THE JOURNAL OF SYMBOLIC LOGIC
Volume 48, Number 1, March 1983

DECISION PROBLEM FOR SEPARATED DISTRIBUTIVE LATTICES

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Abstract. It is well known that for all recursively enumerable sets X_1, X_2 there are disjoint recursively enumerable sets Y_1, Y_2 such that $Y_1 \subseteq X_1$, $Y_2 \subseteq X_2$ and $Y_1 \cup Y_2 = X_1 \cup X_2$. Alistair Lachlan called distributive lattices satisfying this property *separated.* He proved that the first-order theory of finite separated distributive lattices is decidable. We prove here that the first-order theory of all separated distributive lattices is undecidable.

Introduction. A distributive lattice with 0 is *separated* if it satisfies the following separation property: for every x_1, x_2 there are $y_1 \leq x_1$ and $y_2 \leq x_2$ such that y_1, y_2 are disjoint (i.e. $y_1 \wedge y_2 = 0$) and $y_1 \vee y_2 = x_1 \vee x_2$. Alistair Lachlan introduced separated distributive lattices in [La] in connection with his study of the first-order theory of the lattice of recursively enumerable sets. He mentioned to me a question whether the first-order theory of separated distributive lattices is decidable. The answer is negative: in §2 a known undecidable theory is interpreted in the first-order theory of separated distributive lattices. The known undecidable theory is the first-order theory of the following structures: a Boolean algebra with a distinguished subalgebra. About undecidability of it see [Ru].

Actually the first version of the undecidability proof used the closure algebra CACD of Cantor Discontinuum, i.e. the Boolean algebra of subsets of Cantor Discontinuum with the closure operation. CACD is easily interpretable in the separated distributive lattice of functions f from Cantor Discontinuum into $\{0, 1, 2\}$ such that $f^{-1}(2)$ is clopen. By [GS1] a finitely axiomatizable essentially undecidable arithmetic reduces to the first-order theory of CACD, hence to the first-order theory of the mentioned separated distributive lattice of functions, hence to the first-order theory of separated distributive lattices. The last step is somewhat complicated by the fact that [GS1] does not interpret the standard model N of arithmetic in CACD. (Even though [GS2] reduces the second-order theory of N to the first-order theory of CACD, [GS3] proves that N cannot be interpreted in CACD.) However the Boolean algebra of subsets of Cantor Discontinuum with a distinguished subalgebra of clopen (closed and open) sets is easily interpretable in CACD. This way I came to use Rubin's result which made the undecidability proof very simple. From the other side the cited result of [GS1] can be used to reprove Rubin's theorem and

Received October 12 1980; revised August 30, 1981.

[1]The results were obtained and the paper was written during the Logic Year in the Institute for Advanced Studies of Hebrew University.

0022-4812/83/4801-0020/$01.40

to prove undecidability of the first-order theory of the following structures: a complete Boolean algebra with a distinguished subalgebra, see [GKM].

Lachlan proved that the first-order theory of finite separated distributive lattices is decidable by interpreting this theory in the monadic theory of finite trees. Since his proof is not published we bring out it here, see §4.

The author thanks Alistair Lachlan for useful discussions, and the anonymous referee for a thorough report contributing to further simplification of the undecidability proof.

§1. **Boolean pairs.** For the sake of brevity a pair $A \supseteq B$ of Boolean algebras, where B is a subalgebra of A, will be called a *Boolean pair*. We consider Boolean pairs as models for the language of Boolean algebras augmented by a unary predicate symbol.

THEOREM 1 (SEE [Ru]). *The first-order theory of Boolean pairs is undecidable* (*and creative*).

§2. **Interpreting Boolean pairs in separated distributive lattices.** Suppose $A \supseteq B$ is a Boolean pair. Let $L(A, B)$ be the sublattice of the direct product $A \times B$ consisting of pairs (a, b) with $a \geq b$.

Claim 1. $L(A, B)$ is separated.

PROOF. Given $(a_1, b_1), (a_2, b_2)$ in $L(A, B)$ set

$$a_3 = (a_1 - b_2) \vee b_1, \qquad b_3 = b_1,$$
$$a_4 = (a_2 - a_1) \vee (b_2 - b_1), \qquad b_4 = b_2 - b_1.$$

It is easy to check that $(a_3, b_3) \leq (a_1, b_1), (a_4, b_4) \leq (a_2, b_2), (a_3, b_3) \wedge (a_4, b_4) = 0$ and $(a_3, b_3) \vee (a_4, b_4) = (a_1, b_1) \vee (a_2, b_2)$. □

If $x \wedge y = 0$ and $x \vee y = z$ (in arbitrary lattice with 0) we say that x, y *partition* z and we write $x + y = z$.

Claim 2. The diagonal $\{(b, b): b \in B\}$ is definable in $L(A, B)$.

PROOF. This is the set of elements of $L(A, B)$ which have complements. □

Claim 3. $\{(a, 0): a \in A\}$ is definable in $L(A, B)$.

PROOF. The desired formula $\alpha(x)$ says that $y \leq x \rightarrow y = 0$ for any diagonal element y. □

Claim 4. $\{(b, 0): b \in B\}$ is definable in $L(A, B)$.

PROOF. The desired formula $\beta(x)$ says that for some diagonal element y, x is the greatest element such that $x \leq y$ and $\alpha(x)$. □

THEOREM 5. *The first-order theory of Boolean pairs is interpretable in the first-order theory of separated distributive lattices.*

PROOF. Above it has been shown that an arbitrary Boolean pair can be defined in a uniform way in a separated distributive lattice. □

Theorem 1 of §1 and the previous theorem give

COROLLARY 6. *The first-order theory of separated distributive lattices is undecidable* (*and creative*).

§3. **The monadic theory of finite trees.** A *tree* is a partial ordering T such that $\{y: y \geq x\}$ is linearly ordered in T for every $x \in T$. The *monadic language* of order is obtained from the first-order language of order by adding variables for point sets and atomic formulas of the form $x \in Y$. Quantification of the set variables is allowed. The *monadic theory* of a tree T is the theory of T in the monadic language of order when the set variables range over arbitrary subsets of T.

THEOREM 1. *The monadic theory of finite trees is decidable.*

Theorem 1 is well known. It is due to [Do] and [T & W]. It follows easily from the main result of [Ra]. Recall that the infinite binary tree *IBT* is the set of all words in alphabet $\{0, 1\}$ ordered as follows: $x \geq y$ if x is an initial segment of y. Rabin proved that the monadic theory of *IBT* is decidable. Finite subsets are definable in the monadic theory of *IBT*. Every finite submodel of *IBT* is a finite tree, every finite tree is embeddable into *IBT*. Thus the monadic theory of finite trees is decidable.

Let us sketch a more direct proof of Theorem 1. Every finite tree can be constructed from singleton trees by the following two operations:

(i) the disjoint union $T_1 + T_2$ of trees T_1, T_2, and

(ii) forming a new tree T^+ from a given tree T by adding a new element greater than any element in T.

Define n-theories with respect to [Lä]. Check that the n-theory of $T_1 + T_2$ is computable from the n-theories of T_1, T_2, and the n-theory of T^+ is computable from the n-theory of T. That gives two operations on n-theories. The collection of n-theories of finite trees is the least collection containing the n-theory of a singleton tree and closed under these two operations.

§4. Finite separated distributive lattices.

§4. Finite separated distributive lattices. For the sake of brevity a subset X of a tree T will be called a *cone* if $y < x \in X \rightarrow y \in X$ for $y \in T$. The collection Cone(T) of cones of T is closed under intersections and unions. Thus Cone(T) is a sublattice of the Boolean algebra of subsets of T. In particular Cone(T) is distributive.

Claim 1. For every finite tree T the lattice Cone(T) is separated.

PROOF. Suppose X_1, X_2 are cones of T and M is the set of maximal elements of $X_1 \cup X_2$. Let $M_1 = M \cap X_1$ and $M_2 = M - M_1$. Let Y_i be the cone generated by M_i for $i \leq 1$. Then $Y_i \subseteq X_i$ for $i \leq 1$ and Y_1, Y_2 partition $X_1 \cup X_2$. $\square$

An element x of a lattice L is called *join irreducible* if for every y, z in L, $x = y \vee z$ implies either $x = y$ or $x = z$. If L is a lattice with 0 let $JI(L)$ be the set of nonzero join irreducible elements of L ordered with respect to L.

Claim 2. If L is a separated distributive lattice then $JI(L)$ is a tree.

PROOF. Suppose x, y_1, y_2 are nonzero join irreducible elements of L and $x \leq y_i$ for $i \leq 1$. There are $z_1 \leq y_1$ and $z_2 \leq y_2$ such that $y_1 \vee y_2 = z_1 + z_2$. Without loss of generality $y_1 \neq z_1$. But $y_1 = (y_1 \wedge z_1) + (y_1 \wedge z_2)$, hence $y_1 = y_1 \wedge z_2$ and $y_1 \leq z_2 \leq y_2$. $\square$

Claim 3. Suppose L is a finite separated distributive lattice and $T = JI(L)$. Then L is isomorphic to Cone(T). $\square$

PROOF. The desired isomorphism assigns the join of X to each cone X of T.

Claims 1, 3 of this section and Theorem 1 of §3 give

THEOREM 4. *The first-order theory of finite separated distributive lattices is interpretable in the monadic theory of finite trees. Hence it is decidable.*

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