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HOMOGENEOUS OPTIMAL FLEET

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Abstract—The structure of chains in the optimal chain decomposition of a periodic schedule S is investigated. A finite oriented graph termed the Linis Graph (LG) is defined which serves as the key for this investigation. The edges of the LG are trip-types of S and the vertices of the LG represent terminals. It is proved that there is an Euler cycle for a connected LG satisfying natural precedence relations between arrival and departure times. Expansion of this cycle in a real time gives a "master-chain" of trips which, being repeated periodically, gives an infinite periodic chain. Time-shifted periodic replication of this chain allows obtaining a group of twin-type periodic chains forming an optimal fleet over S. It is proved that if the LG has m connected components then there is an optimal fleet consisting of m groups of similar periodic chains. It is shown that if the graph of terminals is connected and the LG is disconnected then it is possible to obtain a twin-type fleet over S by adding to S "dummy" trip-types. A general approach to constructing a twin-type fleet of minimal size for this case is described. The relation of the theory developed to the so-called center problem is discussed.

1. INTRODUCTION

This paper treats some problems arising in the latest stages of planning transportation activities. Assume that the following decisions have been already made: trips which have to be carried out during the planning period (say, a year) are chosen; a preliminary estimation of the fleet size has been made and possible changes in the departure/arrival times in order to minimize the fleet size have been already done. (It is assumed that each vehicle can carry out any trip in the schedule.) The next phase in planning transportation work should be constructing routes for individual vehicles, or more formally, decomposing the schedule into chains each of which represent a sequence of trips carried out by one physical vehicle. If this decomposition is made properly, the minimal fleet size should be equal to a certain constant determined in the following way (see Bartlett (1957), Salzborn (1974), Linis and Maksim (1967), Gertsbakh and Gurevich (1977)): let $d_a(t)$ be an integer valued function defined as the difference between the number of departures and arrivals occuring at terminal a during the time $[0, t]$. Then the minimal fleet size (MFS) is equal to the total deficit, D:

$$D = \sum_{a \in A} \max_{0 \leq t} d_a(t). \tag{1.1}$$

(Here the sum is taken over the set A of all terminals appearing in the schedule.)

The chains in the optimal decomposition constitute an important object for solving further problems connected with the implementation of the schedule, namely, the assignment of crews to trips and vehicles and the maintenance of the vehicles. The structure of chains in the optimal chain decomposition will be the main subject of this paper.

In this paper we deal with purely periodic schedules. This means that there is a time unit called the period (a week, a day, etc.) such that all trips whose departures take place in a period are always repeated in the next one.

Periodic schedules are widely used in different kinds of transportation systems because they attract more passengers: people prefer to use stable, periodically repeated trips to which they get accustomed to in the course of time. In addition, the periodic component of the schedule usually serves as a frame for the whole time-table to which additional trips should be added according to passenger demand during the peak periods.

In this paper we shall investigate the possibility of decomposing the schedule into a number of periodic twin-type chains. Such decomposition makes every vehicle to repeat the same sequence of trips where each sequence contains "twins" of all trips appearing in one period. This implies that the arrival times of any vehicle to any given terminal contain a periodic subsequence of times t_0, t_0+T^*, $t_0+2T^*, \ldots, t_0+kT^*, \ldots$, period T^* being the same for all vehicles. Thus each vehicle does the same amount of transportation work during time T^* (say, the same total mileage) and visits all terminals mentioned in the schedule. Therefore, all vehicles are in similar conditions and the total transportation work is uniformly distributed between them. It is a common rule to send each vehicle to maintenance after completing a fixed amount of work (e.g. an aircraft has to be checked after each 500 hr spent in the air). A twin-type fleet allows to distribute the maintenance times uniformly over any calendar period for each vehicle thus providing a uniform load distribution for repair facilities. Besides, if these facilities are located in a terminal mentioned in the schedule no additional trips will be needed to provide arrival of vehicles to the repair station.

Twin-type fleet might be convenient for crew scheduling because an assignment of one or several crews to each vehicle provides automatically that all crews will have the same load. Also the process of computerized scheduling simplifies considerably when the chains are of twin-type. It is worth noting that practical scheduling experts prefer usually to think about the schedule in terms of chains; here again twin-type representation is convenient for manual and graphical operation.

Let us term the fleet which consists of groups of similar periodic chains as *homogeneous*. The main subject of this paper is to develop a theory for constructing a homogeneous optimal fleet.

Let us consider an example illustrating our results and providing an intuitive background for their derivation.

Example 1

We have a schedule containing six different trip-types named α, β, γ, δ, φ, ψ which has to be carried out in every calendar period of length T (see the upper part of Fig. 1). The letters a, b, c, e designate the departure/arrival terminals. We say that a trip belongs to the kth period if its departure takes place in this period, i.e. if the departure time t satisfies the inequality $(k-1)T \leq t < kT$, $k = 1, 2, \ldots$. A trip belonging to the kth period will be denoted by a lower index k: $\alpha_{(k)}$, $\beta_{(k)}$, etc. The schedule is purely periodic. The lower part of Fig. 1 shows the deficit functions of the schedule. Recall that the deficit function $d_a(t)$ for terminal a increases by one at each departure from a and decreases by one at each arrival at a (see Gertsbakh and Gurevich (1977)).

Note that deficit functions become periodic with period T already after time T. The time span between two neighbouring maxima of a deficit function will be called a "hollow". We see that hollows of $d_b(t)$ form two sequences with period T each. We will say that $d_b(t)$ has two hollow types b_1 and b_2. $d_a(t)$, $d_c(t)$ and $d_e(t)$ have just one hollow type each: a_1, c_1 and e_1.

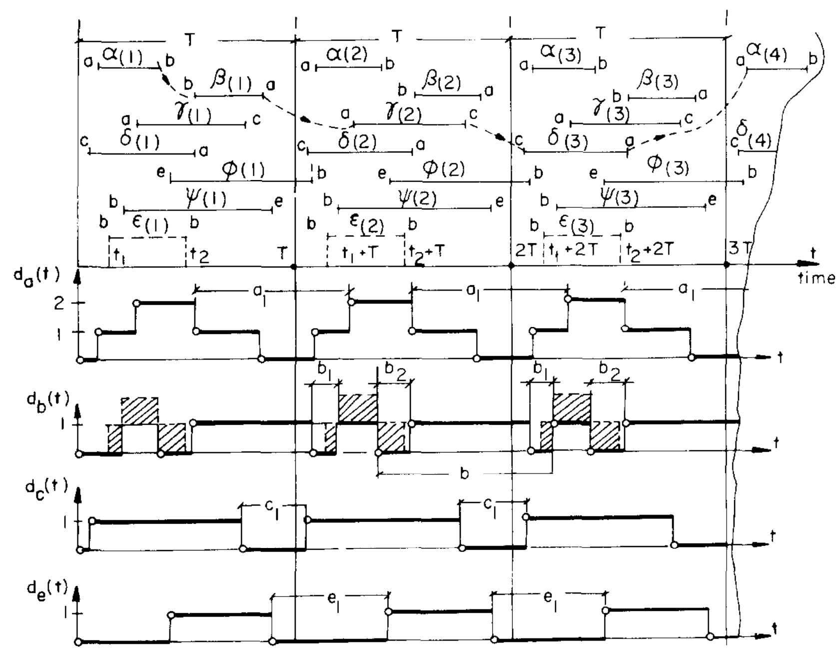

Fig. 1. A periodic schedule and its deficit functions.

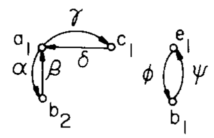

Fig. 2. The Linis Graph of the schedule given on Fig. 1.

We say that φ goes from a_1 to b_1 because its departure is located in the hollow of type a_1 and its arrival is located in the hollow of type b_1. The same holds for other trip-types.

Now let us construct an oriented graph (the Linis Graph) whose vertices are the hollow types and whose oriented edges are the trip-types. This graph is shown on Fig. 2 and we can see the fact that it contains two connected components.

For trip-types α, β, γ, δ corresponding to the larger component of the graph it is possible to construct the following infinite chain having period $3T$ (see Fig. 1):

$$\alpha_{(1)} \to \beta_{(1)} \to \gamma_{(2)} \to \delta_{(3)} \to \alpha_{(4)} \to \beta_{(4)} \to \gamma_{(5)} \to \delta_{(6)} \to \alpha_{(7)} \to \tag{1.2}$$

In order to carry out all the trips of type α, β, γ, δ, except possibly for some trips in a finite number of periods, the chain (1.2) should be "copied" twice, each time with one T shift:

$$\begin{gathered}\alpha_{(2)} \to \beta_{(2)} \to \gamma_{(3)} \to \delta_{(4)} \to \alpha_{(5)} \to \beta_{(5)} \to \gamma_{(6)} \to \delta_{(7)} \to \alpha_{(8)} \to \\ \alpha_{(3)} \to \beta_{(3)} \to \gamma_{(4)} \to \delta_{(5)} \to \alpha_{(6)} \to \beta_{(6)} \to \gamma_{(7)} \to \delta_{(8)} \to \alpha_{(9)} \to\end{gathered} \tag{1.3}$$

The following two infinite similar chains with period $2T$ serve the second component:

$$\phi_{(i)} \to \psi_{(i+1)} \to \phi_{(i+2)} \to \psi_{(i+3)} \to \quad i = 1, 2. \tag{1.4}$$

The system of five chains (1.2), (1.3), (1.4) decomposes the whole schedule, except for trips $\gamma_{(1)}$, $\delta_{(1)}$, $\delta_{(2)}$, $\psi_{(1)}$. It is easy to join them to the existing chains "from the left": put $\delta_{(1)}$ before $\alpha_{(2)}$, $\{\gamma_{(1)} \to \delta_{(2)}\}$ before $\alpha_{(3)}$ and $\psi_{(1)}$ before $\varphi_{(2)}$. Thus we obtained a homogeneous fleet over the given schedule.

Note that the total number of chains is equal to the minimal fleet size which is five, as one can see from the deficit functions. Note that all chains are "good" in the sense defined in Gertsbakh and Gurevich (1977), specifically each pair of neighbouring trips in a chain satisfies the following local property: the arrival of the first one and the departure of the second one are located in the same hollow. It is worth noting that the trip-types in chains (1.2), (1.3) constitute an Euler cycle over the corresponding component of the Linis Graph.

It is convenient now to introduce the notion of a regular chain: it is an infinite, periodic chain containing trips of all types appearing in the schedule; between any two trips of the same type there is exactly one trip of each other type. For example, chains (1.2), (1.3) are regular for the schedule which contains only trip-types α, β, γ, δ.

It is not possible to decompose the given schedule into five regular chains. However, there is a possibility to find a decomposition into six regular chains. Indeed, a regular chain could be constructed if there were edges between b_1 and b_2 (thus providing that the Linis Graph becomes connected). In order to achieve this, let us add to the schedule a new "dummy" trip-type ϵ from b to b shown by dotted lines on the top of Fig. 1. Carrying out this trip means in practical terms that a vehicle staying at b just before t_1 (the "departure" time of ϵ) must stay at b until the "arrival" of ϵ at t_2.

Now $d_b(t)$ has only *one* hollow-type and the corresponding Linis Graph becomes connected (see Figs. 1 and 3). The following Euler cycle for this graph is admissible regarding the precedence relations between arrivals and departures: $\psi \to \varphi \to \epsilon \to \beta \to \gamma \to \delta \to \alpha$. In this way we arrive at a regular chain

$$\psi_{(1)} \to \varphi_{(2)} \to \epsilon_{(3)} \to \beta_{(3)} \to \gamma_{(4)} \to \delta_{(5)} \to \alpha_{(6)} \to \psi_{(7)} \to \tag{1.5}$$

having period $6T$. It is clear that in order to carry out all trips in the schedule, except for some

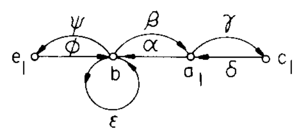

Fig. 3. The Linis Graph after adding dummy trips.

finite number of trips in the beginning, this chain should be "copied" five times by adding $r = 1, 2, 3, 4, 5$ to trip lower indices. The fleet size is now six. So, in order to decompose the schedule into a system of regular chains, it was necessary to increase the minimal fleet size by one and to "glue" together the components of the Linis Graph. This is always possible if both components have vertices representing the same terminal.

Our exposition will be as follows. Section 2 contains a series of definitions and two auxillary propositions which are necessary from proving theorems in Section 3. This section contains Theorem 1 which states that a connected Linis Graph of a periodic schedule permits constructing an Euler cycle which satisfies specific precedence relations. The expansion of this cycle in a real time gives a "master chain" in which the arrival and departure times of each pair of adjacent trips are located in one hollow. This master chain serves as a basis for obtaining an optimal fleet consisting of D regular (and similar) chains (Theorem 2). Section 3 is concluded by Theorem 3 which generalizes Theorem 2 for the case where the Linis Graph has several connected components. Section 4 considers the problem of obtaining a homogeneous fleet consisting of only one group of twin-type chains for the case when the Linis Graph is disconnected. Section 5 contains some concluding remarks concerning the relation between the results obtained and so-called center problem.

2. DEFINITIONS AND AUXILLARY PROPOSITIONS

2.1 *Trip, periodic schedule, deficit function, peak-zones and hollows, regular chains*

For the reader's convenience we repeat with slight changes several definitions from Gertsbach and Gurevich (1977).

Let A be a finite non-empty set of elements (called *terminals*). Letters $a, b, \dots$ will denote terminals. A trip is a 4-tuple $p = (p1, p2, p3, p4)$ where $p1$, $p2 \in A$, $p3$, $p4$ are real numbers such that $0 \leq p3 < p4$; $p1$ and $p2$ represent departure and arrival terminals respectively; $p3$ is the departure time (from $p1$), $p4$ is the arrival time (to $p2$).

For a set X, $\#X$ will denote the power of this set, i.e. the number of elements in it.

Let S be a set of trips (finite or infinite). For this set, define the following three functions for each terminal $a \in A$.

$$\begin{aligned} &\text{departure function: } d_1(t, a) = \#\{p \in S\colon p1 = a \wedge p3 \leq t\},\\ &\text{arrival function: } d_2(t, a) = \#\{p \in S\colon p2 = a \wedge p4 \leq t\}.\\ &\text{deficit function: } d_a(t) = d_1(t, a) - d_2(t, a). \end{aligned} \tag{2.1}$$

The first function, $d_1(t, a)$ is equal to the number of departures from terminal a, up to instant t (including the departure at this finite moment). $d_2(t, a)$ has a similar meaning with respect to arrivals at terminal a.

A set S of trips is called a *schedule* if it satisfies the following conditions:

$$\left.\begin{aligned} &\text{(i) } \{p \in S\colon p1 = a \wedge p3 \leq t\} \text{ is finite for every } a \text{ and } t.\\ &\text{(ii) } d_a(t) \text{ is bounded for every terminal } a. \end{aligned}\right\} \tag{2.2}$$

Both of these conditions are quite natural: (i) restricts the total number of departures and, therefore, the number of arrivals in every terminal on a fixed finite time span; (ii) restricts the difference between arrivals and departures over an infinite time span.

The graph of the deficit function for terminal a can be obtained in the following manner: add one at every departure from a and subtract one at every arrival in a.

The following properties of a deficit function are almost obvious: for every terminal a, $d_a(t)$ is stepwise, right-continuous; it has a maximum value $D(a) = \max_{t\geq 0} d_a(t)$ and $d_a(0) \geq 0$.

We say that trip p belongs to interval $[t_1, t_2)$ if its departure time lies within this interval: $t_1 \leq p3 < t_2$. Schedule S will be called periodic with period T if for each trip $(p1, p2, p3, p4) \in S$ the trip $(p1, p2, p3 + T, p4 + T) \in S$. In other words, each trip is "copied" infinitely many times by shifting it to the right by time T.

Let S be T-periodic. We will number trips belonging to $[0, T)$ by an upper index: $p^1_{(1)}, p^2_{(1)}, \ldots, p^n_{(1)}$; the "copy" of $p^r_{(1)}$ belonging to $[(k-1)T, kT)$ is denoted $p^r_{(k)}$, $k \geq 1$. Trips in S having a fixed upper index i form a trip-type e_i: $e_i = \{p^i_{(k)}, k \geq 1\}$. Precedence relation between two trips p and q (denoted $p \to q$) means that $p2 = q1$ and $p4 \leq q3$.

A set of trips C is called a *chain* if all trips in C can be arranged in a sequence $p \to q \to r \to s \to w \to \cdots$. Physically, each chain is a sequence of trips which might be performed by one vehicle.

A set F of chains is called a *fleet* over S iff each trip in S belongs to one and only one chain in F (the schedule is decomposed into chains constituting the set F). The subject of greatest interest is a decomposition containing the minimal possible number of chains. The *minimal fleet size* (MFS) of the given schedule S is defined as follows:

$$\mathrm{MFS} = \min\{\#F : F \text{ is a fleet over } S\}. \tag{2.3}$$

A chain itself is a schedule. The chain C is called *τ-periodic* iff $(p1, p2, p3, p4) \in C \Rightarrow (p1, p2, p3 + \tau \cdot T, p4 + \tau \cdot T) \in C$. A chain C is called τ-regular in S if (i) it is τ-periodic and infinite; (ii) it contains trips of each type; (iii) between any two trips of the same type there is exactly one trip of each other type. For example, the sequence (1.6) is a 6-regular chain in schedule $S = \{\alpha_{(i)}, \beta_{(i)}, \gamma_{(i)}, \delta_{(i)}, \varphi_{(i)}, \psi_{(i)}, \epsilon_{(i)}, i \geq 1\}$. The deficit function $d_a(t)$ will be termed *T-periodic for $t \geq t^*$* if $d_a(t) = d_a(t + T)$ for $t \geq t^*$.

From Lemma 2.1 in Gertsbach and Gurevich (1977) it follows that for a T-periodic schedule, all deficit functions should be also T-periodic.

Definition of a peak-zone of $d_a(t)$. A peak-zone of $d_a(t)$ is a closed interval L satisfying the following conditions: L has at least two points, there are no arrivals into a nor departure from a during $(\min L, \sup L)$, $d_a(t) = D(a)$ for $\min L \leq t < \sup L$; either $\min L = 0$ or $\min L > 0$ and there is $\epsilon > 0$ with $d_a(t) < D(a)$ for $[\min L - \epsilon, \min L)$; if $\sup L = \max L < \infty$ then there is $\epsilon > 0$ with $d_a(t) < D(a)$ for $t \in [\max L, \max L + \epsilon)$.

Definition of a hollow of $d_a(t)$. If L is the first peak-zone of $d_a(t)$ and $0 < \min L$, then $[0, \min L]$ is a *hollow* of $d_a(t)$. If L_1, L_2 are successive peak-zones of $d_a(t)$ (with L_2 following L_1) then $[\max L_1, \min L_2]$ is a *hollow* of $d_a(t)$ (it may be that $\max L_1 = \min L_2$). If L is the last peak-zone of $d_a(t)$ and $\sup L = \max L < \infty$, then $[\max L, \infty)$ is a *hollow* of $d_a(t)$.

Only T-periodic schedules will be considered in this paper.

We say that two hollows of $d_a(t)$ are of the *same type* if one of them can be obtained from the other by one or several shifts by T. We denote hollows of $d_a(t)$ by symbols a_i, $i \geq 1$, and assign the same symbol to hollows of the same type. Consider a typical interval $[kT, (k+1)T]$ and number the hollows within it from the left to the right by $a_1, a_2, \ldots$. If kT is in a hollow, say h (which implies that also $(k+1)T$ is in a hollow, say h') denote h and h' by the same symbol a_1. Figure 4 illustrates the principle of numbering hollows.

2.2 *The Linis Graph. Balanced functions and balanced graph. f, g-Euler cycle*

The object of our main concern will be a special construction, an oriented graph termed the Linis Graph.

Definition of the Linis Graph. The Linis Graph of schedule S, $LG(S)$ is a finite oriented graph $LG(S) = (V, E)$ where the vertices are hollow types of deficit functions and the edges are

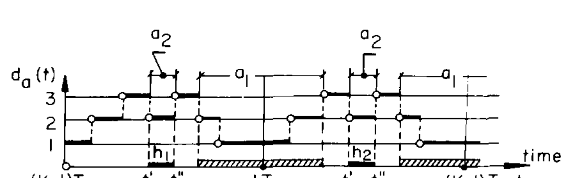

Fig. 4. Hollow types of $d_a(t)$: hollows h_1 and h_2 are of the same type.

trip-types; if trip $e = (e1, e2, e3, e4) \in S$ is of type $e_i \in E$, $e3$ is located in a hollow of type $e1_m$, $e4$ is located in a hollow of type $e2_j$, then edge e_i comes out from vertex $e1_m$ and enters vertex $e2_j$. (See Introduction for examples of $LG(S)$).

A nonformal description of the Linis Graph is given in the paper of Linis and Maksim (1967), Section 4.

In $LG(S)$ each vertex $v \in V$ has even degree: the number of edges entering v is equal to the number of edges coming out from v.

Assume that $LG(S)$ is connected. We want to construct a Euler cycle (i.e. a closed tour containing all edges, each only once) satisfying the following condition: if e, e' are successive edges in the cycle and p, p' are trips of trip-types e, e', respectively, $p4$ and $p'3$ are located in the same hollow, then

$$p4 \leq p'3. \tag{2.4}$$

Specific relationship between arrival-departure times of trips which are located in one hollow motivate the following:

Definition of balanced functions. Let I, J be finite sets, R be the set of reals and $f: I \to R$, $g: J \to R$ (i.e. f, g are finite real-valued functions with domains I, J, respectively). f and g are *balanced* if either I and J are empty or (i) $\#I = \#J$ and (ii) $ld_{fg}(t) = \#\{j: g(j) \leq t, j \in J\} - \#\{i: f(i) \leq t, i \in I\}$ is negative in the interval [min $rngf$, max $rngg$) and zero outside it. $ld_{jg}(t)$ will be termed the *local deficit function* of f, g. In the case when I and J are edges entering vertex v and coming from v, respectively, (i) means that the number of edges entering v is equal to the number of edges leaving v. The values of f, $f(1), \dots, f(k)$, represent arrival times at v and the values of g, $g(1), g(2), \dots, g(k)$ represent departure times from v. (ii) is a precise description that these times are located in one hollow: in any point between the earliest arrival and the latest departure the local deficit function is negative.

Note that if f and g are balanced, then min $rngf \leq$ min $rngg$ and max $rngf \leq$ max $rngg$. Instead of "$ld_{fg}(t)$ is zero outside [min $rngf$, max $rngg$)" it suffices to request "the local deficit function is nonpositive everywhere".

We use the notation $f' = f \mid I - \{\alpha\}$ for the restriction of f onto the domain $I - \{\alpha\}$. Assume a pair of arrival-departure times is deleted from a hollow. It is important for our further exposition that all the arrival-departure times left still belong to one hollow.

Lemma 1. If $f: I \to R$, $g: J \to R$ be finite balanced functions, $\alpha \in I$, $\beta \in J$, $f' = f|I - \{\alpha\}$, $g' = g|J - \{\beta\}$, $f(\alpha) \leq g(\beta)$. Then f', g' are balanced iff $f' = g' = \emptyset$ or $l\,d_{fg}(t) \leq -2$ in the interval $L = [\min rngf', \max rngg') \cap [f(\alpha), g(\beta))$.

The proof is not complicated and we omit it. Let us explain the lemma by an example. Consider a hollow-configuration shown in Fig. 5. After deleting $f(\alpha) = p_14$ and $g(\beta) = p_53$ one obtains $L = [p_24, p_63) \cap [p_14, p_53) = [p_24, p_53)$ and the condition of Lemma 1 is violated. Dotted lines show the deficit function after deleting p_14, p_53 and it is seen that the original hollow had disintegrated into two hollows. If, on the contrary, one takes $f(\alpha) = p_24$, $g(\beta) = p_33$ then $L = [p_24, p_33)$, $l\,d_{fg}(t) \leq -2$ for $t \in L$ and f', g' again create a hollow(the change in the form of the original deficit function is shown by lined area).

The next claim is that it is always possible to delete a pair $f(\alpha)$, $g(\beta)$, $f(\alpha) \leq g(\beta)$ by keeping f', g' balanced.

Corollary 2. Let $f: I \to R$, $g: J \to R$ be finite balanced functions. Then for each $\alpha \in I$ there is $\beta \in J$ such that $f(\alpha) \leq g(\beta)$ and $f' = f \mid I - \{\alpha\}$, $g' = g \mid J - \{\beta\}$ are balanced.

The idea of the proof is simple: take for the arrival time $f(\alpha)$ the departure time $g(\beta)$ which is

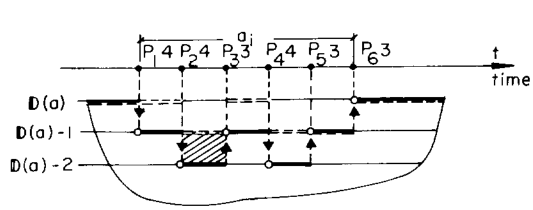

Fig. 5. A fragment of a deficit function $d_a(t)$ and a hollow a_i formed by arrivals at times p_14, p_24, p_44 and by departures at times p_33, p_53, p_63.

the nearest departure time to $f(\alpha)$ from the right; delete $f(\alpha)$, $g(\beta)$ and the original hollow remains one hollow. One can check that by means of Fig. 5.

Definition. Let $G=(V, E)$ be a finite oriented graph, $f: E \to R$, $g: E \to R$. f, g will be called *balanced* on G if for each $v \in V$, $f|\{e \in E: e$ enters $v\}$ and $g|\{e \in E: e$ leaves $v\}$ are balanced.

We will consider often an oriented graph G together with a pair of balanced functions on it. It is convenient to give the following.

Definition. Let $G=(V, E)$ be a finite oriented graph, $f: E \to R$, $g: E \to R$ be balanced on G. Then the triple (G, f, g) will be called a *balanced graph.* If G is connected the above triple is termed a *connected balanced graph.*

Definition. Let $(G=(V, E), f: E \to R, g: E \to R)$ be a balanced graph and $C=(e_1, e_2, \ldots, e_n)$ be a sequence of distinct edges of G. C is called an *f, g-path* if e_i enters the vertex v left by e_{i+1} and $f(e_i) \leq g(e_{i+1})$, $i=1, \ldots, n-1$. If, in addition, e_n enters v left by e_1 and $f(e_n) \leq g(e_1)$ C is called an *f, g-cycle.* If $n=\#E$ (i.e. C contains all edges of E) and C is a cycle then it will be termed an *f, g-Euler cycle.*

3. CONSTRUCTING AN f,g-EULER CYCLE. REGULAR CHAINS. OPTIMAL TWIN-TYPE FLEET.

3.1 *f, g-Euler cycle*

Consider a balanced graph $(G=(V, E), f: E \to R, g: E \to R)$. We assume in subsections 3.1–3.3 that G is connected. Our goal is to find an f, g-Euler cycle whenever it is possible.

The proof of the existence of a Euler cycle for a connected oriented graph with each vertex of even degree is based on the following facts (see, e.g. Berge (1958), Chap. 17): it is always possible to find some cycle C_0; after deleting C_0 from the original graph, the connected components of the remaining graph can be arranged by the inductive hypothesis into cycles $C_1, \ldots, C_n$; these cycles can be "glued" together with C_0 into one cycle which is the desired Euler cycle. Our proof goes along the same lines. A complication arises in proving that it is possible to glue together two f, g-cycles with a common vertex.

Lemma 3. Let (i) $(G=(V, E), f: E \to R, g: E \to R)$ be a balanced graph;

(ii) $C_1=(e_1, e_2, \ldots, e_k)$ and $C_2=(e_1', e_2', \ldots, e_m')$ be two f, g-cycles with no common edges;

(iii) C_1 and C_2 pass through a vertex $v \in V$;

(iv) any edge incidental v belongs to C_1 or C_2.

Then C_1 and C_2 can be concatenated into one f, g-cycle.

The outline of the proof is the following. Assume there are two pairs of adjacent edges (e_i, e_{i+1}) and (e_k', e_{k+1}') belonging to C_1, C_2, respectively, incidental to v such that intervals $[f(e_i), g(e_{i+1}))$ and $[f(e_k'), g(e_{k+1}'))$ have a nonempty intersection (see, e.g. Fig. 6). Assuming without loss of generality that $f(e_i) \leq f(e_k')$ we can obtain one cycle by "jumping" from e_i to e_{k+1}', returning to v along C_2 and "jumping" back to C_1 from e_k' to e_{i+1} (see Fig. 6). It is always possible to find two pairs of edges with the above property. Indeed, assume this is not true and there is a point t_0 and $\epsilon>0$ such that all arrivals to v and departures from v in C_1 are in $(-\infty, t_0-\epsilon)$ and all arrivals to v and departures from v in C_2 are in $(t_0+\epsilon, \infty)$. Let f_0 (respectively g_0) be the restriction of f (respectively g) onto the set of edges entering (respectively leaving) v. Note that f_0, g_0 is a pair of balanced functions. The corresponding local deficit function is $l\,\mathrm{d}_{f_0g_0}(t)$. t_0 is located in the interval between the earliest arrival to v and the latest departure from v. But $l\,\mathrm{d}_{f_0g_0}(t_0)=0$ which is impossible.□

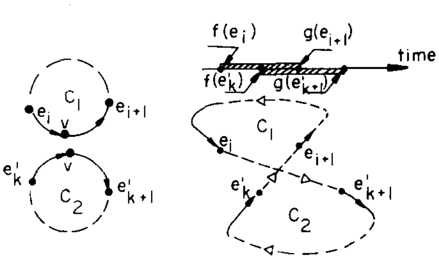

Fig. 6. How to glue together two f, g-cycles.

Theorem 1. Let $(G=(V,E), f:E\to R, g:E\to R)$ be a connected balanced graph. Then there is an f, g-Euler cycle.

Proof. By induction on $\#E$. For $\#E=0$ it is obvious. Let the theorem be true for $\#E\leq n$ and consider a connected balanced graph with $\#E=n+1$. Take an arbitrary $v_0\in V$ and an edge e_0 coming out from v_0. We want to construct an auxillary f, g-cycle containing e_0. Let $(e_0, e_1, \dots, e_m)$ be an f, g-path such that $f\,|\,\{e\in E: e \text{ enters } v,\ e\neq e_i,\ i<m\}$ and $g\,|\,\{e\in E: e \text{ leaves } v,\ e\neq e_i,\ 0<i\leq m\}$ are balanced for each v. e_m enters some vertex v. By Corollary 2 there is always an edge e' leaving v such that $f\,|\,\{e\in E: e \text{ enters } v,\ e\neq e_i,\ i\leq m\}$ and $g\,|\,\{e\in E: e \text{ leaves } v,\ e\neq e_i,\ 1\leq i\leq m,\ e\neq e'\}$ remain balanced. If $e'\neq e_0$, define $e_{m+1}=e'$ and proceed. Otherwise stop and the path $C_0=(e_0,\dots,e_m)$ is the desired auxillary f, g-cycle. If $\#C_0=n+1$ the theorem is proved. Otherwise define $E'=E-C_0$, $V'=\{v: v\in V,\ v \text{ incidental to } e\in E'\}$ and consider $G'=(V',E')$. It splits into connected components $G_i=(V_i,E_i)$, $i=1,\dots,k$. It is easy to see that for each $i=1,\dots,k$ $f\,|E_i$ and $g\,|E_i$ are balanced on G_i. By the induction hypothesis there is an f, g-Euler cycle for each G_i, say C_i. There is a vertex v belonging to C_0 and C_1 because G is connected. Now use Lemma 3 whose conditions are satisfied to concatenate C_0 with C_1. Then concatenate C_2 with $C_0\cup C_1$, etc. until an f, g-balanced Euler cycle is obtained.□

Theorem 1 suggests the following algorithm for constructing an f, g-Euler cycle. Construct an auxillary cycle C_0 and delete all its edges from G; for the remaining graph construct another auxiliary cycle C_1, etc. Assume that a sequence $C_0, C_1, \dots, C_r$ is obtained until G gets exhausted. Then glue together C_r with C_{r-1} into one cycle, say C'_{r-1} using Lemma 3, glue together C'_{r-1} with C_{r-2}, etc. until all C_i will be concatenated into one f, g-Euler cycle.

3.2 *Constructing a balanced Linis Graph*

Let us construct the Linis Graph $LG(S)$ for the T-periodic schedule S. Our goal is to define balanced functions $f:E\to R$, $g:E\to R$ representing the arrival/departure times.

Let h be some hollow of hollow type a_i of the deficit function $d_a(t)$ and I_h, J_h be the sets of trips whose arrival and departure times, respectively, are located in h. Let $t_0=\min\{p4: p\in I_{a_i}\}$. For each $p\in I_h$ let us define the *local arrival time at* a_i as $p4^*=p4-t_0$ and for each $q\in J_h$, let us define the *local departure time from* a_i as $q3^*=q3-t_0$. It is clear that if h_1, h_2 are different hollows of hollow type a_i and $p^k_{(l)}$, $p^k_{(m)}$ are different trips of the same trip type p^k with arrival (departure) times located in h_1, h_2, respectively, then their local arrival (departure) times are equal. Therefore, the local arrival/departure times are defined for each hollow type a_i in a unique way. Define now I_{a_i}, J_{a_i} as the sets of trip-types whose arrival and departure times, respectively, are located in some hollow h of hollow type a_i. Define functions $f_{a_i}:I_{a_i}\to R$, $g_{a_i}:J_{a_i}\to R$ as follows: if $p^k\in I_{a_i}$, then $f_{a_i}(p^k)=p^k4^*$; if $q^k\in J_{q_i}$ then $g_{a_i}(q^k)=q^k3^*$.

Denote $I=\cup_{a\in A}\cup_i I_{a_i}$, $J=\cup_{a\in A}\cup_i J_{a_i}$ and define $f:I\to R$, $g:J\to R$ in such a way that their restriction onto I_{a_i}, J_{a_i} coincide with f_{a_i}, g_{a_i}. Clearly, f, g will be balanced on $LG(S)$. The triple $(LG(S)=(V,E), f:I\to R, g:J\to R)$ is a f, g-balanced graph termed a *balanced Linis Graph.*

3.3 *Constructing regular chains*

Let us construct a balanced Linis Graph and a f, g-Euler cycle. Without loss of generality it might be written in the following chain form

$$p^1\to p^2\to\cdots\to p^n, \tag{3.1}$$

where p^i is a trip-type. Each trip-type in S appears in (3.1) once and only once.

Now construct an infinite periodic chain C^0 according to the following rules.

(1) Take trip $p^1_{(i_1)}$ (belonging to the i_1th period on the periodic part of the deficit functions); let $p^1_{(i_1)}4\in h$; join to $p^1_{(i_1)}$ the trip of trip-type p^2 whose departure time is located in the same hollow h. Let this be trip $p^2_{(i_2)}$. Clearly, $p^1_{(i_1)}4\leq p^2_{(i_2)}3$ because (3.1) is an f, g-Euler cycle. Proceed until a trip of trip-type p^n is picked. Denote the obtained sequence of trips by $C_1{}^0$:

$$C_1{}^0=\{p^1_{(i_1)}\to p^2_{(i_2)}\to\cdots\to p^n_{(i_n)}\}. \tag{3.2}$$

$C_1{}^0$ is a chain. We call it a *master chain.* Let $p^n_{(i_n)}4\in h'$. Consider the trip-type p^1 departing from h' and let it be $p^1_{(i_1+\tau)}$.

(2) Define

$$C_r^0 = \{p^1_{(i_1+\tau\cdot r)} \to \cdots \to p^n_{(i_n+\tau\cdot r)}\},\ r \geq 0, \tag{3.3}$$

and

$$C^0 = \{C_1^0 \to C_2^0 \to \cdots \to C_r^0 \to C_{r+1}^0 \to \cdots\}. \tag{3.4}$$

Obviously, C^0 is a τ-regular chain in S (see Subsection 1.2). Now obtain $\tau - 1$ additional τ-regular chains in S, $C^1, \ldots, C^{\tau-1}$, where C^k is obtained from C^0 by adding the integer k to all lower indices of trips in C^0, $k = 1, \ldots, \tau - 1$:

$$C^k = \{C_1^k \to C_2^k \to \cdots \to C_r^k \to C_{r+1}^k \to \cdots\}, \tag{3.5}$$

$$C_r^k = \{p^1_{(i_1+k+\tau\cdot r)} \to \cdots \to p^n_{(i_n+k+\tau\cdot r)}\}. \tag{3.6}$$

In this way we arrive at a set F^0 of τ τ-regular chains in S: $F^0 = \{C^i, 0 \leq i \leq \tau - 1\}$. Their union is a schedule denoted S^0: $S^0 = \cup_{i=0}^{\tau-1} C^i$. Clearly, ($S$–$S^0$) is finite. We say that S^0 contains *almost all* trips of S.

It is possible to add the trips in S–S^0 to chains of F^0 "from the left" to obtain a fleet over S of size τ. It can be done by extending to the left the chains in F^0 by adding to them the missing trips of S–S^0 (see Example 1). In this way we arrive at a system of chains $F = \{\hat{C}^k, 0 \leq k \leq \tau - 1\}$, where $\hat{C}^k$ is obtained from C^k by extending it to the left. Clearly, F is a fleet over S and each chain in F is τ-regular.

F will be termed a set of *twin-type* chains with respect to S or a *twin-type* fleet over S. $\hat{C}^0, \ldots, \hat{C}^{\tau-1}$ will be called *twin-type* chains.

Theorem 2. Suppose (i) S be a T-periodic schedule; (ii) The Linis Graph of S is connected. Then:

(1) There are a positive integer τ and a set F^0 of τ τ-regular chains $F^0 = \{C^i, 0 \leq i \leq \tau - 1\}$ such that $\cup_{i=0}^{\tau-1} C^i$ contains almost all trips in S;

(2) C^k is obtained from C^{k-1} by adding 1 to lower indices of trips in C^{k-1}, $k = 1, \ldots, \tau - 1$;

(3) F^0 can be extended to an optimal twin-type fleet F over S of size τ.

Proof. We must prove only that τ is equal to the total deficit D (see (1.1)).

(1°) Assume that $\max_{t \geq 0} d_a(t) = D(a) > 0$ and let $F(a) \subseteq F$ be the set of chains whose first trip departs from a. Define t^* as the latest departure time of the first trips in $F(a)$ and let

$$t_a = \min\{t\colon d_a(t) = D(a) \text{ and } t \geq t^*\}$$

(t_a is the left end of a peak-zone of $d_a(t)$ which is the nearest to t^* from the right).

Let h be a hollow of $d_a(t)$. Define $\mathrm{Arr}(h) = \{p\colon p2 = a, p4 \in h\}$, $\mathrm{Dep}(h) = \{p\colon p1 = a, p3 \in h\}$.

Let $p_1 \in \hat{C}^i$, $p_12 = a$, $p_14 \leq t_a$ and p_2 be the direct successor of p_1 in $\hat{C}^i$. Let us prove that $p_23 \leq t_a$. Assume that $p_23 > t_a$. Then $p_23 \in h_1$ where h_1 is a hollow of $d_a(t)$ which is "balanced": $\#\mathrm{Arr}(h_1) = \#\mathrm{Dep}(h_1)$. Therefore, there is a trip $p_3 \in \mathrm{Arr}(h_1)$ such that its direct successor in F, say $p_4 \notin \mathrm{Dep}(h_1)$. Continuing this reasoning one obtains an infinite sequence of trip pairs $\{(p_k, p_{k+1}), k \geq 1\}$ such that p_{k+1} is the direct successor of p_k in F but the arrival time of p_k and the departure time of p_{k+1} are not located in one hollow. This means that in one of the chains of F^0 there will appear a pair of adjacent trips, say $p \to q$ such that $p4$ and $q3$ are located in *different* hollows. But this contradicts properties of F^0.

(2°) By the definition of $d_a(t)$ and F $d_a(t_a) = \Sigma_{\hat{C}^i \in F(a)}\{\#$ of dep. from a on $[0, t_a]$ in $\hat{C}^i$—$\#$ of arr. to a in $\hat{C}^i$ on $[0, t_a]\} + \Sigma_{\hat{C}^i \in F - F(a)}\{\#$ of dep. from α on $[0, t_a]$ in $\hat{C}^i$—$\#$ of arr. to a on $[0, t_a]$ in $\hat{C}^i\}$. Each $\hat{C}^i \in F - F(a)$ *begins* with an arrival at a; if it takes place at t', $t' \leq t_a$, then it is followed in F by a departure from a at $t'' \leq t_a$, as proved in 1°. Thus the second sum is zero.

If a chain begins with p, $p1 = a$ and each arrival to a on $[0, t_a]$ in it is followed by a departure from a on $[0, t_a]$ (which is exactly our case as proved in 1°), then each bracket in the first sum is equal to 1. Thus, $d_a(t_a) = \#F(a)$.

Let $b \in A$ be such that no chain begins with a trip departing from b. Then $d_b(t) \leq 0$ and thus $\max_{t \geq 0} d_b(t) = d_b(0) = 0$. Now

$$D = \sum_{a \in A} D(a) = \sum_{\{a:\, D(a)>0\}} D(a) = \sum_{\{a:\, D(a)>0\}} d_a(t_a) = \sum_{a \in A} (\#F(a)) = \#F = \tau. \square$$

3.4 *Disconnected Linis Graph*

This case does not pose difficulties and the following generalization of Theorem 2 is valid:

Theorem 3. Suppose (i) S be a T-periodic schedule; (ii) The Linis Graph of S is disconnected and contains s connected components $LG_i(S) = (V_i, E_i)$, $i = 1, \dots, s$.

Then:

(1) S is partitioned into s subschedules $\hat{S}_1, \dots, \hat{S}_s$, $\hat{S}_i$ containing all trips in S of trip-types constituting the set E_i, $i = 1, \dots, s$;

(2) For each i, $i = 1, \dots, s$, there is a positive integer τ_i and a system of τ_i chains $F^0(i) = \{C^0(i), \dots, C^{\tau_i - 1}(i)\}$ such that (a) $S_i = \cup_{k=0}^{\tau_i - 1} C^k(i)$ contains almost all trips in $\hat{S}_i$; (b) Each chain in $F^0(i)$ is τ_i-regular with respect to S_i; (c) $C^k(i)$ is obtained from $C^{k-1}(i)$ by adding 1 to all lower indices of trips in $C^{k-1}(i)$, $k = 1, \dots, \tau_i - 1$;

(3) For each $i = 1, \dots, s$ there is a twin-type fleet $F(i)$ over $\hat{S}_i$ consisting of τ_i chains;

(4) $F = \{F(i), i = 1, \dots, s\}$ is an optimal fleet over S, i.e. $\Sigma_{i=1}^{s} \tau_i = \text{MFS}$.

To prove Theorem 3 let us define s pairs of balanced functions f_m: $I_m \to R$, $J_m \to R$, $m = 1, 2, \dots, s$, by restricting f and g onto the domains I_m and J_m representing local arrival and departure times for the subschedule $\hat{S}_m$ containing trip-types of the set E_m. Triples $(LG_i(S) = (V_i, E_i), f_i, g_i)$, $i = 1, \dots, s$ are f_i, g_i-balanced graphs. We construct for them twin-type fleet $F(i)$ over $\hat{S}_i$ exactly as it was described in Theorem 2. Claims (1), (2) and (3) follow immediately. To prove (4) we repeat first word for word (1°) of Theorem 2 and (2°) of this Theorem with obvious changes in formulas for $d_a(t_a)$. Denote by $F(i, a)$ the subset of chains of $F(i)$ which start with departure from a. Similarly for 2°, one obtains that $d_a(t_a) = \#F(a)$ where $F(a) = \cup_{i=1}^{s} F(i, a)$ and that the total deficit $D = \#F = \Sigma_{i=1}^{s} \tau_i$. $\square$

Remark 1. It is clear that if an optimal twin-type fleet over S does exist then the Linis Graph of S is connected because each chain in the fleet "visits" hollows of all types. Therefore, the connectedness of the Linis Graph is a necessary and sufficient condition for the existence of an optimal twin-type fleet over S.

Remark 2. The optimal twin-type fleet is not unique if the Linis Graph has more than one f, g-Euler cycle: each such cycle can serve as a basis for a master chain which, being expanded in real time and replicated, produces an optimal twin-type fleet.

4. MORE ABOUT TWIN-TYPE FLEET OVER S WHEN $GL(S)$ IS DISCONNECTED

Let us examine when a twin-type fleet over S can exist if $GL(S)$ is disconnected and has several connected components $GL_i(S) = (V_i, E_i)$, $i = 1, \dots, s$. It is clear that a twin-type fleet (not necessary an optimal one) over a schedule exists if and only if this schedule has a connected Linis Graph. So, we can ask if there is a method to change the given schedule S in order to obtain a new schedule with a connected Linis Graph. Of course, such a method does exist if we are allowed to add to S new *real* trips. We do not consider in this paper such an option. The only tool we have at our disposal is adding to S *fictitious*, "dummy" trips, as was done in Example 1 (see Introduction). Let us explain this point.

In this section we denote terminals by $a(1), a(2), \dots, a(m)$.

Assume that each of the sets $V_1, V_2, \dots, V_r$ contains at least one hollow type of terminal $a(j)$. Let $a_1(j), \dots, a_n(j)$ be hollow types of deficit function $d_{a(j)}(t)$ and let $[t(j), t'(j)]$ be a peak-zone of $d_{a(j)}(t)$ in its periodic part. Define the set $S(j)$ of dummy trips as follows:

$$S(j) = \{p(k) = (a(j), a(j), t(j) + kT, t'(j) + kT), k \geq 0\} \tag{4.1}$$

and consider the schedule $S' = S \cup S(j)$. All deficit functions of S' remain as they were for S except for $d_{a(j)}(t)$. The new deficit function of $a(j)$, $d_{a(j)}(t)$, has changed and now its maximum is $\hat{D}(a(j)) = \max d_{a(j)}(t) + 1$. One can easily check that for $t \in [t'(j) + kT, t(j) + kT + T)$, $\hat{d}_{a(j)}(t) < \hat{D}(a(j))$. This means that $d_{a(j)}(t)$ has in its periodic part hollows of *only one type* denoted $a_1(j)$.

What are the changes in $GL(S')$ in comparison with $GL(S)$? All vertices of type $a_i(j)$ within

$GL_i(S)$ are replaced by vertex $a_1(j)$; each edge $e \in E_i$ from $a_u(j) \in V_i$ to $v \in V_i$ is replaced by edge from $a_1(j)$ to v. The new subgraphs of $GL(S')$ corresponding to $GL_i(S)$, $i = 1, \ldots, r$ will have now one common vertex $a_1(j)$ and will form one connected component in $GL(S')$; the sets $V_1, V_2, \ldots, V_r$ now are "glued" together.

So, we have a tool to glue together connected components of the disconnected Linis Graph. Assume that $p \to w \to q$ is a fragment of some chain in the fleet over S' where $p, q \in S$ and w is a dummy trip. "To carry out" w means that a vehicle arrived in $p2$ at time $p4$ must remain in terminal $p2$ until time $q3$. S' is essentially the same schedule S; adding dummy trips to S means only increasing the time spent by vehicles in terminals. More formally, adding $S(j)$ to S forces us to build chains by overlapping some peak-zones of the deficit functions of the schedule S, which is, in fact, a violation of rules for constructing an optimal fleet over S.

Now we need to define an oriented graph called the graph of terminals.

Definition. The graph of terminals of schedule S, $GT(S)$ is a finite oriented graph $GT(S) = (V, E)$ where the vertices are terminals and the edges are trip-types; if trip $e = (a(j), a(k), e3, e4) \in S$ is of trip-type $e_i \in E$ then the edge e_i comes out from vertex $a(j)$ and enters vertex $a(k)$.

Now define a set $B \subseteq A$, $B = \{a(i)$: $d_{a(i)}(t)$ has more than one hollow type$\}$. For each $a(r) \in B$ define the corresponding sequence of dummy trips $S(r)$, similar to (4.1) and consider a new schedule $S^* = S \cup (\cup_{\{a(r) \in B\}} S(r))$. Its Linis Graph is $GL(S^*) = (V, E)$, where V is the set of terminals and E is the set of trip-types of S^*. The only difference between $GL(S^*)$ and $GT(S)$ is that in $GL(S^*)$ there are loop-type edges from $a(i)$ to $a(i)$, $a(i) \in B$, created by dummy trip-types. Therefore, $GL(S^*)$ is connected if and only if $GT(S)$ is connected. Therefore, there is a way to obtain a twin-type fleet over S iff $GT(S)$ is connected.

From now on let $GT(S)$ be connected. One can look for a most economic way to complement S by sets of dummy trips. This is illustrated by the following.

Example 2. Let $GL(S)$ have three connected components $GL_i(S) = (V_i, E_i)$, $i = 1, 2, 3$, and $V_1 = \{a_1(1), a_1(2)\}$, $V_2 = \{a_2(1), a_2(2), a_1(3)\}$, $V_3 = \{a_3(1), a_2(3)\}$. If we add to S the sequence $S(2)$ then V_1 and V_2 are glued together; adding $S(3)$ will result in a connected Linis Graph. On the other hand, adding only one sequence of dummy trips $S(1)$ would immediately lead to a connected Linis Graph.

In most practical problems the Linis Graph has not too many components and the smallest set of dummy trips can be found easily by means of a simple enumeration. Note that the set of potential candidates for T^* can be reduced by excluding terminals which appear in only one component of the Linis Graph.

Assume that adding Q dummy trip types in terminals $\{a(i_1^*), \ldots, a(i_Q^*)\} = A(Q)$ provides a connected Linis Graph and there is no other set of terminals of smaller size with the same property. We call $A(Q)$ the core set. In example 2 $A(Q) = \{a(1)\}$.

The above discussion can be summarized in the following

Theorem 4. Suppose (i) S is a T-periodic schedule; (ii) $GT(S)$ is connected; (iii) $GL(S)$ is disconnected; (iv) The core set of the Linis Graph of S is $\{a(1), \ldots, a(Q)\}$.
Then the optimal twin-type fleet over S contains $D + Q$ twin-type chains.

Proof. For each terminal $a(j)$ in the core set define the sequence of dummy trips (4.1) where $[t(j), t'(j)]$ is a peak zone of the deficit function $d_{a(j)}(t)$ in its periodic part. Consider the schedule

$$S^* = S \cup (\bigcup_{j=1}^{Q} S(j)). \tag{4.4}$$

By the definition of the core set, the Linis Graph of S^* is connected. Then by Theorem 2 it allows construction of a twin-type fleet whose size is equal to the total deficit of S^*, $D + Q$. It follows from the definition of the core set that the minimal number of sequences of dummy trips added to S in order to obtain a schedule with a connected Linis Graph is Q. Thus, the fleet of size $D + Q$ is optimal along all twin-type fleets over S.□

5. CONCLUDING REMARKS: THE CENTER PROBLEM; ADDING TRIPS TO THE SCHEDULE.

The following problem is of interest for planning transportation systems (see Gertsbach and Gurevich (1977), Section 3.2). Assume that there is a set of terminals A^* called *center*. It is necessary to construct a fleet of minimal size having the following property: each chain in the fleet must go through a terminal belonging to the center.

If there are connected components of the Linis Graph which do not contain representatives of A^*, then one can try to use the dummy trip technique (see Section 4) to glue such components to those which contain representatives of the center set. If it is possible to obtain a new Linis Graph with center representatives in each of its connected component then a fleet satisfying the center property can be found.

It can be said that the center problem may have no solution at all if the graph of terminals is disconnected. An example is $GL_i(S) = (V_i, E_i)$, $i = 1, 2$, $V_1 = \{a_1, a_2, b_1\}$, $V_2 = \{c_1, c_2, e_1\}$. $A^* = \{a, b\}$. It is clear, that it is not possible to obtain any fleet which would contain chains visiting terminals a, c, e, b. Of course, such chains could be obtained if we were allowed to add new trips to the original schedule, say from a to c and from c to a, thus providing a connection between components of the Linis Graph. Here we enter a new circle of problems connected with adding real trips to the schedule. We will make only some brief remarks.

Adding new trips can give many surprising results which might be classified into two main groups: changing the fleet size, including the decrease of it; changing the structure of the optimal chain decomposition. The only work in this area we know is a paper of Ceder and Stern (1982) in which the influence of adding deadheading trips on the fleet size has been investigated for a nonperiodic schedule.

Remark. In November 1981 we became aware about the existence of a recent paper by Linis and Maksim (1980), "The number of transportation units needed for a schedule", published in Moldavian Math. Collection (in Russian). Independently of us, Linis and Maksim obtained some of the results presented in this paper and, in particular, a theorem which is similar to our Theorem 1.

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