*

Proof: We describe a winning strategy for us (the player II) against the devil (the player I) in the Ehrenfeucht game $G_n(U, V)$.

If $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$ were chosen during the first $m$ moves consider all intersections $\pm X_1 \cap \ldots \cap \pm X_m$ and $\pm Y_1 \cap \ldots \cap \pm Y_m$ where $+Z = Z$ and $-Z$ is the complement of $Z$ in the respective Boolean algebra of sandy sets. It gives $A_1, \ldots, A_l$ in $U$ and $B_1, \ldots, B_l$ in $V$, respectively, where $l = 2^m$. Our strategy is to satisfy the following requests: $A_k' = 0$ iff $B_k' = 0$, and either both $|A_k|$, $|B_k|$ are $\geqslant 2^{n-k}$ or $|A_k| = |B_k|$.

## 6. The theory of sum

In order to analize theory $T$ we use the method of GUREWICH (1979).

CLAIM 1. *The universal fragment of $T$ is decidable.*

PROOF: It suffices to prove satisfiability in $T$ of any quantifier-free formula $\varphi \wedge \psi$ where $\varphi$ states that $v_1, \ldots, v_l$ partition 1 and $\psi$ describes a quasi-order on $0, v_1, \ldots, v_l$ with $0 \leqslant v_k$ for each $k$. Consider the discrete sum $U_1 + \ldots + U_l$ where $U_k$ is a copy of $\omega$ if $\psi \vdash v_k \leqslant 0$ and of $\omega + 1$ if $\psi \vdash 0 < v_k$. Choose disjoint infinite sandy $U_{k1}, \ldots, U_{kl}$ in each $U_k$ and interprete each $v_k$ as $\bigcup \{U_{jk}: \psi \vdash v_j \leqslant v_k\}$. □

A crumbly space $U$ with nwd $U'$ augmented by a sequence $P = \langle P_1, \ldots, P_l \rangle$ of sandy subsets of $U$ will be called an *ausp* of weight $l$. If $V$ is a clopen subspace of $U$, then $\langle V, P|V \rangle$ will be called a *subausp* of $\langle U, P \rangle$.

The 0-*theory* $\mathrm{Th}^0(U, P)$ of an ausp$\langle U, P \rangle$ of weight $l$ is a quantifier-free description of that ausp in variables $v_1, \ldots, v_l$ (see a more rigorous definition in GUREVICH, 1979). Let $\xi = \langle \xi n : n < \omega \rangle$ be a sequence of natural numbers with a tail of zeros. The $n - \xi$-*theory* $\mathrm{Th}^n_\xi(U, P)$ of an ausp$\langle U, P \rangle$ is defined by inductions:

$$\mathrm{Th}^0_\xi(U, P) = \mathrm{Th}^0(U, P),$$
$$\mathrm{Th}^{n+1}_\xi(U, P) = \{\mathrm{Th}^n_\xi(U, P\hat{\ } Q) : lh(Q) = \xi n\}.$$

The true value of $U \models \varphi(P)$ with elementary $\varphi$ is computable from an appropriate $\mathrm{Th}^n_\xi(U, P)$ (with $n$, $\xi$ computable from $\varphi$).

Sets $\mathrm{Tr}^n_\xi(l)$ and $l$-traces are defined by induction. $\mathrm{Tr}^0(l) = \mathrm{Tr}^0_\xi(l) = \{\mathrm{Th}^0(M) : M$ is an ausp of weight $l\}$. $s \in \mathrm{Tr}^0(l)$ is an *$l$-trace* of $t \in \mathrm{Tr}^0(l+m)$ if there are $U$, $P$, $Q$ such that $s = \mathrm{Th}^0(U, P)$ and $t = \mathrm{Th}^0(U, P\hat{\ } Q)$. $\mathrm{Tr}^{n+1}_\xi(l) = \{t \subseteq \mathrm{Tr}^n_\xi(l + \xi n)$: all members of $t$ have the same $l$-trace$\}$. $s \in \mathrm{Tr}^0(l)$ is an *$l$-trace* of $t \in \mathrm{Tr}^{n+1}_\xi(l+m)$ if $s$ is the $l$-trace of each member of $t$. Check that $\mathrm{Tr}^n_\xi(l)$ is recursive in $n$, $\xi$, $l$ and contains the $n - \xi$-theory of any ausp of weight $l$.

Let $\tilde{U} = \langle U, P \rangle$ be an ausp of weight $l$, let $E$ be a congruence on $U$, $X$ range over open fibers of $E$ and let $\tilde{X}$ be the subausp $\langle X, P|X \rangle$. We introduce some more notation. For each $t \in \mathrm{Tr}^n_\xi(l)$ the set $\{X : \mathrm{Th}^n_\xi(\tilde{X}) = t\}$ will be denoted by $(\tilde{U}/E)^n_\xi t$. The sequence $\langle (\tilde{U}/E)^n_\xi t : t \in \mathrm{Tr}^n_\xi(l) \rangle$ (we consider $\mathrm{Tr}^n_\xi(l)$ being linearly ordered in a standard way) will be denoted by $(\tilde{U}/E)^n_\xi$. Finally, $[\tilde{U}/E]^n_\xi$ is the ausp $\langle U/E, (\tilde{U}/E)^n_\xi \rangle$. $\xi$ may be omitted everywhere if $n = 0$.

CLAIM 2. $\mathrm{Th}^0(\tilde{U})$ *is computable from* $\mathrm{Th}^0[\tilde{U}/E]^0$.

PROOF: If $\tau$ is a Boolean term in variables $v_1, \ldots, v_l$, then $\tau(P) \cap X = \tau(P|X)$ (check by induction on $\tau$). Given $\mathrm{Th}^0[\tilde{U}/E]^0$, we compute whether $\tau(P) = 0$ and whether $\tau_1(P) \leqslant \tau_2(P)$.

$\tau(P) = 0$ iff $\tau(P|X) = 0$ for each $X$ iff $(\tilde{U}/E)^0 t \neq 0$ only for $t$ implying $\tau = 0$.

$\tau_1(P) \leqslant \tau_2(P)$ iff $\tau_1(P|X) \leqslant \tau_2(P|X)$ for each $X$ and $\{X: \tau_1(P|X) \neq 0\}' \subseteq \{X: \tau_2(P|X) \neq 0\}'$ in $U/E$ iff $(\tilde{U}/E)^0 t \neq 0$ only for $t$ implying $\tau_1 \leqslant \tau_2$ and

$$\bigcup \{(\tilde{U}/E)^0 t: \ t \vdash \tau_1 \neq 0\} \leqslant \bigcup \{(\tilde{U}/E)^0 t: \ t \vdash t_2 \neq 0\}.$$

$\eta$ denotes below a sequence $\langle \eta n: n < \omega \rangle$ of natural numbers with a tail of zeros.

THEOREM 3. *There is a recursive function* $\eta = T(n, \xi, l)$ *such that* $\mathrm{Th}^n_\xi(\tilde{U})$ *is computable from* $n$, $\xi$, $l$ *and* $\mathrm{Th}^n_\eta[\tilde{U}/E]^n_\xi$.

PROOF: See Theorem 2.2 in GUREWICH (1979) and Claim 2.

If $V$ is the discrete sum of spaces $V_j$, then $\mathrm{Th}^n_\xi(V, Q)$ will be called the *discrete sum* of $\mathrm{Th}^n_\xi(V_j, Q|V_j)$. That defines the commutative semigroup of $n-\xi$-theories of ausps of weight $l$. By Theorem 3 the sum $s+t$ of elements of that semigroup is computable from $n$, $\xi$, $l$, $s$, $t$.

If $I$ is a crumbly space with nwd $I'$, $U = \sum \{U_i: i \in I\}$, $P$ is a sequence of $l$ sandy subsets of $U$ and $\mathrm{Th}^n_\xi(U_i, P|U_i) = s$ for each isolated $i \in I$ we write $\mathrm{Th}^n_\xi(U, P) = s \cdot I$. (In particular, $s2 = s+s$.) By Theorem 3, $s \cdot I$ is computable from $n$, $\xi$, $l$, $s$ and the $n-T(n, \xi, l)$-theory of $I$.

CLAIM 4. *Let $S$ be the semigroup of $n-\xi$-theories of ausps of weight $l$.*

(i) *There is an integer $m$ such that $tm = t\omega = t \cdot I$ for every $t \in S$ and every infinite discrete space $I$.*

(ii) *The semigroup generated by a non-empty subset $S_0$ of $S$ is the closure of $S_0$ under discrete sums.*

(iii) $t(\omega+1) = t \cdot I$ *for any crumbly space $I$ with singleton derivative.*

PROOF: (i) is clear. (ii) follows from (i). To prove (iii) use Claim 5 in § 5.

## 7. Uniformity

Two ausps are *0-equivalent* (*$n-\xi$-equivalent*) if they have the same 0-theory ($n-\xi$-theory). An ausp $\tilde{U} = \langle U, P \rangle$ is 0-uniform if $0 \neq U' = U''$, and either $P$ is the empty sequence or $P$ partitions 1, and $U' = P_i'$ for each non-empty member $P_i$ of $P$. Check that all non-sandy subausps of a 0-uniform ausp are 0-equivalent. $\tilde{U}$ is *$n-\xi$-uniform* if all non-sandy subausps of $\tilde{U}$ are $n-\xi$-equivalent. Below $\tilde{U} = \langle U, P\rangle$ is an ausp. If $X \subseteq U$ is non-empty and clopen, then $\tilde{X} = \langle X, P|X\rangle$.

CLAIM 1. *Suppose that $0 \neq U' = U''$ and either $P$ is the empty sequence or $P$ partitiones 1. For every $n$, $\xi$ there is an $n-\xi$-uniform subausp of $\tilde{U}$.*

PROOF: Let $M$, $N$ range over non-sandy subausps of $\tilde{U}$ and $S(M) = \{\mathrm{Th}^n_\xi(N)\colon N \subseteq M\}$. Choose a minimal $S = S(M_0)$ and form $t = \sum\{s\omega\colon s \in S\}$. Check that $s_1 + s_2 \in S$ if $s_1, s_2 \in S$. In virtue of Claim 4 in § 6 that implies $t \in S$. For any $s \in S$ there is $s' \in S$ such that $s = s' + t = t$. Hence $M_0$ is $n-\xi$-uniform.

A congruence $E$ on a 0-uniform ausp $\tilde{U}$ will be called *normal* if every sandy fiber of $E$ is singleton and for each non-empty member $P_i$ of $P$ the derivative of $\{x \in P_i\colon x/E \text{ is singleton}\}$ is equal to $\bigcup (U/E)'$.

CLAIM 2. *For each family $A$ of open sets in a 0-uniform ausp $\tilde{U}$ there is a normal congruence on $\tilde{U}$ crumbling $A$.*

PROOF: Crumble $A$ by an arbitrary congruence $E$, split every sandy non-singleton fiber of $E$ into new singleton fibers, and for each fiber $F$ of $E$ and each non-empty $P_i$ pinch out a new singleton fiber from $F \cap P_i$.

We say that $t \in \mathrm{Tr}^n_\xi(l)$ is *sandy* if the 0-trace of $t$ implies $l \leqslant 0$. The recursive function $\eta = T(n, \xi, l)$ from Theorem 3 in § 6 is used below.

LEMMA 3. *Let $K$ be a class of ausps of weight $l$ closed under subausps. A set $S \subseteq \mathrm{Tr}^n_\xi(l)$ includes $\mathrm{Th}^n_\xi(K) = \{\mathrm{Th}^n_\xi(M)\colon M \in K\}$ if*

(0) *$S$ contains the $n-\xi$-theory of any singleton member of $K$,*

(1) *$S$ is closed under discrete sums,*

(2) *$t(\omega+1) \in S$ if $t \in S$, and*

(3) *$\mathrm{Th}^n_\xi(\tilde{U}) \in S$ if $\tilde{U} \in K$ and there is a normal congruence $E$ on $\tilde{U}$ s.t. $\mathrm{Th}^n_\xi(\tilde{X}) \in S$ for any open fiber of $E$ and $[\tilde{U}/E]^n_\xi$ is $n-T(n, \xi, l)$-uniform.*

*If $U' = U''$ for every $\tilde{U} \in K$, then* (2) *can be replaced by a weaker condition.*

(2′) *$t(\omega+1) \in S$ if $t \in S$ and $t$ is not sandy.*

PROOF: Let $\tilde{U} \in K$, and $X$, $Y$ range over clopen subspaces of $U$, and $\mathrm{th}(X) = \mathrm{Th}^n_\xi(\tilde{X})$. Call $X$ *good* if $\mathrm{th}(Y) \in S$ for each $Y \subseteq X$. By contradiction suppose that $U$ is not good.

By Claim 2 the family of good sets can be crumbled by a normal congruence $E$. Let $I = U/E$. Use (0) to check that each open $i \in I$ is good. If $I$ is discrete then each $X$ is the discrete sum of non-empty $i \cap X$; hence, by (1), $S$ contains each $\mathrm{th}(X)$, hence $U$ is good. Therefore $I' \neq 0$.

If $i \in I' - I''$, choose a clopen nbd $J$ of $i$ whose derivative is singleton. Given $X \subseteq \bigcup J$, use (1) and (2) to check that $\text{th}(X) \in S$. Hence $\bigcup J$ is good. But then $i$ should be isolated in $I$. Therefore $I' = I''$.

By Claim 1 there is a clopen $J \subseteq I$ such that the corresponding subausp of $[\tilde{U}/E]^n_\xi$ is $n - T(n, \xi, l)$-uniform. By (3) $S$ contains $\text{th}(\bigcup J)$. It is easy to see that $\text{th}(X) \in S$ for any $X \subseteq \bigcup J$, i.e. $\bigcup J$ is good. But then $J$ should be sandy which is impossible. □

We define now uniform $n - \xi$-theories of an ausp $\tilde{U}$ of weight $l$. $UT^0_\xi(\tilde{U}) = \text{Th}^0(\tilde{U})$ and $UT^{n+1}_\xi(\tilde{U}) = \{UT^n_\eta(M): M \in Q^{n+1}_\xi(\tilde{U})\}$ where $\eta = T(n, \xi, l)$ and $Q^{n+1}_\xi(\tilde{U})$ is the collection of $n - \eta$-uniform ausps $\langle V/E, R\rangle$ where $V$ is a clopen subspace of $U$, $E$ is a normal congruence of $V$, $R = \langle Rt: t \in \text{Tr}^n_\xi(l + \xi n)\rangle$ and for every $X \in Rt$, $\text{Th}^0(\tilde{X})$ is the $l$-trace of $t$.

THEOREM 4. *There is an algorithm computing* $\text{Th}^n_\xi(\tilde{U})$ *from* $n$, $\xi$, $l$ *and* $UT^n_\xi(\tilde{U})$ *whenever* $\tilde{U}$ *is an* $n - \xi$*-uniform ausp of weight* $l$.

PROOF: The algorithm uses a recursion in $n$. Let $\tilde{U} = \langle U, P\rangle$ be an $(n+1) - \xi$-uniform ausp of weight $l$. Given $n+1$, $\xi$, $l$ and $UT^{n+1}_\xi(\tilde{U})$, compute $\eta = T(n, \xi, l)$ and $B = \{\text{Th}^n_\eta(M): M \in Q^{n+1}_\xi(\tilde{U})\}$. Let $K$ be the set of ausps $\langle \tilde{V}, Q\rangle = \langle V, (P|V)^\frown Q\rangle$ where $V$ is a clopen subspace of $U$ and $lh(Q) = \xi n$. It suffices to compute $\text{Th}^n_\xi(K) = \{\text{Th}^n_\xi(M): M \in K\}$, because $\text{Th}^{n+1}_\xi(\tilde{U}) = \{t \in \text{Th}^n_\xi(K): t$ is not sandy$\}$. The set $S_0$ of $n - \xi$-theories of singleton members of $K$ is computable from $n$, $\xi$, $l$ and $B$.

If $S = \text{Th}^n_\xi(K)$, then

(a) $S_0 \subseteq S$, $S$ is closed under discrete sums, and $t(\omega+1) \in S$ for every non-sandy $t \in S$, and

(b) For every $\langle V/E, R\rangle \in Q^{n+1}_\xi(\tilde{U})$ with $\{t: Rt \neq 0\} \subseteq S$ there is $Q$ such that $R = (\langle \tilde{V}, Q\rangle/E)^n_\xi$.

Clause (a) is clear. To check (b), for every $X \in Rt$ choose $Q_X$ such that $\text{Th}^n_\xi(\tilde{X}, Q_X) = t$. The desired $Q$ satisfies $Q|X = Q_X$ for each open fiber $X$ of $E$.

Given $b = \text{Th}^n_\eta(\langle V/E, R\rangle) \in B$, we can compute $\{t: Rt \neq 0\}$, the *spectrum* of $b$. If $R = (\langle \tilde{V}, Q\rangle/E)^n_\xi$ then the algorithm of Theorem 3 in § 6 computes $\text{Th}^n_\xi(V, Q)$ from $n$, $\xi$, $l + \xi n$ and $b$. Therefore (b) implies

(c) If $b \in B$ and $S$ includes the spectrum of $b$, then the algorithm of Theorem 3 in § 6 is applied to $\langle n, \xi, l + \xi n, b\rangle$ and the result belongs to $S$.

By Lemma 3, $\text{Th}^n_\xi(K)$ is the least subset of $\text{Tr}^n_\xi(l + \xi n)$ satisfying (a), (c). Hence it is computable from $n$, $\xi$, $l$ and $B$.

## 8. Elimination of quantifiers

Let $\tilde{U} = \langle U, P\rangle$ be a 0-uniform ausp and let $\tilde{V} = \langle V, P|V\rangle$ be a subausp of $\tilde{U}$. It is easy to see that if $\langle V/E, R\rangle \in Q_0^1(\tilde{U})$ (where 0 denotes the zero sequence), then $R = (\tilde{V}/E)^0$ and $\langle V/E, R\rangle = [\tilde{V}/E]^0$.

CLAIM 1. *$UT_0^1(\tilde{U})$ is computable from* $\mathrm{Th}^0(\tilde{U})$.

PROOF: The identity relation id on $U$ gives $[\tilde{U}/\mathrm{id}]^0 \in Q_0^1(\tilde{U})$. By Claim 4 in § 3 there is $C \subseteq U'$ perfect and nwd in $U'$. By Claim 2 in § 7 $U-C$ can be crumbled by a congruence $e$ normal on $\tilde{U}$. $[\tilde{U}/e]^0$ is another member of $Q_0^1(\tilde{U})$. Let $[\tilde{V}/E]^0 \in Q_0^1(\tilde{U})$. If $E$ is the identity relation on $V$, then $\mathrm{Th}^0[\tilde{V}/E]^0 = \mathrm{Th}^0[\tilde{U}/\mathrm{id}]^0$. Otherwise $\mathrm{Th}^0[\tilde{V}/E]^0 = \mathrm{Th}^0[\tilde{U}/e]$. So $UT_0^1(\tilde{U})$ comprises exactly two elements both computable from $\mathrm{Th}^0(\tilde{U})$.

CLAIM 2. *$UT_\xi^n(\tilde{U})$ is computable from $n$, $\xi$ and* $\mathrm{Th}^0(\tilde{U})$.

PROOF: The algorithm uses a recursion in $n$. Given $n+1$, $\xi$ and $\mathrm{Th}^0(\tilde{U})$ compute the weight $l$ of $U$ and $\eta = T(n, \xi, l)$. For $M = [\tilde{V}/E]^0 \in Q_0^1(\tilde{U})$ let $KM = \{\langle W/e, R\rangle \in Q_\xi^{n+1}(\tilde{U})$: $W$ is a clopen subspace of $V$ and $e = E|W\}$ and $UT_\eta^n(KM) = \{UT_\eta^n(N)$: $N \in KM\}$. It suffices to prove that $UT_\eta^n(KM)$ is computable from $\mathrm{Th}^0(M)$, because

$$UT_\xi^{n+1}(\tilde{U}) = \bigcup \{UT_\eta^n(KM)\colon M \in Q_0^1(\tilde{U})\}$$

and, by Claim 1, $\{\mathrm{Th}^0(M)\colon M \in Q_0^1(\tilde{U})\}$ is computable from $\mathrm{Th}^0(\tilde{U})$.

Given $M = [\tilde{V}/E]^0 \in Q_0^1(\tilde{U})$, compute $S = \{\mathrm{Th}^0(X, P|X)$: $X$ is an open fiber of $E\}$. Let $s^* = \{t \in \mathrm{Tr}_\xi^n(l+\xi n)$: $s$ is the $l$-trace of $t\}$ and $F$ be the set of functions $f$ with $S = \mathrm{dom}(f)$ and $0 \neq fs \subseteq s^*$ for each $s \in S$. By the induction hypothesis the uniform $n-\eta$-theory of $N = \langle W/e, R\rangle \in KM$ is computable from $\mathrm{Th}^0(N)$ which is computable from the function $f_N$ associating $\{t \in s^*\colon Rt \neq 0\}$ with each $s \in S$. It suffices to prove that for each $f \in F$ there is $N \in KM$ with $f_N = f$. Using (Cr2), split each non-empty $(\tilde{V}/E)^0 s$ into disjoint subsets $Qt$ with the same derivative where $t \in fs$. By Claim 1 in § 7, there is an $n-\eta$-uniform subausp $N$ of the ausp $\langle \tilde{V}/E, Q\rangle$. Clearly, $N \in KM$ and $f_N = f$.

CLAIM 3. *$\tilde{U}$ is $n-\xi$-uniform.*

PROOF: By Claim 1 in § 7 there is an $n-\xi$-uniform subausp $\tilde{V}$ of $\tilde{U}$, by Claim 2 and Theorem 4 in § 7 all $n-\xi$-uniform subausps of $\tilde{U}$ have the same $n-\xi$-theory $u$. Let $X$, $Y$ range over clopen subspaces of $U$, let $K$ be the collection of subausps of $\tilde{U}$, $\mathrm{th}(X) = \mathrm{Th}_\xi^n(\tilde{X})$, $S_1 = \{\mathrm{th}(X)$: $X$ is singleton$\}$, $S_2$ is the closure of $S_1$ under discrete sums and $S = S_2 \cup \{u\}$. It suffices to verify conditions (0), (1), (2′) and (3) of Lemma 3 in § 7.

(0) is trivial. In order to verify (1) check that $u+u = u$ and $u+s = u$ for each $s \in S_1$. The latter implies $u+s = u$ for each $s \in S_2$.

*Checking* (2′). Pick $x \in V'$ and crumble $V-\{x\}$ by a normal congruence of $\tilde{V}$. Together with the previous paragraph it gives $u(\omega+1) = u$.

*Checking* (3). Let $E$ be a normal congruence on $\tilde{X}$ such that $\mathrm{th}(Y) \in S$ for each open fiber $Y$ of $E$ and $[\tilde{X}/E]^n_\xi$ is $n-\eta$-uniform where $\eta = T(n, \xi, l)$ and $l$ is the weight of $\tilde{U}$. By Claim 2 $\tilde{U}$ and $\tilde{V}$ have the same $(n+1)-\xi$-uniform theory; hence there is $\langle Y/e, R\rangle \in Q^{n+1}_\xi(V)$ whose $n-\eta$-uniform theory coincides with that of $[\tilde{X}/E]^n_\xi$. It is easy to see that $\langle Y/e, R\rangle = [\tilde{Y}/e]^n_\xi$. By Theorem 3 in § 6 $\mathrm{th}(X) = t(Y) = u \in S$.

THEOREM 4. *$\mathrm{Th}^n_\xi(\tilde{U})$ is computable from $n$, $\xi$ and $\mathrm{Th}^0(\tilde{U})$.*

PROOF: Use Claims 2, 3 and Theorem 4 in § 7.

COROLLARY 5.

(i) *The adjusted first-order theory of crumbly spaces is decidable.*

(ii) *The first-order theory of crumbly spaces is decidoble.*

(iii) *Every two 0-uniform crumbly spaces have the same adjusted first-order theory.*

(iv) *Every two crumbly spaces without isolated points have the same first-order theory.*

PROOF: (i) and (iii) follow from Theorem 4. Now use Corollary 4 in § 5.

## References

GUREVICH, Y., 1979, *Modest theory of short chains I*, Journal of Symbolic Logic, vol. 44, pp. 481–490

HENSON, C. W., C. G. JOCKUSCH, JR., L. A. RUBEL, G. TAKEUTI, 1977, *First order topology*, Dissertationes Mathematicae, vol. 143